LOSS OF RECURRENCE IN REINFORCED RANDOM WALK

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Abstract

We show that a necessary and sufficient condition on the initial weighting w_i, $-\infty < i < \infty$, of a reinforced random walk $X_0, X_1, \ldots$, to guarantee that $P(\lim_{n\to\infty} |X_n| = \infty) = 0$, is that both $\sum_{i=1}^{\infty} w_i^{-2} = \infty$ and $\sum_{i=1}^{\infty} w_{-i}^{2} = \infty$. Together with an old result of T. E. Harris, this characterizes those initial weightings which, if unreinforced, correspond to a recurrent process, but which, if suitably reinforced, yield a process converging to infinity with positive probability, and in particular shows that there are such initial weightings.

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LOSS OF RECURRENCE IN REINFORCED RANDOM WALK

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In this note it is shown that a necessary and sufficient condition on the initial weighting a_i, $-\infty < i < \infty$, of a reinforced random walk $X_0, X_1, \ldots$, to guarantee that $P(\lim_{n\to\infty} |X_n| = \infty) = 0$, is that both $\sum_{i=1}^{\infty} a_i^{-2} = \infty$ and $\sum_{i=1}^{\infty} a_{-i}^{-2} = \infty$. Together with an old result of T. E. Harris, this characterizes those initial weightings which, if unreinforced, correspond to a recurrent process, but which, if suitably reinforced, yield a process converging to infinity with positive probability, and in particular shows that there are such initial weightings.

It is easily seen that any Markov process, with state space the integers and stationary transition probabilities $p_{i,j}$ which satisfy $p_{i,i+1} + p_{i,i-1} = 1$, $p_{i,i+1} > 0$, and $p_{i,i-1} > 0$, is associated with a set w_i, $-\infty < i < \infty$, of positive numbers, unique up to multiplication by constants, by the relations

$$p_{i,i+1} = 1 - p_{i,i-1} = \frac{w_i}{w_{i-1} + w_i}. \tag{1}$$

We will refer to w_i as the weight of the interval $(i, i+1)$. We have $p_{i,i+1}/p_{i,i-1} = w_i/w_{i-1}$, that is, the transition probabilities from i to $i+1$ and $i-1$ are proportional to the weights of the connecting intervals. For example, to get ordinary fair coin tossing random walk, the weights of all intervals must be equal.

Recently Coppersmith and Diaconis introduced a model in which the weight of each interval $(i, i+1)$ is initially 1 and is increased by 1 each time the process jumps across it, so that its weight at time n is one plus the number of indices $k \leq n$ such that (X_k, X_{k+1}) is either $(i, i+1)$ or $(i+1, i)$, where X_n is the state of the process at time n. Given $\{X_0 = i_0, \ldots, X_n = i_n\}$, X_{n+1} is either $i_n + 1$ or $i_n - 1$ with probabilities proportional to the weights at time n of $(i_n, i_n + 1)$ and $(i_n - 1, i_n)$ respectively. See [2] or [4] for a more detailed description of this and related processes, as well as a description of the methods of exchangeability theory and random walk in a random environment which may be used to study them.

In [2] the author considers the more general setting in which any positive initial weights are permitted and any nonnegative reinforcement which does not depend on the future is allowed. In this situation the methods mentioned above can not be used, and martingales are the principal tool. Formally, let I stand for the integers. We define a reinforced random walk (RRW) on the integers to be a sequence $X_0, X_1, \ldots = \boldsymbol{X}$ of integer valued random variables and a collection $\boldsymbol{w} = \{w(n,j),\ n = 0,1,2,\ldots,\ j \in I\}$ of positive random variables such that

i) $w(n,i) \geq w(n-1,i)$, $0 < n < \infty$, $i \in I$, with equality unless (X_{n-1}, X_n) is either $(i, i+1)$ or $(i+1, i)$, and

ii)
$$\begin{aligned} P(X_{n+1} = i+1 | X_n = i, \mathcal{F}_n) &= 1 - P(X_{n+1} = i-1 | X_n = i, \mathcal{F}_n) \\ &= w(n,i)/[w(n,i-1) + w(n,i)],\ n \geq 0,\ i \in I, \end{aligned}$$

where $\mathcal{F}_n = \sigma\{X_j,\ 0 \leq j \leq n,\ w(k,i),\ 0 \leq k \leq n,\ i \in I\}$. Usually we will just use $\boldsymbol{X}$ to designate a RRW, and do not explicitly mention the weights. We call $\{w(0,j)\colon\ j \in I\}$ the initial weights of $\boldsymbol{X}$, which from now on are assumed to be constants; if they are not, $\boldsymbol{X}$ may be analyzed by conditioning on them. We call $\boldsymbol{X}$ *initially fair* if all the initial weights are equal and *initially recurrent* if the Markov process with transition probabilities $p_{i,j}$ satisfying $p_{i,i+1} = 1-p_{i,i-1} = w(0,i)/(w(0,i-1)+w(0,i))$, $i \in I$, is recurrent. We set $R(\boldsymbol{X}) = \{j \in I\colon\ X_n = j \text{ for some } n\}$, and call $\{R(\boldsymbol{X}) = I\}$ the set where $\boldsymbol{X}$ is recurrent and $\{R(\boldsymbol{X}) \text{ has finite cardinality}\}$ the set where $\boldsymbol{X}$ has finite range. In [2] initially fair processes are studied, and the following theorem is proved.

Theorem 1. *If $\boldsymbol{X}$ is initially fair then*

$$P(\text{range of } \boldsymbol{X} \text{ is finite}) + P(\boldsymbol{X} \text{ is recurrent}) = 1.$$

Of course, fair random walk is recurrent, so one way to interpret this result is to say that this property is preserved under reinforcement unless the reinforcement is so strong as to trap the process in a finite number of states. This paper addresses the question of whether this remains true for any initially recurrent RRW. Explicit criteria are available for initial recurrence. An old result of T. E. Harris (see [2], or [3] p. 109) is equivalent to the statement that a Markov process like that discussed in the first paragraph of this paper is recurrent if and only if both the sums $\sum_{i=1}^{\infty} w_i^{-1}$ and $\sum_{i=1}^{\infty} w_{-i}^{-1}$ are infinite, where the w_i are given by (1). Thus a RRW $\boldsymbol{X}$ is initially recurrent if and only if both $\sum_{i=1}^{\infty} w(0,i)^{-1}$ and $\sum_{i=1}^{\infty} w(0,-i)^{-1}$ are infinite.

The purpose of this paper is to prove the following theorem.

Theorem 2. *Let* $\{a_i,\ -\infty < i < \infty\}$ *be a set of positive numbers. A necessary and sufficient condition in order that for each* RRW $\boldsymbol{X}$ *with* $w(0,i) = a_i,\ -\infty < i < \infty$, *we have*

$$P(\boldsymbol{X} \text{ is recurrent}) + P(\boldsymbol{X} \text{ has finite range}) = 1,$$

is that both $\sum_{i=1}^{\infty} a_i^{-2}$ *and* $\sum_{i=1}^{\infty} a_{-i}^{-2}$ *are infinite.*

Proof. Our proof of sufficiency resembles the proof of Theorem 1 given in [2], although that proof did not use the martingale square function while this one, necessarily as far as we can tell, does. See also the proof of Theorem 3.3 of [2]. We assume that $EX_0 < \infty$. If this is not true we can just condition on the value of X_0.

Let $a_i,\ -\infty < i < \infty$, satisfy $\sum_{i=1}^{\infty} a_i^{-2} = \sum_{i=1}^{\infty} a_{-i}^{-2} = \infty$, and let $\boldsymbol{X}$ have initial weighting $w(0,i) = a_i,\ -\infty < i < \infty$. We will show

$$P(\overline{\lim}_{n\to\infty} X_n = +\infty, \underline{\lim}_{n\to\infty} X_n > -\infty) = 0, \tag{2}$$

which immediately, by reflection, implies $P(\underline{\lim}_{n\to\infty} X_n = -\infty, \overline{\lim}_{n\to\infty} X_n < \infty) = 0$, which together with (2) establishes sufficiency. To prove (2) it suffices to show that, given $n \in \{0,1,2\ldots\}$ and $k \in I$,

$$P(X_n > k,\ \sup_i X_i = \infty,\ \inf_{i\geq 0} X_{n+i} > k) = 0, \tag{3}$$

and to prove (3) it suffices to prove it in the special case $k = 0$, since our condition on $a_i,\ i \in I$, is equivalent to $\sum_{i=1}^{\infty} w(0,i)^{-2} = \infty$ and $\sum_{i=1}^{\infty} w(0,-i)^{-2} = \infty$, which is clearly invariant under change of origin.

Now let n be fixed and put $A = \{X_n > 0\}$, $T = \inf\{k \geq n\colon\ X_k \leq 0\}$, $B = A \cap \{T = \infty\} \cap \{\sup_i X_i = \infty\}$. To prove (3) in the case $k = 0$ we must show

$P(B) = 0$. Note that $\sum_{i=0}^{\infty} w(n,i)^{-2} = \infty$ a.s. since $w(n,k) = w(0,k)$ for all but at most n integers k.

For $m \geq 0$ put

$$G(m,j) = \sum_{i=0}^{j-1} w(n+m,i)^{-1}, \quad j \geq 1,$$
$$= 0, \quad j \leq 0,$$

and define

$$M_\lambda = G((n+\lambda) \wedge T, X_{(n+\lambda)\wedge T}), \quad \lambda = 0,1,2,\ldots$$

where $\wedge$ denotes minimum.

For $\lambda \geq 0$ put

$$H_\lambda = M_\lambda + \sum_{i=n+1}^{n+\lambda} [w(i-1, X_{i-1})^{-1} - w(i, X_{i-1})^{-1}] I(X_i > X_{i-1}, i < T), \qquad (4)$$

where the sum is taken to be zero if $\lambda = 0$. Then H_λ, $\lambda = 0,1,2,\ldots$ is a nonnegative martingale, and furthermore, if $j \geq X_n$, $H_{i+1} - H_i = w(n,j)^{-1}$ on

$$D_{i,j} = \{X_{n+i+1} = j+1, X_{n+i} = j, \ 1 \leq X_\gamma \leq j, \ n \leq \gamma \leq n+i\}.$$

For a proof of this see the end of the proof of Lemma 3.1 of [2], although the reader will probably be able to construct the proof, which is just a calculation. Note since we assume $j \geq X_n$, $\bigcup_{i=0}^{\infty} D_{i,j} \supset B$. Thus

$$
\begin{aligned}
S(H) &= H_0^2 + \sum_{i=1}^{\infty}(H_{i+1} - H_i)^2 \\
&\geq \sum_{i=0}^{\infty} \sum_{j=X_n}^{\infty} w(n,j)^{-2} I(D_{i,j}) \\
&= \sum_{j=X_n}^{\infty} \sum_{i=0}^{\infty} w(n,j)^{-2} I(D_{i,j}) \\
&\geq \sum_{j=X_n}^{\infty} w(n,j)^{-2} I(B) \\
&= \infty I(B),
\end{aligned}
$$

where I denotes indicator function. Now since H is a nonnegative it is L^1 bounded, and thus a result of D. G. Austin ([1]) gives $P(S(H) = \infty) = 0$, implying $P(B) = 0$, concluding the proof of sufficiency.

To prove necessity it suffices, with no loss of generality, to let $b_0, b_1, b_2 \ldots$ be a sequence of positive numbers satisfying $\sum_{i=1}^{\infty} b_i^{-2} < \infty$ and to construct a RRW, such that for $k \geq 0$ the initial weighting of $(k, k+1)$ is b_k, which converges to $+\infty$ with positive probability. We will use $\boldsymbol{Y} = Y_0, Y_1, \ldots$ to stand for this example, and the associated interval weights will be designated $\gamma(n,k)$, defined by

$$\gamma(0,k) = b_k, \quad k \geq 0,$$

$$\gamma(n,k) - \gamma(n-1,k) = b_k I(Y_{n-1} = k+1, Y_n = k), \quad k \geq 0, \ n \geq 1,$$

$$\gamma(n,k) = 1 \quad \text{if } k < 0.$$

That is, if $k \geq 0$ the weight of $(k, k+1)$ is increased by b_k each time the process downcrosses this interval. We also specify $Y_0 = 1$. In the discussion below, the weights of $(k-1, k)$, $k \leq 0$, are irrelevant.

To show this example has the desired properties we first prove the following lemma. Much stronger results are known.

Lemma 1. *Let* $f_0, f_1, \ldots$ *be a nonnegative martingale satisfying* $P(f_0 = 1) = 1$ *and* $\lim_{n\to\infty} f_n = 0$. *Then* $P(\sum_{i=1}^{\infty}(f_i - f_{i-1})^2 > \lambda) > 0,\ \lambda > 0$.

Proof. Suppose to the contrary that there is a $\lambda_0 > 0$ such that $P(\sum_{i=1}^{\infty}(f_i - f_{i-1})^2 > \lambda_0) = 0$. Then $E(f_n - f_i)^2 = E\sum_{i=1}^{n}(f_i - f_{i-1})^2 \le \lambda_0^2$, so that $f_0, f_1, \ldots$ is an L^2 bounded martingale, implying $1 = Ef_0 = \lim_{n\to\infty} Ef_n = E\lim_{n\to 0} f_n = 0$, a contradiction.

Now put $\tau = \inf\{j\colon\ Y_j = 0\}$, define

$$
\begin{aligned}
G(m,j) &= \sum_{i=0}^{j-1}\gamma(m,i)^{-1}, \quad j > 0\\
&= 0, \quad j = 0,
\end{aligned}
$$

and put $\Gamma_n = G(n\wedge\tau, Y_{n\wedge\tau}),\ n \ge 0$. Then $\Gamma_i,\ i \ge 0$, is a nonnegative martingale. The proof of this follows in a manner similar to the proof that the process $H_0, H_1, \ldots$ is a nonnegative martingale together with the fact that on $\{Y_n > Y_{n-1}\}$, $\gamma(n, Y_{n-1}) = \gamma(n-1, Y_{n-1})$, so that the expression corresponding to the sum in (4) is zero. See [2], especially the comment after the proof of Theorem 3.1, for a fuller discussion of this.

Now

$$
\begin{aligned}
S(\Gamma)^2 &= \Gamma_0^2 + \sum_{i=1}^{\infty}(\Gamma_i - \Gamma_{i-1})^2 \qquad (5)\\
&= 1 + \sum_{i=1}^{\infty}(\Gamma_i - \Gamma_{i-1})^2\\
&\le 1 + \sum_{k=0}^{\infty}\sum_{i=1}^{\infty}(\Gamma_i - \Gamma_{i-1})^2 I\{(Y_{i-1}, Y_i) = (k, k+1) \text{ or } (k+1, k)\}
\end{aligned}
$$

$$= 1 + \sum_{k=0}^{\infty} Z_k,$$

where Z_k is just defined to be the second sum in the inequality above, and the inequality holds since $\Gamma_n = 0$ if $Y_k = 0$ for some $k < n$, so that Z_j for $j < 0$, if we defined it analogously, would be 0.

Now

$$Z_k \le \sum_{i=0}^{\infty} \gamma(i,k)^{-2} I\{(Y_i, Y_{i+1}) = (k, k+1) \text{ or } (k+1, k)\}$$
$$= \gamma(\tau_1, k)^{-2} + \gamma(\tau_2, k)^{-2} + \dots,$$

where Y crosses $(k, k+1)$ for the i^{th} time between times τ_i and $\tau_i + 1$. For $k \ge 1$ we have, recalling that $Y_0 = 1$, that $\gamma(\tau_1, k) = b_k$, $\gamma(\tau_2, k) = b_k$, $\gamma(\tau_3, k) = 2b_k$ (for in fact $\gamma(\tau_2 + 1, k) = 2b_k$, since the first downcrossing of $(k, k+1)$ occurs between times τ_2 and $\tau_2 + 1$), and in general $\tau_{2n-1} = \tau_{2n} = nb_k$. Thus

$$Z_k \le 2 \sum_{n=1}^{\infty} (nb_k)^{-2} < 4b_k^{-2}, \quad k \ge 1,$$

and, similarly, $Z_0 < 4b_0^{-2}$, so that by (5) we have $S(\Gamma)^2 < 4 \sum_{k=0}^{\infty} b_k^{-2}$. Lemma 1 now implies that

$$P(\lim_{n \to \infty} \Gamma_n = 0) < 1,$$

so that

$$P(Y_n > 0 \text{ for all } n) > 0. \tag{6}$$

We will now prove

$$P(0 < \overline{\lim}_{n \to \infty} Y_n < \infty) = 0, \tag{7}$$

which, together with (6), proves $P(\lim Y_n = \infty) > 0$. To prove (7) it suffices to show that, if $\lambda > 0$ is an integer and n is an integer, and $A(n,\lambda) = A = \{Y_{n+i} = \lambda$ for infinitely many i, $Y_{n+i} \leq \lambda$, $i \geq 0\}$, then $P(A) = 0$. To establish this we assume that $\lambda \geq Y_n$, and note that there is a (random) integer ψ such that $\gamma(n, \lambda - 1) = \psi b_{\lambda-1}$ and a (random) integer θ such that $\gamma(n,\lambda) = \theta b_\lambda$. Let $T_1 = \inf\{j \geq n\colon Y_j = \lambda\}$, and $T_i = \inf\{j \geq T_{i-1}\colon Y_j = \lambda\}$, $i > 1$, and put $A_k = \{T_k < \infty\} \cap \{\max_{n \leq j \leq T_k} Y_j = \lambda\}$ so that $A_1 \supseteq A_2 \supseteq \ldots$ and $\bigcap_{i=1}^{\infty} A_i = A$. Now $\gamma(T_k, \lambda) = \theta b_\lambda$ on A_k, while

$$\gamma(T_k, \lambda - 1) = \psi b_{\lambda-1} + (k-1) b_{\lambda-1} \text{ on } A_k.$$

Thus

$$P(Y_{T_k+1} = \lambda + 1 | A_k, \psi, \theta) = \frac{\theta b_\lambda}{\psi b_{\lambda-1} + (k-1) b_{\lambda-1}}, \quad k \geq 1,$$

and, since $Y_{T_k+1} = \lambda - 1$ on A_{k+1}, this implies

$$P(A_{k+1}^c | A_k, \psi, \theta) \geq \frac{\theta b_\lambda}{\psi b_{\lambda-1} + (k-1) b_{\lambda-1}},$$

where E^c stands for the complement of E, so that

$$P(\bigcap_{k=1}^{\infty} A_k | \psi, \theta) \leq \prod_{k=1}^{\infty} \left(1 - \frac{\theta b_\lambda}{\psi b_{\lambda-1} + (k-1) b_{\lambda-1}}\right) = 0.$$

Thus $P(A) = 0$, completing the proof of (7) and thus the proof that Y_n approaches ∞ with positive probability.

We remark that it is not difficult to show that in fact $P(\lim_{n\to\infty} Y_n = \infty) = 1$.

References

[1] Austin, D. G. (1966). A sample function property of martingales. *Ann. Math. Statist.* **37**, 1396–1397.

[2] Davis, B. Reinforced random walk. Purdue Department of Statistics Technical Report #88-22.

[3] Karlin, S., and Taylor. A first course in stochastic processes, second edition. Academic Press, New York, 1975.

[4] Pemantle, R. Phase transition in reinforced random walk and RWRE. To appear, *Ann. Prob.*