On The Convergence Rates Of A Monotone Empirical Bayes Test For Uniform Distributions*

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ON THE CONVERGENCE RATES OF A MONOTONE EMPIRICAL BAYES TEST FOR UNIFORM DISTRIBUTIONS*

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ABSTRACT

We investigate the convergence rates of a sequence of monotone empirical Bayes tests for the two–action decision problems involving uniform distributions. It is found that the sequence of monotone empirical Bayes tests under study is asymptotically optimal, and the order of associated convergence rate is $O(\beta(n))$, where $\beta(n)$ is such that $n^{-1/2} < \beta(n) < (\ln n/n)^{1/2}$ and n is the number of accumulated past experience at hand.

Abbreviated title: Monotone Empirical Bayes Test

AMS 1980 Subject Classification: 62C12.

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1. Introduction

Let X be a random variable having uniform distribution with *pdf* $f(x|\theta) = \theta^{-1} I_{(0,\theta)}(x)$, $\theta > 0$. Consider the following testing: $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$, where θ_0 is a known positive constant. For each $i = 0, 1$, let i denote the action deciding in favor of H_i. For the parameter θ and action i, the loss function $L(\theta, i)$ is defined as

$$L(\theta, i) = (1-i)(\theta_0 - \theta) I_{(0,\theta_0)}(\theta) + i(\theta - \theta_0) I_{[\theta_0,\infty)}(\theta). \tag{1.1}$$

In (1.1), the first item is the loss due to taking action 0 when $\theta < \theta_0$, and the second item is the loss of taking action 1 when $\theta \geq \theta_0$. It is assumed that θ is the value of a random variable Θ having a prior distribution G.

For a decision rule d, let $d(x)$ be the probability of taking action 0 when $X = x$ is observed. Let D be the class of all decision rules. For each $d \in D$, let $r(G, d)$ denote the associated Bayes risk. Then $r(G) = \inf_{d \in D} r(G, d)$ is the minimum Bayes risk among all decision rules in the class D. A decision rule, say d_G, such that $r(G, d_G) = r(G)$ is called a Bayes rule.

Based on the statistical model described above, the Bayes risk associated with the decision rule d is:

$$r(G, d) = \int_{x=0}^{\infty} [\theta_0 - \varphi(x)] d(x) f(x) dx + C, \tag{1.2}$$

where

$$f(x) = \int f(x|\theta) dG(\theta) = \int_x^{\infty} \frac{1}{\theta} dG(\theta), \tag{1.3}$$

$$\varphi(x) = E[\theta|x] = \frac{1 - F(x)}{f(x)} + x, \tag{1.4}$$

$F(x) = \int_0^x f(y)dy$ is the marginal accumulated distribution function of X, and

$$C = \int_{x=0}^{\infty} \int_{\theta=\max(\theta_0,x)}^{\infty} (\theta - \theta_0) f(x|\theta) dG(\theta) dx. \tag{1.5}$$

We only consider those priors such that $\int_0^\infty \theta dG(\theta) < \infty$ to insure that the Bayes risk is always finite. Note that C is a constant, which is independent of the decision rule d. Thus, from (1.2), a nonrandomized Bayes rule, denoted by d_G, is clearly given by

$$d_G(x) = \begin{cases} 1 & \text{if } \varphi(x) \geq \theta_0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

<u>Definition 1.1</u>. A decision rule d is said to be monotone if $0 < x < y$ then $d(x) \leq d(y)$.

Since the class of uniform distributions $\{f(x|\theta)|\theta > 0\}$ has monotone likelihood ratio in x, the posterior mean $\varphi(x)$ is a nondecreasing function of x. Thus, from (1.6), one can see that the Bayes rule d_G is a monotone decision rule. Recall that the class of all monotone decision rules is essentially complete; see Berger (1985).

When the prior distribution G is unknown, it is not possible to apply the Bayes rule for the decision problem at hand. In this situation, an empirical Bayes approach will be used. Although Gupta and Hsiao (1983) have studied the above decision problem via an empirical Bayes approach, however, the empirical Bayes decision rule they proposed is not monotone. Later, Van Houwelingen (1987) proposed a monotone empirical Bayes test for this decision problem. He derived an empirical Bayes test from a monotone empirical Bayes estimator for the unknown $\varphi(x)$, based on the observed sampling spacings. The empirical Bayes test was proved to possess the monotone property. Van Houwelingen (1987) has also studied some asymptotic properties of the proposed test.

The usefulness of empirical Bayes decision rules in practical applications clearly depends on the convergence rates with which the risks for the successive decision problems approach the minimum Bayes risk. In this paper, a sequence of monotone empirical Bayes decision rules $\{d_n^*\}$ is proposed for the above described two–action decision problem. We investigate the asymptotic optimality property of $\{d_n^*\}$. It is found that the order of the convergence rates of $\{d_n^*\}$ is $O(\beta(n))$ where n is the number of accumulated past experience at hand, and $\beta(n)$ is such that $n^{-1/2} < \beta(n) < (\ln n/n)^{1/2}$.

2. A Monotone Empirical Bayes Decision Rule

For each $j = 1, \ldots, n$, let (X_j, Θ_j) be a pair of random variables, where X_j is observable but Θ_j is not. Conditional on $\Theta_j = \theta, X_j$ has a uniform distribution with $pdf\, f(x|\theta)$. It is assumed that $\Theta_j, j = 1, \ldots, n$, are independently distributed with common unknown prior distribution G. Therefore, $X_1, \ldots, X_n$ are iid with $pdf\, f(x)$. Let $\underset{\sim}{X}_n = (X_1, \ldots, X_n)$ denote the n past observations and let $X_{n+1} = X$ denote the present random observation.

Let $\{\alpha_n\}$ be a sequence of decreasing positive numbers such that $\lim_{n\to\infty} \alpha_n = 0$. For each $n = 1, 2, \ldots$, we partition the half line $(0, \infty)$ into equal length subintervals $(C_{n,m}, C_{n,m+1}]$, where $C_{n,m} = m\alpha_n, m = 0, 1, 2, \ldots$. Define

$$f_n(C_{n,m}) = \frac{1}{n} \sum_{j=1}^{n} I_{(C_{n,m-1}, C_{n,m}]}(X_j), m = 1, 2, \ldots$$

and

$$F_n(x) = \frac{1}{n} \sum_{j=1}^{n} I_{(0,x]}(X_j), x > 0.$$

Note that $F_n(C_{n,m}) = \sum_{i=1}^{m} f_n(C_{n,i}), m = 1, 2, \ldots$.

Since $f(x)$ is decreasing in x (see (1.3)), thus the marginal distribution function $F(x)$ of the random variable X is a concave function. However, for each n, the empirical frequency function $f_n(C_{n,m})$ may not be decreasing in m and therefore the function $F_n(C_{n,m})$ may not be concave. Following Van Houwelingen (1987), we let $\{F_n^*(C_{n,m})\}$ be the least–

concave majorant of $\{F_n(C_{n,m})\}$ and let

(2.1)
$$f_n^*(t) = \frac{F_n^*(C_{n,m}) - F_n^*(C_{n,m-1})}{C_{n,m} - C_{n,m-1}} \text{ if } t \in (C_{n,m-1}, C_{n,m}], m = 1, 2, \ldots$$
$$F_n^*(x) = \int_0^x f_n^*(t)dt, \quad \text{for } x > 0.$$

From Barlow, et. al. (1972), $f_n^*(t)$ is a nonincreasing function of t. Now, define

(2.2)
$$\varphi_n^*(x) = \frac{1 - F_n^*(x)}{f_n^*(x)} + x, x > 0,$$

where $\frac{0}{0} \equiv 0$. By the definition $\varphi_n^*(x) = \varphi_n^*(C_{n,m})$ if $x \in (C_{n,m-1}, C_{n,m}]$.

We use $\varphi_n^*(x)$ to estimate the posterior mean $\varphi(x)$, and propose the following decision rule d_n^*:

(2.3)
$$d_n^*(x) = \begin{cases} 1 & \text{if } \varphi_n^*(x) \geq \theta_0, \\ 0 & \text{otherwise.} \end{cases}$$

By the nonincreasing property of the function $f_n^*(x)$, one can see that the function $\varphi_n^*(x)$ is nondecreasing in x, and therefore d_n^* is a monotone decision rule. Also, it should be noted that the past data $\underset{\sim}{X}_n$ is implicitly contained in the subscript n.

3. Asymptotic Optimality of $\{d_n^*\}$

For an empirical Bayes decision rule d_n with $d_n(x) \equiv d_n(x, X_1, \ldots, X_n)$ being a function of the current observation x and the past data $X_1, \ldots, X_n$, let $r(G, d_n)$ denote the associated Bayes risk and $E[r(G, d_n)]$ the associated overall expected Bayes risk. Then,

$$r(G, d_n) = \int_{x=0}^{\infty} (\theta_0 - \varphi(x))d_n(x)f(x)dx + C, \text{ and}$$
$$E[r(G, d_n)] = \int_{x=0}^{\infty} (\theta_0 - \varphi(x))E[d_n(x)]f(x)dx + C,$$

where C is given in (1.5) and the expectation $E[d_n(x)]$ is taken with respect to $\underset{\sim}{X}_n$. Since $r(G)$ is the minimum Bayes risk, thus, $r(G, d_n) - r(G) \geq 0$ and hence $E[r(G, d_n)] - r(G) \geq 0$, for all $n \geq 1$. Van Houwelingen (1987) has investigated the asymptotic properties of an empirical Bayes test, say d_n^H, and shown that $r(G, d_n^H) - r(G) = O_p(n^{-1/2})$. The nonnegative difference $E[r(G, d_n)] - r(G)$ has been used as a measure of the optimality of the decision rule d_n in many other empirical Bayes problems. For examples, see Johns and Van Ryzin (1971,1972). In this paper, we are concerned only with the difference $E[r(G, d_n)] - r(G)$.

Definition 3.1. A sequence of empirical Bayes decision rules $\{d_n\}$ is said to be asymptotically optimal in E at least of order β_n relative to the prior distribution G if $E[r(G, d_n)] - r(G) \leq O(\beta_n)$ as $n \to \infty$, where $\{\beta_n\}$ is a sequence of positive numbers such that $\lim_{n \to \infty} \beta_n = 0$.

It should be noted that $E[r(G, d_n)] - r(G) \leq O(\beta_n)$ implies $r(G, d_n) - r(G) \leq O_p(\beta_n)$. However, in general, the converse is not true. In the following, we investigate the asymptotic optimality of the sequence of the proposed empirical Bayes decision rules $\{d_n^*\}$. Our evaluation is based on the following assumptions A1 and A2.

Assumption A1. $\lim_{x \downarrow 0} \varphi(x) < \theta_0$ and $\lim_{x \uparrow \theta_0} \varphi(x) > \theta_0$.

Let $C_0 = \inf\{x > 0 | \varphi(x) \geq \theta_0\}$. By the nondecreasing property of $\varphi(x)$ and Assumption A1, $0 < C_0 < \theta_0$.

Assumption A2. There exist positive numbers, say $0 < m < M$, and an open interval $(C_1, C_2) \subset (0, \theta_0)$ such that $C_0 \in (C_1, C_2), \int_y^x \frac{1}{\theta} dG(\theta) \geq m(x - y)$ for all $C_1 < y < x < C_2$ and $\int_y^x \frac{1}{\theta} dG(\theta) \leq M\ (x - y)$ for all $0 < y < x < \theta_0$.

For each fixed positive integer n, consider the function $h(x) = x - e^{-nx^4}, x \geq 0$. Then $h(x)$ is strictly increasing in $x, h(0) = -1$ and $h(1) = 1 - e^{-n} > 0$. Thus, there is a unique solution of the equation $h(x) = 0$ in the interval (0,1). We denote this solution by $x_0(n)$. Furthermore, $h(n^{-1/4}) = n^{-1/4} - e^{-1}$, which is less than zero as $n \geq 81$; also, $h((\ln n/n)^{1/4}) > 0$ as $n \geq 3$. Hence, by the increasing property of the function $h(x), n^{-1/4} < x_0(n) < (\ln n/n)^{1/4}$ for $n \geq 81$, which implies $x_0(n) \to 0$ as $n \to \infty$.

The following theorem is our main result.

Theorem 3.1. Let $\{d_n^*\}$ be the sequence of empirical Bayes decision rules constructed in Section 2, at which the sequence of positive numbers $\{\alpha_n\}$ is chosen such that $\alpha_n = x_0(n)(2\theta_0^2)^{1/4}$ for sufficiently large n. Then, under Assumption A, $E[r(G, d_n^*)] - r(G) \leq O(\alpha_n^2)$ as $n \to \infty$.

The proof of this theorem is given in the next section.

4. Proof of Theorem 3.1

In this section, we first give some useful preliminary results to present a concise proof of Theorem 3.1.

By the nondecreasing property of $\varphi(x)$ and the definition of C_0, the Bayes rule d_G can be written as

$$d_G(x) = \begin{cases} 1 & \text{if } x \geq C_o, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, from (2.2) and (2.3), we obtain

$$\begin{aligned}(4.1)\quad & E[r(G, d_n^*)] - r(G) \\ &= \int_0^{C_0} (\theta_0 - \varphi(x)) P\{\varphi_n^*(x) \geq \theta_0\} f(x)dx + \int_{C_0}^{\theta_0} (\varphi(x) - \theta_0) P\{\varphi_n^*(x) < \theta_0\} f(x)dx \\ &= \int_0^{C_0} H(x) P\{\varphi_n^*(x) \geq \theta_0\} dx + \int_{C_0}^{\theta_0} [-H(x)] P\{\varphi_n^*(x) < \theta_0\} dx\end{aligned}$$

where $H(x) = (\theta_0 - \varphi(x))f(x) = (\theta_0 - x)f(x) - 1 + F(x), x \in (0, \theta_0)$. $H(x)$ is decreasing in x for $x \in (0, \theta_0)$. By Assumption A2, $f(x)$ is continuous on the interval $(0, \theta_0)$. Thus,

$F(x)$ and $H(x)$ are continuous on $(0, \theta_0)$. Hence, $H(C_0) = 0$ by the definition of C_0 and the continuity property of $H(x)$ on $(0, \theta_0)$.

<u>Lemma 4.1</u>. For $0 < y < x < \theta_0, H(y) - H(x) \leq M(x-y)(\theta_0 - y)$; also, for $C_1 < y < x < C_2, m(x-y)(\theta_0 - x) \leq H(y) - H(x)$, where the constants $0 < m < M$ are given in Assumption A2.

Proof: By Assumption A2 and the decreasing property of $f(x)$, for $0 < y < x < \theta_0, H(y) - H(x) \leq (\theta_0 - y)[f(y) - f(x)] = (\theta_0 - y) \int_y^x \frac{1}{\theta} dG(\theta) \leq M(x-y)(\theta_0 - y)$.

Similarly, for $C_1 < y < x < C_2, H(y) - H(x) \geq (\theta_0 - x)[f(y) - f(x)] = (\theta_0 - x) \int_y^x \frac{1}{\theta} dG(\theta) \geq m(x-y)(\theta_0 - x)$.

Thus, the proof of the lemma is complete. □

Let $\alpha_n = x_0(n)(2\theta_0^2)^{1/4}$ be the positive number defined in Section 3. For each n, define

$$\text{(4.2)} \qquad \begin{cases} A_n = \{y | 0 \leq y \leq C_0, 0 \leq H(y) \leq \alpha_n\}, & a_n = \inf A_n, \\ B_n = \{y | C_0 \leq y \leq \theta_0, 0 \geq H(y) \geq -\alpha_n\}, & b_n = \sup B_n. \end{cases}$$

Since $\alpha_n = o(1)$, by Assumption A2, for sufficiently large n, $C_1 < a_n < b_n < C_2$, $H(a_n) = \alpha_n$ and $H(b_n) = -\alpha_n$. By noting that $H(C_0) = 0$, from (4.2) and Lemma 4.1, following some simple algebraic computation, we have the following lemma.

<u>Lemma 4.2</u>. As n sufficiently large, under Assumptions A1 and A2, we have,

a) $\frac{\alpha_n}{M(\theta_0 - a_n)} \leq C_0 - a_n \leq \frac{\alpha_n}{m(\theta_0 - C_0)}$, and

b) $\frac{\alpha_n}{M(\theta_0 - C_0)} \leq b_n - C_0 \leq \frac{\alpha_n}{m(\theta_0 - b_n)}$.

<u>Lemma 4.3</u>. Let $p(y|c) = ye^{-\frac{cy^2}{2}}$ for $y \geq c^{-\frac{1}{2}}$, where $c > 0$. Then $p(y|c)$ is decreasing in y for $y \geq c^{-\frac{1}{2}}$.

Proof: Straightforward computation will yield the result. The detail is omitted here. □

For each $x \in (0, C_0), x \in (C_{n,j-1}, C_{n,j}]$ for some $j = 1, 2, \ldots$. By (2.1) and (2.2), following some computation, we have

$$
\begin{aligned}
&P\{\varphi_n^*(x) \geq \theta_0\} \\
&= P\{1 - F_n^*(C_{n,j}) + f_n^*(C_{n,j})(C_{n,j} - \theta_0) \geq 0\} \\
&= P\{F_n^*(C_{n,j})(C_{n,j-1} - \theta_0) - F_n^*(C_{n,j-1})(C_{n,j} - \theta_0) \geq -\alpha_n\} \\
&= P\{(\theta_0 - C_{n,j})[F_n^*(C_{n,j-1}) - F(C_{n,j-1})] - (\theta_0 - C_{n,j-1})[F_n^*(C_{n,j}) - F(C_{n,j})] \\
&\qquad \geq \alpha_n[(\theta_0 - C_{n,j})[F(C_{n,j}) - F(C_{n,j-1})]\alpha_n^{-1} - [1 - F(C_{n,j})]]\} \\
(4.3)\quad &\leq P\{(\theta_0 - C_{n,j})[F_n^*(C_{n,j-1}) - F(C_{n,j-1})] - (\theta_0 - C_{n,j-1})[F_n^*(C_{n,j}) - F(C_{n,j})] \\
&\qquad \geq \alpha_n H(C_{n,j})\} \\
&\leq P\{(\theta_0 - C_{n,j})[F_n^*(C_{n,j-1}) - F(C_{n,j-1})] \geq \frac{\alpha_n H(C_{n,j})}{2}\} \\
&\qquad + P\{(\theta_0 - C_{n,j-1})[F_n^*(C_{n,j}) - F(C_{n,j})] \leq -\frac{\alpha_n H(C_{n,j})}{2}\} \\
&\leq 2P\{\sup_{i \geq 1} |F_n^*(C_{n,i}) - F(C_{n,i})| \geq \frac{\alpha_n H(C_{n,j})}{2\theta_0}\} \\
&\leq 2P\{\sup_{i \geq 1} |F_n(C_{n,i}) - F(C_{n,i})| \geq \frac{\alpha_n H(C_{n,j})}{2\theta_0}\}.
\end{aligned}
$$

(Since $F(x)$ is concave and $\{F_n^*(C_{n,i})\}$ is the least concave majorant of $\{F_n(C_{n,i})\}$, see Barlow, et. al. (1972))

$$
\begin{aligned}
&\leq 2P\{\sup_{x \geq 0} |F_n(x) - F(x)| \geq \frac{\alpha_n H(C_{n,j})}{2\theta_0}\} \\
&\leq 2ce^{-\frac{n\alpha_n^2 H^2(C_{n,j})}{2\theta_0^2}}
\end{aligned}
$$

where the last inequality is obtained due to Lemma 2.1 of Schuster (1969) and c is a positive constant independent of the distribution function F, see Schuster (1969).

Also, for $x \in (C_0, \theta_0), x \in (C_{n,j-1}, C_{n,j}]$ for some $j \geq 1$. Now

$$P\{\varphi_n^*(x) < \theta_0\} \tag{4.4}$$

$$=P\{\varphi_n^*(x) < \theta_0, \max(X_1, \ldots, X_n) \leq C_{n,j-1}\}$$

$$+ P\{\varphi_n^*(x) < \theta_0, \max(X_1, \ldots, X_n) > C_{n,j-1}\}$$

where

$$P\{\varphi_n^*(x) < \theta_0, \max(X_1, \ldots, X_n) \leq C_{n,j-1}\} \leq [F(C_{n,j-1})]^n \leq [F(\theta_0)]^n, \tag{4.5}$$

and

$$P\{\varphi_n^*(x) < \theta_0, \max(X_1, \ldots, X_n) > C_{n,j-1}\} \leq P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}.$$

Analogous to (4.3), we can obtain

$$\begin{aligned} &P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\} \\ &\leq 2P\{\sup_{i \geq 1} |F_n^*(C_{n,i}) - F(C_{n,i})| \geq \frac{\alpha_n |H(C_{n,j-1})|}{2\theta_0}\} \\ &\leq 2P\{\sup_{x \geq 0} |F_n(x) - F(x)| \geq \frac{\alpha_n |H(C_{n,j-1})|}{2\theta_0}\} \\ &\leq 2ce^{-\frac{n\alpha_n^2 H^2(C_{n,j-1})}{2\theta_0^2}}. \end{aligned} \tag{4.6}$$

For each n, let a_n and b_n be those defined in (4.2). Also, let m_n and M_n be the integers such that $m_n \alpha_n < a_n \leq (m_n + 1)\alpha_n$, and $(M_n - 1)\alpha_n \leq b_n < M_n \alpha_n$.

<u>Lemma 4.4</u>. Under Assumptions A1 and A2, $\int_0^{C_0} H(x) P\{\varphi_n^*(x) \geq \theta_0\} dx \leq O(\alpha_n^2)$.

Proof : Note that

$$\begin{aligned} &\int_0^{C_0} H(x) P\{\varphi_n^*(x) \geq \theta_0\} dx \\ &= \int_0^{m_n \alpha_n} H(x) P\{\varphi_n^*(x) \geq \theta_0\} dx + \int_{m_n \alpha_n}^{a_n} H(x) P\{\varphi_n^*(x) \geq \theta_0\} dx \\ &+ \int_{a_n}^{C_0} H(x) P\{\varphi_n^*(x) \geq \theta_0\} dx \end{aligned} \tag{4.7}$$

where,

$$(4.8)\qquad \begin{aligned} &\int_{a_n}^{C_0} H(x)P\{\varphi_n^*(x) \geq \theta_0\}dx \\ &\leq \int_{a_n}^{C_0} \alpha_n P\{\varphi_n^*(x) \geq \theta_0\}dx \qquad (0 \leq H(x) \leq \alpha_n \text{ for } x \in (a_n, C_0)) \\ &\leq \alpha_n(C_0 - a_n) \\ &\leq \frac{\alpha_n^2}{m(\theta_0 - C_0)} \qquad \text{(by Lemma 4.2)}, \\ &= O(\alpha_n^2), \end{aligned}$$

(4.9)

$$\begin{aligned} &\int_{m_n\alpha_n}^{a_n} H(x)P\{\varphi_n^*(x) \geq \theta_0\}dx \\ &= \int_{m_n\alpha_n}^{a_n} [H(x) - H(a_n)]P\{\varphi_n^*(x) \geq \theta_0\}dx + \int_{m_n\alpha_n}^{a_n} H(a_n)P\{\varphi_n^*(x) \geq \theta_0\}dx \\ &\leq \int_{m_n\alpha_n}^{a_n} M(a_n - x)(\theta_0 - x)dx + \int_{m_n\alpha_n}^{a_n} \alpha_n dx \text{ (by Lemma 4.1 and the definition of } a_n) \\ &\leq O(\alpha_n^2), \end{aligned}$$

and

$$(4.10)\qquad \begin{aligned} &\int_0^{m_n\alpha_n} H(x)P\{\varphi_n^*(x) \geq \theta_0\}dx \\ &= \sum_{j=1}^{m_n} \left[\int_{C_{n,j-1}}^{C_{n,j}} [H(x) - H(C_{n,j})]P\{\varphi_n^*(x) \geq \theta_0\}dx + \int_{C_{n,j-1}}^{C_{n,j}} H(C_{n,j})P\{\varphi_n^*(x) \geq \theta_0\}dx \right] \\ &\leq \sum_{j=1}^{m_n} \left[\int_{C_{n,j-1}}^{C_{n,j}} 2M(C_{n,j} - x)(\theta_0 - x)ce^{-\frac{n\alpha_n^2 H^2(C_{n,j})}{2\theta_0^2}} dx \right. \\ &\qquad \left. + \int_{C_{n,j-1}}^{C_{n,j}} 2cH(C_{n,j})e^{-\frac{n\alpha_n^2 H^2(C_{n,j})}{2\theta_0^2}} dx \right] \qquad \text{(by 4.3)} \\ &\leq \sum_{j=1}^{m_n} \left[\int_{C_{n,j-1}}^{C_{n,j}} 2M\alpha_n\theta_0 ce^{-\frac{n\alpha_n^4}{2\theta_0^2}} dx + \int_{C_{n,j-1}}^{C_{n,j}} 2c\alpha_n e^{-\frac{n\alpha_n^4}{2\theta_0^2}} dx \right] \end{aligned}$$

(by the fact that $H(C_{n,j}) \geq \alpha_n$ and Lemma 4.3 for n sufficiently large)

$$\begin{aligned} &= O(\alpha_n e^{-\frac{n\alpha_n^4}{2\theta_0^2}}) \\ &= O(\alpha_n^2). \end{aligned}$$

Therefore from (4.7)-(4.10), the proof of the lemma is complete. □

Lemma 4.5. Under Assumptions A1 and A2, $\int_{C_0}^{\theta_0}[-H(x)]P\{\varphi_n^*(x) < \theta_0\}dx \leq O(\alpha_n^2)$.

Proof: From (4.4) and (4.5),

$$\int_{C_0}^{\theta_0}[-H(x)]P\{\varphi_n^*(x) < \theta_0\}dx$$
$$\leq \int_{C_0}^{\theta_0}[-H(x)][F(\theta_0)]^n dx + \int_{C_0}^{\theta_0}[-H(x)]P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}dx,$$

where

$$\int_{C_0}^{\theta_0}[-H(x)][F(\theta_0)]^n dx = O([F(\theta_0)]^n) = O(\exp(-n\ell n(F(\theta_0))^{-1})),$$

and where $F(\theta_0) < 1$ by Assumption A1.

Thus, it suffices to consider the asymptotic behavior of

$$\int_{C_0}^{\theta_0}[-H(x)]P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}dx.$$

Analogous to Lemma 4.4, we have

$$\int_{C_0}^{\theta_0}[-H(x)]P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}dx$$
$$= \int_{C_0}^{b_n}[-H(x)]P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}dx$$
$$+ \int_{b_n}^{M_n\alpha_n}[-H(x)]P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}dx$$
$$+ \int_{M_n\alpha_n}^{\theta_0}[-H(x)]P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}dx$$

where

$$\int_{C_0}^{b_n}[-H(x)]P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}dx \leq O(\alpha_n^2)$$
$$\int_{b_n}^{M_n\alpha_n}[-H(x)]P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}dx \leq O(\alpha_n^2)$$

and

$$\int_{M_n\alpha_n}^{\theta_0}[-H(x)]P\{1 - F_n^*(x) - f_n^*(x)(\theta_0 - x) < 0\}dx \leq O(\alpha_n^2).$$

Hence, we complete the proof of this lemma. □

Now, Theorem 3.1 is a direct consequence of Lemma 4.4 and Lemma 4.5.

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