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Technical Report#87-2

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January 1987

* The research of this author was supported in part by the Office of Naval Research Contract N00014-84-C-0167 at Purdue University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

ON THE ASYMPTOTIC LEAST FAVORABLE CONFIGURATION OF A SELECTION PROCEDURE BASED ON RANKS

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ABSTRACT

The problem of selection of the population with the largest parameter is considered using the subset selection as well as the indifference zone approach for distributions that belong to a location or a scale parameter family. The procedures are based on the sums of combined (Wilcoxon type) ranks and vector (Friedman type) ranks. The least favorable configurations are obtained in an asymptotic framework under certain order relations between the "gaps" of parameters. The asymptotic theory is based on exact moments of the rank sum statistics.

1. INTRODUCTION

Let $\Pi_1, \Pi_2, \ldots, \Pi_k$ be k independent populations, where Π_i has the associated cumulative distribution function (c.d.f) $G_{\theta_i}(y)$, i = 1,2,...,k. It is assumed that $\{G_\theta\}$ is a location or a scale parameter family, i.e. $G_\theta(y) = G(y - \theta)$, $-\infty < \theta < \infty$, or $G_\theta(y) = G(y/\theta)$, $\theta > 0$. Let the ordered θ_i be denoted by $\theta_{[1]} \leq \theta_{[2]} \leq \ldots \leq \theta_{[k]}$. The population associated with $\theta_{[k]}$ is defined to be the <u>best</u>. Our procedures for selecting the best population are

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based on ranks of observations from these populations in the location parameter case, and on ranks of the absolute values of the observations in the scale parameter case. For any observation Y from $G_\theta(y)$ in the scale parameter case, the c.d.f. of $X = |Y|$ is $F_\theta(x) = G(x/\theta) - G(-x/\theta)$. For convenience of presentation, we use $F_\theta(x)$ to denote the c.d.f. of $|Y|$ in the scale parameter case as well as that of Y itself in the location parameter case [i.e. $F_\theta(x) = G(x-\theta)$]. Then we have

$$F_{\theta_{[1]}}(x) \geq F_{\theta_{[2]}}(x) \geq \ldots \geq F_{\theta_{[k]}}(x) \tag{1.1}$$

for all x. We assume without loss of generality, that Π_k is the best population (i.e. $\theta_k \geq \theta_i$, $i = 1,2,\ldots,k-1$).

Let $Y_{i1}, Y_{i2}, \ldots, Y_{in}$ be n independent observations from Π_i, $i = 1,2,\ldots,$ k. Let $X_{ij} = Y_{ij}$ in the location parameter case, and $X_{ij} = |Y_{ij}|$ in the scale parameter case. We consider rank sum statistics based on combined (Wilcoxon type) ranks as well as vector (Friedman type) ranks. Let $R_{ij}^{(1)}$ denote the rank of X_{ij} among all kn observations in the combined sample and $R_{ij}^{(2)}$ denote the rank of X_{ij} among $X_{1j}, X_{2j}, \ldots, X_{kj}$. In ranking observations in a group, the smallest is given rank 1. For $\alpha = 1,2$, define

$$H_i^{(\alpha)} = c_{n\alpha} \sum_{j=1}^{n} R_{ij}^{(\alpha)}, \quad i = 1,2,\ldots,k, \tag{1.2}$$

and

$$\underline{H}^{(\alpha)} = (H_1^{(\alpha)}, H_2^{(\alpha)}, \ldots, H_k^{(\alpha)})', \tag{1.3}$$

where $c_{n1} = 1/n$ and $c_{n2} = 1$.

For selecting the best population, we consider both the subset selection approach of Gupta (see Gupta and Panchapakesan [5]) and the indifference zone approach of Bechhofer [3]. For selecting a subset containing the best, we define the following procedures:

$$R(\alpha,\beta,1): \text{ Select } \Pi_i \text{ if and only if } H_i^{(\alpha)} \geq \max H_j^{(\alpha)} - d_\beta$$

$$i = 1,2, \ldots,k; \quad d_\beta \geq 0; \quad \alpha,\beta = 1,2. \tag{1.4}$$

Here $\beta = 1$ and 2 correspond to the location and scale cases, respectively. The use of these rule is justified by Theorem 4.2. A correct selection (CS) is said to occur if and only if the best population (in our case Π_k) is included in the selected subset. Our aim is to select a subset satisfying

$$\inf_{\Omega} P(CS|R(\alpha,\beta,1)) \geq P^* \tag{1.5}$$

where $\alpha,\beta = 1,2$; $1/k < P^* < 1$; $\Omega = \{\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_k); \theta_i \in \Theta,\ i = 1, 2,\ldots,k\}$, Θ is a real line. The constant d_β is the smallest nonnegative number satisfying (1.5), the so-called P^*-condition.

Using the indifference zone approach, we define the procedures:

$$R(\alpha,\beta,2): \text{ Select the population associate with } H_{[k]}^{(\alpha)} \text{ as the best.} \tag{1.6}$$

In this case, the rule $R(\alpha,\beta,2)$, $\alpha,\beta = 1,2$ are required to satisfy the following probability condition:

$$P(CS|R(\alpha,\beta,2)) \geq P^* \text{ whenever } \varphi_\beta(\theta_k, \theta_i) \geq c_\beta + \delta_\beta^* \tag{1.7}$$

where $\alpha,\beta = 1,2$, $1/k < P^* < 1$, $\delta_\beta^* > 0$ is a given constant,

$$\varphi_\beta(\theta_i, \theta_j) = \begin{cases} \theta_i - \theta_j & \text{when } \beta = 1 \\ \theta_i/\theta_j & \text{when } \beta = 2 \end{cases} \tag{1.8}$$

and

$$c_\beta = \begin{cases} 0 & \text{when } \beta = 1 \\ 1 & \text{when } \beta = 2. \end{cases} \tag{1.9}$$

Selection procedures (using both subset selection and indifference zone approaches) based on this statistic $\underline{H}^{(1)}$ have been studied by many authors including Lehmann [9], Bartlett and Govindarajulu [2], Gupta and McDonald [4], Puri and Puri [15,16], and Alam and Thompson [1]. Also procedures based on $\underline{H}^{(2)}$ have been studied by McDonald [12,13], Mätsui [10] and Lee and Dudewicz [8]. A review of procedures based on ranks is given by Gupta and McDonald [6], and Gupta and Panchapakesan [7].

A parametric configuration which gives the infimum of PCS, the probability of a correct selection, is called the least favorable configulation (LFC). It is fairly difficult to establish the LFC for both rules $R(\alpha,\beta,1)$ and $R(\alpha,\beta,2)$ using statistics $\underline{H}^{(1)}$, $\underline{H}^{(2)}$ and is still an open question in general (α,β = 1, 2). The counter examples of Rizvi and Woodworth [17] and Lee and Dudewicz [8] show that the configuration $\theta_1 = \theta_2 = \dots = \theta_k$ in the case of subset selection rules, and the configuration $\theta_1 = \theta_2 = \dots = \theta_{k-1}$; $\varphi_\beta(\theta_k, \theta_{k-1}) = c_\beta + \delta^*_\beta$ in the case of indifference zone procedures, do not yield, in general the infimum of the PCS. A discussion on the LFC can be found in Gupta and McDonald [6].

Our purpose in this paper is to discuss the LFC in an asymptotic framework. An order relation is assumed to hold between the "gaps" of parameters. This assumption is similar to those considered by Puri and Puri [15,16], and Alam and Thompson [1]. The LFC's of the procedure are studied by using the exact moments of the combined and the vector rank statistics $\underline{H}^{(\alpha)}$, α = 1,2, for location and scale parameter cases (β = 1,2) for both subset selection and indifference zone approaches.

In Section 2, asymptotic distribution of $\underline{H}^{(\alpha)}$, $\alpha = 1,2$, are considered under the assumption of order relation between gaps of parameters. The PCS and LFC are investigated in Section 3. Moments results are given in Section 4 as an appendix.

2. ASYMPTOTIC PROPERTY

2.1 Moments of Ranks

Let us define the mean vector and variance-covariance matrix of $\underline{H}^{(1)}$ by $\underline{\mu}_\beta^{(1)}$ and $\underline{\Lambda}_\beta^{(1)}$ according as we are dealing with location ($\beta = 1$) or scale ($\beta = 2$) parameters. Under the population model we considered in Section 1, the elements of $\underline{\mu}_\beta^{(1)}$ and $\underline{\Lambda}_\beta^{(1)}$ are given as follows. These relations are obtained from more general results given in Theorem 4.1 of Appendix

$$\mu_{\beta i}^{(1)} = kn \int G^* dF_i + 1/2, \quad i = 1,2,\dots,k, \tag{2.1}$$

$$\lambda_{\beta ij}^{(1)} = \begin{cases} k(3n-1) \int G^* dF_i - 2k(2n-1) \int F_i G^* dF_i + k^2 n \int G^{*2} dF_i \\ \quad - k \int H^* dF_i - k^2 n (\int G^* dF_i)^2 - (n-1) \sum_{m=1}^{k} (\int F_m dF_i)^2 - 1/12, & i = j, \\ kn(2 - \int F_j dF_i) \int G^* dF_j + kn(2 - \int F_i dF_j) \int G^* dF_i \\ \quad - n \sum_{m=1}^{k} \int F_m dF_i \int F_m dF_j - 2kn \int F_j G^* dF_i - 2kn \int F_i G^* dF_j \\ \quad + \int F_i dF_j \int F_j dF_i + \int F_i^2 dF_j + \int F_j^2 dF_i - 1, & i \neq j, \end{cases} \tag{2.2}$$

where

$$G^*(x) = (1/k) \sum_{j=1}^{k} F_j(x), \tag{2.3}$$

$$H^*(x) = (1/k) \sum_{j=1}^{k} F_j^2 (x). \tag{2.4}$$

In case of vector rank $R_{ij}^{(2)}$, the moments results are given in Matsui [11] from which we obtain mean vector $\underline{\mu}_{\beta}^{(2)}$, variance-covariance matrix $\underline{\Lambda}_{\beta}^{(2)}$ of statistic $\underline{H}^{(2)}$ as follows;

$$\mu_{\beta i}^{(2)} = kn \int G^{*}dF_i + n/2, \quad i = 1,2,\ldots,k, \tag{2.5}$$

$$\lambda_{\beta ij}^{(2)} = \begin{cases} n[2k \int G^{*}dF_i - 2k \int F_i G^{*}dF_i + k^2 \int G^{*2} dF_i \\ \qquad - k \int H^{*}dF_i - k^2 (\int G^{*}dF_i)^2 - 1/12], & i = j, \\ n[k(2 - \int F_j dF_i) \int G^{*}dF_j + k(2 - \int F_i dF_j) \int G^{*}dF_i \\ \quad - \sum_{m=1}^{k} \int F_m dF_i \int F_m dF_j - 2k \int F_j G^{*}dF_i - 2k \int F_i G^{*}dF_j \\ \quad + \int F_i dF_j \int F_j dF_i + \int F_i^2 dF_j + \int F_j^2 dF_i - 1], & i \neq j, \end{cases} \tag{2.6}$$

2.2 Assumption

We assume the following relation to hold between the gaps of parameters:

$$\varphi_{\beta}(\theta_i, \theta_j) = c_{\beta} + \kappa_{\beta ij} n^{-1/2} + o(n^{-1/2}), \quad \beta = 1,2. \tag{2.7}$$

where c_{β} is given by (1.9); for each pair (i,j), $\kappa_{\beta ij}$ depends on θ_i and θ_j and is increasing in θ_i when θ_j is fixed, and decreasing in θ_j when θ_i is fixed; also, $\kappa_{\beta ij} = c_{\beta}$ when $\theta_i = \theta_j$.

Then putting

$$I_{\beta ij} \equiv \sqrt{n}\{\int F_j(x)dF_i(x) - \int F_i(x)dF_j(x)\}, \tag{2.8}$$

we obtain the following lemma.

Lemma 2.1

For $\varphi_{\beta}(\theta_i, \theta_j)$ $(\beta = 1,2)$ given by (2.7), we have the following:

$$I_{\beta ij} = K_{\beta ij} + o(1), \tag{2.9}$$

where

$$K_{\beta ij} = \begin{cases} \kappa_{1ij} \int f^2(x)dx, & \text{when } \beta = 1, \\ \kappa_{2ij} \int x f^2(x)dx, & \text{when } \beta = 2, \end{cases} \tag{2.10}$$

$$i,j = 1,2,\ldots,k;\ i \neq j.$$

Example:

When $F(x)$ is $N(0,1)$, $\varphi_\beta(\theta_i, \theta_j)$ given by (2.8), we get

$$I_{1ij} = (1/2\sqrt{\pi})\,\kappa_{1ij} + o(1) \tag{2.11}$$

and

$$I_{2ij} = (1/\pi)\,\kappa_{2ij} + o(1). \tag{2.12}$$

2.3 Asymptotic Distribution

Let us define

$$W_i^{(\alpha)} = (1/\sqrt{n})(H_k^{(\alpha)} - H_i^{(\alpha)}),\quad \alpha = 1,2. \tag{2.13}$$

Then

$$\underline{W}^{(\alpha)} = (1/\sqrt{n})\ \underline{A}\ \underline{H}^{(\alpha)},\quad \alpha = 1,2, \tag{2.14}$$

where $\underline{W}^{(\alpha)} = (W_1^{(\alpha)}, W_2^{(\alpha)}, \ldots, W_k^{(\alpha)})'$, $\underline{A} = (-\underline{E}_{(k-1)}, \underline{J}_{(k)})_{(k-1)\times k}$ and $\underline{E}_{(k-1)}$ is a unit matrix of order k-1, $\underline{J}_{(k)} = (1, 1, \ldots, 1)'_{k\times 1}$. $\underline{W}^{(\alpha)}$ has mean vector $\underline{\eta}_\beta^{(\alpha)}$, variance-covariance matrix $\underline{\Sigma}_\beta^{(\alpha)}$ such that

$$\underline{\eta}_\beta^{(\alpha)} = (1/\sqrt{n})\ \underline{A}\ \underline{\mu}_\beta^{(\alpha)} \tag{2.15}$$

and

$$\underline{\Sigma}_\beta^{(\alpha)} = (1/n)\ \underline{A}\ \Lambda_\beta^{(\alpha)}\ \underline{A}'. \tag{2.16}$$

Elements of $\underline{\eta}_\beta$ and $\underline{\Sigma}_\beta$ are given by

$$\eta_{\beta i}^{(\alpha)} = (1/\sqrt{n})(\mu_{\beta k}^{(\alpha)} - \mu_{\beta i}^{(\alpha)}),\quad i = 1,2,\ldots,k-1, \tag{2.17}$$

$$\sigma_{\beta ij}^{(\alpha)} = (1/n)(\lambda_{\beta ij}^{(\alpha)} - \lambda_{\beta ik}^{(\alpha)} - \lambda_{\beta kj}^{(\alpha)} + \lambda_{\beta kk}^{(\alpha)}),\quad i,j = 1,2,\ldots,k-1 \tag{2.18}$$

where $\mu_{\beta i}^{(\alpha)}$ and $\lambda_{\beta ij}^{(\alpha)}$ are given by (2.1) through (2.6).

Now, under the assumption (2.7), using Lemma 2.1, we have for $\beta = 1,2$ and $\alpha = 1,2$,

$$\eta_{\beta i}^{(\alpha)} = (1/\sqrt{n})\{ n \int \sum_{j=1}^{k} F_j dF_k - n \int \sum_{j=1}^{k} F_j dF_i \}$$

$$\rightarrow \sum_{j=1}^{k-1} K_{\beta kj} - \sum_{\substack{j=1 \\ j \neq i}}^{k} K_{\beta ij} \quad (\equiv \gamma_{\beta i}^{(\alpha)}) \tag{2.19}$$

as $n \rightarrow \infty$, where $K_{\beta ij}$ is given by (2.10). Also, under (2.7), we have

$$\lambda_{1ij} \rightarrow \begin{cases} -k/12 & \text{for } i \neq j \\ (k^2 - k)/12 & \text{for } i = j \end{cases}$$

$$\lambda_{2ij} \rightarrow \begin{cases} -(k + 1)/12 & \text{for } i \neq j \\ (k^2 - 1)/12 & \text{for } i = j. \end{cases}$$

Consequently,

$$\sigma_{\beta ij}^{(\alpha)} \rightarrow \begin{cases} v_\alpha & \text{for } i \neq j \\ 2v_\alpha & \text{for } i = j \end{cases} \tag{2.20}$$

where

$$v_\alpha = \begin{cases} k^2/12 & \text{when } \alpha = 1 \\ k(k + 1)/12 & \text{when } \alpha = 2. \end{cases} \tag{2.21}$$

Thus by applying the central limit theorem, we have the following asymptotic distribution of $\underline{W}^{(\alpha)}$:

$$\underline{W}^{(\alpha)} \sim N(\underline{\gamma}_\beta^{(\alpha)}, \underline{\Sigma}_\beta^{(\alpha)}), \ \beta = 1,2 \tag{2.22}$$

where $\underline{\gamma}_\beta^{(\alpha)} = (\gamma_{\beta 1}^{(\alpha)}, \gamma_{\beta 2}^{(\alpha)}, \ldots, \gamma_{\beta (k-1)}^{(\alpha)})'$ with elements given by (2.19) and

$$\underline{\Sigma}_\beta^{(\alpha)} = v_\alpha (\underline{E}_{(k-1)} + \underline{G}_{(k-1)}) \tag{2.23}$$

where $\underline{G}_{(k-1)} = \underline{J}_{(k-1)} \underline{J}'_{(k-1)}$.

3. PCS AND LFC

Using the asymptotic distribution of $\underline{W}_{\beta}^{(\alpha)}$ $(\alpha, \beta = 1,2)$ given by (2.22), the PCS for the rule $R(\alpha, \beta, m)$ $(\alpha, \beta, m = 1,2)$ is given by

$$P(CS|R(\alpha,\beta,m)) = \Pr(\underline{W}_{\beta}^{(\alpha)} \geq -\delta(\beta,m)\ \underline{J}_{(k-1)})$$

$$\approx \Pr\{\underline{U}_{\beta} \leq (\underline{\eta}_{\beta}^{(\alpha)} + \delta(\beta,m)\underline{J}_{(k-1)})/\sqrt{v_{\alpha}}\} \qquad (3.1)$$

where

$$\delta(\beta,m) = \begin{cases} d_{\beta}/\sqrt{n} & \text{when } m = 1 \\ 0 & \text{when } m = 2 \end{cases} \qquad (3.2)$$

$$\underline{U}_{\beta}^{(\alpha)} = (\underline{W}_{\beta}^{(\alpha)} - \underline{\eta}_{\beta}^{(\alpha)})/\sqrt{v_{\alpha}}, \qquad (3.3)$$

$$\underline{U}_{\beta}^{(\alpha)} \sim N(\underline{0}_{(k-1)}, \underline{E}_{(k-1)} + \underline{G}_{(k-1)}), \qquad (3.4)$$

and $a_n \approx b_n$ means that $\lim a_n/b_n = 1$ as $n \to \infty$.

For the subset selection approach $(m = 1)$, since

$$\kappa_{\beta kj} - \kappa_{\beta ij} \geq 0$$

and

$$\kappa_{\beta kj} \geq 0$$

for large n, we have

$$\underline{\eta}_{\beta}^{(1)} \geq \underline{0}_{(k-1)}.$$

Also, for indifference zone approach $(m = 2)$, with the specification

$$\varphi_{\beta}(\theta_k, \theta_i) \geq c_{\beta} + \delta_{\beta}^{*},$$

we have

$$\underline{\eta}_{\beta}^{(2)} \geq \begin{cases} (k \int f^2(x)dx)\sqrt{n}\ \delta_1^{*}\ \underline{J}_{(k-1)} & \text{when } \beta = 1 \\ (k \int x f^2(x)dx)\sqrt{n}\ (\delta_2^{*}/(1 + \delta_2^{*}))\ \underline{J}_{(k-1)} & \text{when } \beta = 2. \end{cases}$$

Thus we have the following

Theorem 3.1

Under the assumption of order restriction (2.7) and for large n, the (asymptotic) LFC of the PCS for rules $R(\alpha, \beta, 1)$ $(\alpha, \beta = 1,2)$ are given by

$$\kappa_{\beta ji} = 0, \quad i,j = 1,2,\ldots,k; \ \alpha, \beta = 1,2 \tag{3.5}$$

and for rules $R(\alpha, \beta, 2)$ $(\alpha, \beta = 1,2)$ are given by

$$\kappa_{\beta ji} = c_\beta, \quad i,j = 1,2,\ldots,k-1, \ i \neq j;$$

$$\kappa_{\beta ki} = c_\beta + \delta_\beta^*, \quad i = 1,2,\ldots,k-1; \ \alpha, \beta = 1,2. \tag{3.6}$$

Under the asymptotic LFC

$$P(CS|R(\alpha, \beta, m) \gtrsim \Pr\{\underline{U}_\beta^{(\alpha)} \leq ((\gamma(\beta, m) + \delta(\beta, m))/\sqrt{v_\alpha})\ \underline{J}_{(k-1)}\} \tag{3.7}$$

where v_α is given by (2.21), $\delta(\beta, m)$ is given by (3.2) and $\gamma(\beta, m)$ is defined by

$$\begin{cases} \gamma(\beta,1) = 0 & \text{for } \beta = 1,2 \\ \gamma(\beta,2) = \begin{cases} (k \int f^2(x)dx)\sqrt{n}\,\delta_1^* & \text{for } \beta = 1 \\ (k \int xf^2(x)dx)\sqrt{n}(\delta_2^*/(1+\delta_2^*)) & \text{for } \beta = 2. \end{cases} \end{cases} \tag{3.8}$$

The expression in (3.7) can be rewritten as

$$P(CS|R(\alpha, \beta, m) \gtrsim \int \Phi^{k-1}\{x + (\gamma(\beta, m) + \delta(\beta, m))/\sqrt{v_\alpha}\}d\Phi(x) \tag{3.9}$$

for α, β, $m = 1,2$, where $\Phi(x)$ is the standard normal c.d.f.

Determination of the d_β Values

By Theorem 3.1, the d_β values can be asymptotically expressed as

$$d_\beta(n) = \sqrt{n/2}\, d\sqrt{v_\alpha} + o(n^{1/2}), \quad \alpha, \beta = 1,2 \tag{3.10}$$

where d is the solution of the following equation:

$$Q(d/\sqrt{2}, d/\sqrt{2}, \ldots, d/\sqrt{2}) = P^*, \tag{3.11}$$

Q is the joint cdf of a normally distributioned vector $(V_1, V_2, \ldots, V_{k-1})$ with $E(V_i) = 0$, $Var(V_i) = 1$ and $Cov(V_i, V_j) = 1/2$, $i \neq j$.

Determination of the Minimum Common Sample Size

In the subset selection approach, let S denote the size of the selected subset and $E_{\underline{\theta}}[S|R]$ the corresponding expected value when subset selection rule R is applied and $\underline{\theta}$ is the true nature of status. Then

$$E_{\underline{\theta}}[S|R] = \sum_{j=1}^{k} P\{\Pi_j \text{ is selected } |R\} \qquad (3.12)$$

Having determined $d_1(n)$ from (3.10), one may determine the common sample size n by imposing the additional requirement that

$$E_{\underline{\theta}}[S|R] \leq 1 + \varepsilon \qquad (3.13)$$

for some $0 < \varepsilon < k-1$, where $\underline{\theta}$ lies in a given proper subset of the parameter space under study, for example, the subset defined by $\theta_{[1]} = \theta_{[2]} = \ldots = \theta_{[k-1]} = \theta_{[k]} - \varepsilon^{(n)}$ for location parameters case and $\theta_{[1]} = \theta_{[2]} = \ldots = \theta_{[k-1]} = \theta_{[k]}/(1 + \Delta^{(n)})$ for scale parameters, where $\varepsilon^{(n)} > 0$ and $\Delta^{(n)} > 0$.

Asymptotically Relative Efficiency

Given two selection procedures R_1 and R_2, the (asymptotic) efficiency of R_2 relative to R_1 is defined by

$$\text{Eff}(R_1, R_2) = n_1/n_2 \qquad (3.14)$$

where the n_i are the sample sizes required to satisfy $P(CS|R_i)_{LFC} = P^*$, $i = 1,2$. Then, by Theorem 3.1 and (3.10), we have the following result:

$$\text{Eff}(R(1,\beta,1), R(2,\beta,1)) = 1, \quad \beta = 1,2,$$

$$\text{Eff}(R(\alpha,1,1), R(\alpha,2,1)) = 1, \quad \alpha = 1,2,$$

$$\text{Eff}(R(1,\beta,2), R(2,\beta,2)) = k/(k+1), \quad \beta = 1,2,$$

$$\text{Eff}(R(\alpha,1,2), R(\alpha,2,2))$$

$$= (\delta_1^* \delta_2^*/(1 + \delta_2^*))^2 \left(\int x f^2(x)dx / \int f^2(x)dx\right)^2, \quad \alpha = 1,2. \qquad (3.15)$$

4. APPENDIX

Let k population Π_1, Π_2, ..., Π_k be given, where Π_i has the associated continuous distribution $F_s(x)$ (s = 1,2,...,k). Take n_s observations X_{s1}, X_{s2}, ..., X_{sn_s} from population π_s (s = 1,2,...,k) and consider the combined (Wilcoxon type) rank R_{sj} of X_{sj} as stated in Section 1. Then the means, variances and co-variances of the ranks R_{sj} are given in the following.

Theorem 4.1

$$E(R_{sj}) = N \int G dF_s + 1/2 \tag{4.1}$$

$$V(R_{sj}) = 2N \int G dF_s - 2N \int F_s G dF_s + N^2 \int G^2 dF_s - N \int H dF_s - N^2\left(\int G dF_s\right)^2 - 1/12 \tag{4.2}$$

$$\mathrm{Cov}(R_{si}, R_{sj}) = 3N \int G dF_s - 4N \int F_s G dF_s - \sum_{m=1}^{k} n_m \left(\int F_m dF_s\right)^2 - 1/12 \tag{4.3}$$

$$\begin{aligned}\mathrm{Cov}(R_{si}, R_{tj'}) = {} & N\left(2 - \int F_t dF_s\right) \int G dF_t + N\left(2 - \int F_s dF_t\right) \int G dF_s \\ & - \sum_{m=1}^{k} n_m \int F_m dF_s \int F_m dF_t - 2N \int F_t G dF_s - 2N \int F_s G dF_t \\ & + \int F_s dF_t \int F_t dF_s + \int F_s^2 dF_t + \int F_t^2 dF_s - 1\end{aligned} \tag{4.4}$$

where s,t = 1,2,...,k, $s \neq t$; i,j = 1,2,...,n_s, $i \neq j$; j' = 1,2,...,n_t and

$$N = \sum_{m=1}^{k} n_m \tag{4.5}$$

$$G(x) = (1/N) \sum_{m=1}^{k} n_m F_m(x) \tag{4.6}$$

$$H(x) = (1/N) \sum_{m=1}^{k} n_m F_m^2(x) \tag{4.7}$$

Proof

We sketch the proofs for (4.1) and (4.3) above. The remaining results are obtained similarly.

Mean :

$$\Pr(R_{11} = s) = \sum_A \Pr(a_1 \text{ of } X_1\text{'s}, a_2 \text{ of } X_2\text{'s}, \ldots, a_k \text{ of } X_k\text{'s} \leq X_1 \leq$$

$$(n_1-a_1-1) \text{ of } X_1\text{'s}, (n_2-a_2) \text{ of } X_2\text{'s}, \ldots, (n_k-a_k) \text{ of } X_k\text{'s}) \quad (4.8)$$

where a_i $(i = 1,2,\ldots,k)$ is an integer such that

$$0 \leq a_1 \leq n_1 - 1, \quad 0 \leq a_i \leq n_i \quad (i = 2,3,\ldots,k) \quad (4.9)$$

$$\sum_{j=1}^{k} a_j = s - 1 \quad (4.10)$$

and "a_i of X_i's", "(n_i-a_i) of X_i's" should be read as "a_i variables out of $(X_{i1}, X_{i2}, \ldots, X_{in_i})$ and remaining (n_i-a_i) variables", and so forth. Further, summation $\sum_A$ is taken over all k-tuples $(a_1,a_2,\ldots,a_k)$ of integers which satisfy the relations (4.9) and (4.10). From (4.8), we have

$$E(R_{11}) = \int \sum_{s=1}^{N} \sum_A s \binom{n_1-1}{a_1}\binom{n_2}{a_2} \cdots \binom{n_k}{a_k} F_1^{a_1} F_2^{a_2} \cdots F_k^{a_k}$$

$$\times (1-F_1)^{(n_1-a_1-1)} (1-F_2)^{(n_2-a_2)} \cdots (1-F_k)^{(n_k-a_k)} dF_1 \quad (4.11)$$

By changing the order of summation, we have

$$E(R_{11}) = \int \sum_{s=1}^{N} \sum_{A_1} s \binom{n_1-1}{a_1}\binom{n_2}{a_2} \cdots \binom{n_{k-1}}{a_{k-1}} F_1^{a_1} F_2^{a_2} \cdots F_{k-1}^{a_{k-1}}$$

$$\times (1-F_1)^{(n_1-a_1-1)} (1-F_2)^{(n_2-a_2)} \cdots (1-F_{k-1})^{(n_{k-1}-a_{k-1})} (n_k F_k + \sum_{i=1}^{k-1} a_i + 1) dF_1$$

where the summation $\sum_{A_1}$ is taken over all (k-1)-tuples $(a_1,a_2,\ldots,a_{k-1})$ of integers which satisfy the relation (4.9). Adding in turn over a_{k-1}, a_{k-2}, $\ldots$, a_1, we obtain the result for $E(R_{11})$.

Covariance :

For $s < t$, we have

$$\Pr(R_{11} = s, R_{21} = t) = \sum_B \Pr(a_1 \text{ of } X_1\text{'s}, a_2 \text{ of } X_2\text{'s}, \ldots, a_k \text{ of } X_k\text{'s}$$

$$\leq X_{11} \leq b_1 \text{ of } X_1\text{'s}, b_2 \text{ of } X_2\text{'s}, \ldots, b_k \text{ of } X_k\text{'s}$$

$$\leq X_{21} \leq c_1 \text{ of } X_1\text{'s}, c_2 \text{ of } X_2\text{'s}, \ldots, c_k \text{ of } X_k\text{'s}) \quad (4.12)$$

where a_i, b_i, c_i $(i = 1,2,\ldots,k)$ are integers such that

$$a_i + b_i + c_i = \nu_i, \quad i = 1,2,\dots,k \tag{4.13}$$

$$\sum_{j=1}^{k} a_j = s - 1, \quad \sum_{j=1}^{k} b_j = t - s - 1, \quad \sum_{j=1}^{k} c_j = n - t \tag{4.14}$$

and $\nu_i = n_i - 1$ for $i = 1,2$, $\nu_i = n_i$ for $i = 3,4,\dots,k$.

Summation $\sum_B$ is taken over all tuples $(a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_k)$ which satisfy the relations (4.13) and (4.14). Then

$$I_1 \equiv \sum_{s<t} s\, t\, Pr(R_{11} = s, R_{12} = t)$$
$$= \iint_{x<y} \sum_{s<t} \sum_B \prod_{i=1}^{k} P_i(x,y)dF_1(x)dF_2(y) \tag{4.15}$$

where

$$P_i(x,y) = \binom{\nu_i}{a_i, b_i, c_i} F_i^{a_i}(x) \ (F_i(y)-F_i(x))^{b_i} \ (1-F_i(y))^{c_i}, \quad i = 1,2,\dots,k.$$

By changing the order of summation, first for s then for t, we have

$$I_1 = \iint_{x<y} \sum_{s<t} \sum_{B_1} C_1 \prod_{i=1}^{k-1} P_i(x,y)dF_1(x)dF_2(y)$$

where

$$C_1 = \alpha_1 + \beta_1 \sum_{j=1}^{k-1} a_j + \gamma_1 \sum_{j=1}^{k-1} b_j + (\sum_{j=1}^{k-1} a_j)^2 + (\sum_{j=1}^{k-1} a_j)(\sum_{j=1}^{k-1} b_j)$$

and

$$\alpha_1 = n_k(n_k-1)\ F_k(x)F_k(y) + 3n_k\ F_k(x) + n_k\ F_k(y) + 2$$

$$\beta_1 = n_k\ F_k(x) + n_k\ F_k(y) + 3$$

$$\gamma_1 = n_k\ F_k(x) + 1.$$

Summation $\sum_{B_1}$ is taken over all tuples $(a_1, \dots, a_{k-1}, b_1, \dots, b_{k-1}, c_1, \dots, c_{k-1})$ which satisfies the condition (4.13). By carrying out the addition in turn for a set (a_i, b_i, c_i), $i = k-1,k-2,\dots,1$, we have a reduced form of I_1. By proceeding on similar steps for $\sum_{s>t} s\, t\, Pr(R_{11} = s, R_{21} = t)$, we obtain $Cov(R_{11},R_{21})$.

For rank sums

$$T_s = \sum_{j=1}^{n_s} R_{sj}, \quad s = 1,2,\dots,k \tag{4.16}$$

we have

$$E(T_s) = n_s\, E(R_{sj}), \quad s = 1,2,\ldots,k \tag{4.17}$$

$$Cov(T_s,T_t) = n_s\, n_t\, Cov(R_{sj},R_{tj'}), \quad s,t = 1,2,\ldots,k,\ s \neq k, \tag{4.18}$$

and

$$V(T_s) = \sum_{j=1}^{n_s} V(R_{sj}) + \sum_{i\neq j}^{n_s} Cov(R_{si},R_{sj})$$

$$= Nn_s(3n_s-1) \int GdF_s - 2Nn_s(2n_s-1) \int F_sGdF_s + N^2n_s \int G^2dF_s$$

$$- Nn_s \int HdF_s - N^2n_s(\int GdF_s)^2 - n_s(n_s-1) \sum_{m=1}^{k} n_m(\int F_m dF_s)^2 - n_s^2/12. \tag{4.19}$$

Especially, if $F_i(x) = F(x)$ for all i, then we have

$$E(T_s) = n_s(N+1)/2, \tag{4.20}$$

$$V(T_s) = n_s(N-n_s)(N+1)/12, \tag{4.21}$$

$$Cov(T_s,T_t) = -n_sn_t(N+1)/12. \tag{4.22}$$

Also for k = 2, we have the following:

$$E(T_i) = n_i(n_i+1)/2 + n_in_j \int F_jdF_i, \quad i,j = 1,2;\ j \neq i \tag{4.23}$$

$$V(T_i) = n_in_j(2n_i-1) \int F_jdF_i + n_in_j(n_j-1) \int F_j^2\, dF_i$$

$$+ n_in_j(n_i-1) \int F_i^2\, dF_j - n_in_j(n_i+n_j-1)(\int F_jdF_i)^2 - n_in_j(n_i-1)$$

$$i,j = 1,2;\ i \neq j, \tag{4.24}$$

$$Cov(T_1,T_2) = n_1n_2[n_1 \int F_1dF_2 + n_2 \int F_2dF_1 - (n_1+n_2-1) \int F_1dF_2 \int F_2dF_1$$

$$- (n_1-1) \int F_1^2\, dF_2 - (n_2-1) \int F_2^2\, dF_1 - 1]. \tag{4.25}$$

Finally, we state an order relation for the expected rank sum. Let $\{F_\theta(x)\}$ be a family of distributions stochastically increasing in θ. Then we have the following.

<u>Theorem 4.2</u>

$E(R_s) \geq E(R_t)$ if and only if $F_s \leq F_t$ where $s,t = 1,2,\ldots,k$.

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER Technical Report #87-2	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) ON THE LEAST FAVORABLE CONFIGURATION OF A SELECTION PROCEDURE BASED ON RANKS	5. TYPE OF REPORT & PERIOD COVERED Technical
	6. PERFORMING ORG. REPORT NUMBER Technical Report #87-2
7. AUTHOR(s) Shanti S. Gupta and Takashi Matsui	8. CONTRACT OR GRANT NUMBER(s) N00014-84-C-0167
9. PERFORMING ORGANIZATION NAME AND ADDRESS Purdue University Department of Statistics West Lafayette, IN 47907	10. PROGRAM ELEMENT, PROJECT, TASK, AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Washington, DC	12. REPORT DATE January 1987
	13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report) Unclassified
	15a. DECLASSIFICATION, DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Selection Procedures, Combined Ranks, Least Favorable Configuration, Moments of Ranks, Wilcoxon Type, Friedman Type

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We consider two types of statistics based on the sums of combined (Wilcoxon type) ranks and vector (Friedman type) ranks. Underlying populations are supposed to belong to the location or scale parameter family of distributions.

Two approaches - subset selection and indifference zone - of ranking and selection procedures based on these statistics are considered in an asymptotic framework for selecting the population with the largest parameter value. The least favorable configurations of parameters are discussed by computing the

exact moments of these statistics and introducing an assumption of order relation between the gaps of parameters.