A BOOK REVIEW OF: THE MALLIAVIN CALCULUS

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The Malliavin Calculus refers to a part of Probability theory which can loosely be described as a type of calculus of variations for Brownian motion. It is intimately concerned with the interplay between Markov processes with continuous paths (i.e., diffusions) and partial differential equations.

A time homogeneous diffusion X with values in $\mathbb{R}^n$ can be represented as a solution of a stochastic integral equation of the form

$$X_t = x + \int_0^t a(X_s)dB_s + \int_0^t b(X_s)ds, \tag{1}$$

where B is a Brownian motion on $\mathbb{R}^m$ (also known as a Wiener process), provided X solves mild regularity conditions. From a statistical standpoint, the diffusion X is determined by its transition probabilities, since it is a Markov process : $P_t(x,A) = P(X_{u+t} \in A \mid X_u = x)$, all $u \geq 0$, all $t > 0$. The measures $P_t(x,dy)$ induce operators on bounded Borel functions : $P_t f(x) = \int f(y)\, P_t(x,dy)$, and since they are a semigroup of operators there is an infinitesimal generator ($P_0 = I$) :

$$Lf(x) = \lim_{t \downarrow 0} \frac{P_t f(x) - f(x)}{t},$$

for an appropriate class of smooth functions f. This operator L is given by

$$(Lf)(x) = \frac{1}{2} \sum_{\lambda=1}^{m} a_\lambda^i\, a_\lambda^j\, D_{ij} f + b^i D_i f$$

where the sums over i and j are implicit, and where the a_λ^i and b^i are from (1) which can be alternatively written ($1 \leq i \leq n$) :

$$X_t^i = x^i + \sum_{\lambda=1}^{m} \int_0^t a_\lambda^i(X_s)dB_s^\lambda + \int_0^t b^i(X_s)ds. \tag{2}$$

Here D_{ij} denotes $\frac{\partial^2}{\partial x_i \partial x_j}$; D_i denotes $\frac{\partial}{\partial x_i}$.

If, for example, f has bounded partial derivations of first and second order (and a and b are at least Lipschitz continuous) then

$$\frac{\partial}{\partial t} P_t f = Lf.$$

Moreover if the measure $P_t(x,dy)$ has a density $p_t(x,y)$ with respect to Lebesgue measure, then Kolmogorov realized sixty years ago that $p_t(x,y)$ satisfies (for fixed y) :

$$\frac{\partial}{\partial t} p_t(x,y) = L_x p_t(x,y) \; ; \tag{3}$$

and if L^* is the adjoint of L then (for fixed x) :

$$(4) \qquad \frac{\partial}{\partial t} p_t(x,y) = L_y^* \, p_t(x,y).$$

The adjoint L^* can be calculated :

$$L^* g = \frac{1}{2} \sum_{\lambda} D_{ij} \left(a_\lambda^i \, a_\lambda^j \, g \right) - D_i (b^i g).$$

Equations (3) and (4) are known respectively as Kolmogorov's backward and forward equations. In the case where the diffusion is simply Brownian motion itself (in $\mathbb{R}^1$), the density $p_t(x,y)$ equals $\frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$, which is the fundamental solution of the heat equation $\frac{\partial p}{\partial t} = \frac{1}{2} \Delta p$.

In principle all knowledge of a diffusion is contained in the transition probabilities, and hence one wishes to study the regularity of the measures $P_t(x,dy)$. There are basically two approaches : the first one is to assume very little smoothness of a and b, and to use lots of ellipticity of the operator L. This approach is known as the martingale problem approach, and is presented in the book by Stroock and Varadhan [10]. The coefficients need not be Lipschitz continuous, as continuity alone often suffices. The second approach is to assume that a and b are very smooth (e.g., C^∞), but to allow the operator L to be degenerate. This second approach is the framework for the Malliavin calculus.

In equations (1) and (2) we represented the diffusion X in terms of the Itô integral. This was important to allow the consideration of the martingale problem approach. Henceforth a and b will be assumed C^∞, and thus we can represent X in the form :

$$(5) \qquad X_t = x + \int_0^t a(X_s) \circ dB_s + \int_0^t a_0(X_s) ds$$

where the stochastic integral is a Stratonovich integral (the a and a_0 of (5) are different from the a and b of (1) and (2) if X is the same ; however they are easily computed by simple transformation rules). The operator L can now be written :

$$L = a_0 + \frac{1}{2} \sum_{\lambda=1}^{m} a_\lambda^2$$

where a_0 is a first order operator and a_λ^2 is a second order operator.

One of the goals of the Malliavin Calculus is to show that if a and a_0 are sufficiently smooth, then $P_t(x,dy)$ has a density which is also smooth. The perfect tool for this is Hörmander's theorem. Let Y_j denote the differential operator

$$Y_j = \sum_{i=1}^{m} a_{ij} \frac{\partial}{\partial x_i}, \quad 0 \le j \le r,$$

where U is an open set and $a_{ij} \in \mathcal{C}^{\infty}(U)$.

Define

$$Gu = \sum_{j=1}^{r} Y_j^2 \; u + Y_0 \; u + cu \; .$$

Then G is <u>hypoelliptic</u> if whenever $Gu = f$ is satisfied for two distributions u,f (i.e., generalized functions) on U, then the following holds : for any open subset V of U such that $f|_V \in \mathcal{C}^{\infty}(V)$, then $u|_V \in \mathcal{C}^{\infty}(V)$. Let $[Y_i, Y_j]$ denote the Lie bracket of Y_i and Y_j . Hörmander's theorem states the following : <u>if every vector field on</u> U <u>can be expressed as a linear combination</u> (<u>with</u> $\mathcal{C}^{\infty}$ <u>coefficients</u>) <u>of</u>

$$\{(Y_i)_{i\geq 0} \; , \; [Y_i, Y_j]_{i,j\geq 0} \; , \; [Y_i \; , \; [Y_j, Y_k]], \; \ldots\} \; ,$$

<u>then</u> G <u>is hypoelliptic</u>. Hörmander's theorem, applied to the operator L (or its adjoint), gives conditions such that the transition density is $\mathcal{C}^{\infty}$.

Indeed, Hömander's theorem translates as follows : <u>if for each</u> $x \in \mathbb{R}^n$ <u>the rank of the (infinite) system of vectors</u>

$$\{(a_\lambda)_{\lambda\geq 1} \; , \; [a_\lambda, a_\mu]_{\lambda,\mu\geq 0} \; , \; [[a_\lambda, a_\mu], a_\nu]_{\lambda,\mu,\nu\geq 0} \; , \; \ldots\}$$

<u>is equal to</u> n, <u>then for each</u> x <u>the measure</u> $P_t(x,dy)$ <u>has a</u> $\mathcal{C}^{\infty}$ <u>density</u>. Note that $\lambda \geq 1$ for the first term : The inclusion of a_0 leads to a slightly weaker result.

What Malliavin did was to provide a probabilistic proof of Hörmander's theorem by constructing a kind of calculus of variations for Brownian motion. This in turn gave probabilistic proofs of the smoothness of the transition densities. This has the advantage of giving probabilistic insight and intuition into what is seen as a fundamental probabilistic result ; it has the disadvantage of giving a longer and perhaps harder proof of Hörmander's theorem than is available in the PDE literature (e.g., [4]). However Malliavin's methods (credit should also be given to those whose work he built upon such as Gross, Kree, Kuo, Eels, Elworthy, ...) are profound, and they are already having ramifications in other areas of probability.

For example, one important operation that has emerged from the Malliavin calculus is known (colloquially) as <u>Malliavin's derivative</u> .(It could also be called Kree's derivative, as it existed in the literature before Malliavin's work). This is an operator that maps random variables into processes : if F is an L^2 random variable on Wiener space, let $(D_s F)_{s\geq 0}$ denote the process that is the Malliavin derivative of F. Ocone [8] has shown that for nice F,

$$F = E\{F\} + \int_0^1 E\{D_s F \mid \mathcal{F}_s\} \, dB_s \; ,$$

where F is a random variable on the Wiener space of a Brownian motion $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, B)$. Also, Nualart, Pardoux, Zakai and the reviewer have used the Malliavin derivative in a series of articles [6], [9], [7] to understand further the Skorohod integral, building on work of Gaveau and Trauber

[3]. (See also the work of Ustunel [11]).

Let us turn now to Denis Bell's book, which is the first of its kind : it is an attempt to treat the Malliavin calculus in a pedagogic manner, bringing together the two basic approaches that have developed since Malliavin's fundamental papers : The first approach, close in spirit to that of Malliavin, is identified with Stroock, S. Watanabe, Ikeda, Shigekawa, Kusuoka, Meyer, ... ; the second approach is identified with Bismut, Michel, Bichteler, Jacod, ... The first chapter contains "background material". The sophistication that the author assumes of the reader is strange. For example, the book begins with abstract Wiener space, which is unnecessary for an introductory treatment ; one does not really need abstract Wiener space - simply Wiener space would suffice. Also if the reader actually needs a two page summary of stochastic integration, there is not much hope ; especially if he takes Bell's advice and looks to the work of Mc Shane "for a more general treatment". Finally, there is one lemma which is key to every treatment of the Malliavin calculus : a measure μ is absolutely continuous if its firstderivatives (in the sense of distributions) are measures. This lemma (Lemma 1.12. on p. 13) should be proved.

Bell then presents, quite concisely, the Stroock et al approach in Chapter 2, followed by Bismut's approach (using Girsanov's theorem) in Chapter 3. The two approaches are related (following the work of Zakai) in Chapter 5. Chapter 6 is the heart of the book. Here the author makes use of Norris'simplifications to give a proof that Σ^{-1} is in L^p for all $p \in \mathbb{N}$, where Σ is the famous "Malliavin covariance matrix". Chapter 6 could have been expanded. Chapter 4 is a treatment of Bell's own contribution to the subject. The key idea of Bell is to examine what happens in a finite dimensional setting (i.e., $\mathbb{R}^d$) and then take limits to derive some of Malliavin's results. This has the advantage over the function-space approach of being easy and perhaps more intuitive, albeit less elegant.

Chapter 7 is, perhaps, the most provocative part of Bell's book. The author is no longer concerned with the smoothness of transition densities, but rather with novel applications of the tools of the Malliavin calculus. While the applications using the Malliavin derivative (already discussed) are not mentioned by Bell, he does nevertheless present diverse applications in Chapter 7, including such disparate subjects as filtering theory and infinite particle systems. Here he could be a bit more authoritative : For example, in the filtering theory section he should mention further work, at least at the bibliographic level (e.g., [1], [2] and [5]).

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