REGULAR EXTENSIONS OF MEASURES

by

Herman Rubin†
Purdue University

Technical Report #86-57

Department of Statistics
Purdue University

December 1986

† Research supported by the National Science Foundation under Grant DMS-8401996.

◇

Regular Extensions of Measures

Herman Rubin†

Purdue University

ABSTRACT

We obtain conditions for the "natural" method for the extension of a (not necessarily regular) measure from the field generated by a lattice base to a larger field to be valid. We show that the same method allows us to obtain limit theorems for weak convergence of measures on topological spaces, and we obtain weak convergence of measurable measure-valued functions. as a corollary.

Let m be a (not necessarily regular) measure on the field generated by the elements of a lattice base A. We define a lattice base B to be k-almost compact relative to m, A if for every $\epsilon > 0$ and every B-open cover U of at most k elements there is a finite collection C of A-closed sets (complements of elements of A) such that each element of C is contained in some element of U and the C-sets cover all except m-measure ϵ. The values of k we shall use are finite (equivalently 2), countable, and arbitrary.

THEOREM 1.*If B is* 2*-almost compact relative to* m, A*, then there is a natural "extension" of* m *to a regular measure* n *on B; if the almost compactness is countable,* n *will be countably additive; and if the almost compactness is arbitrary,* n *will be almost Lindelöf.*

Since m need not be regular, we define n to be an extension of m if n does not increase (decrease) the measure of A-open (closed) sets.

PROOF: Let F be a B-closed set. Define

$$n(F) = \inf\{m(U) : F \subset U \text{ and } U \in \text{A}\}.$$

We need to show that $n(F) + n(G) = n(F \cup G) + n(F \cap G)$. It is only necessary to show that the left side is bounded by the right. Let $V \in$ A contain $F \cup G$ and W contain $F \cap G$. We may assume V contains W and that the approximations are within ϵ in the definition of n. Consider the open cover consisting of $V \backslash F$, $V \backslash G$, W, and $\sim(F \cup G)$. Let A, B, C, and D be A-closed sets contained in the respective open sets, and let Z, $m(Z) < \epsilon$, be

† Research supported by the National Science Foundation under Grant DMS-8401996.

the remainder. Since A is contained in $V\backslash F$ and F is contained in V, $V\backslash A$ is an A-open set containing F; similarly, $V\backslash B$ contains G. Now $(V\backslash A) \cup (V\backslash B)$ is a subset of V, but $(V\backslash A) \cap (V\backslash B)$ need not be a subset of W. We only need show that it is not too much larger than W.

We see that $(V\backslash A) \subset B \cup C \cup D \cup Z$ and we can interchange A and B. Thus $(V\backslash A) \cap (V\backslash B) \subset C \cup (V \cap D) \cup Z$. Now W contains C and Z has small measure, so we need only show that $V \cap D$ also has a small measure. However, D is a closed set disjoint from $F \cup G$, and hence its intersection with V cannot have measure as large as ϵ.

Since n is modular on the B-closed sets, it it easy to show that by defining $n(E\backslash F) = n(E) - n(E \cap F)$ we obtain a finitely additive function on the semi-ring of sets we have just described. Then by the usual method of defining n for disjoint finite unions to be the sum, it can be uniquely extended to a measure on the field F generated by B. It is also clear that n is at least as large as m on A-closed sets. We need only show that for every element A of F and $\epsilon > 0$, there is a C whose complement is in B, which is contained in A, and $n(C) > n(A) - \epsilon$. It is easily seen that we need do this only for B open sets. That is, we must show that any open A has a closed subset C with $n(C \cup {\sim}A) > m(X) - \epsilon$. Now let U be any A-open superset of ${\sim}A$ with $m(U) < n({\sim}A) + \epsilon$. Then $C = {\sim}U$ is such a closed set.

To show countable additivity in the countably-almost compact case, let E_i be a disjoint sequence of elements of F with union $H \in \mathcal{F}$. Let $U_i \in \mathcal{B}$ be such that $E_i \subset U_i$ and $n(E_i) > n(U_i) - \epsilon/2^i$. Let ${\sim}F \in \mathcal{B}$ such that $F \subset H$ and if $T = H\backslash F$, $n(T) < \epsilon$. Apply countable-almost compactness to the cover by ${\sim}F$ and the U's. This yields closed sets C_i, $i \leq n$, a closed set K, and an open set Z which are together a cover and $C_i \subset U_i$, $K \subset {\sim}F$, and $n(Z) < \epsilon$. Let D be the union of the C's. Then $H \subset D \cup K \cup Z$, and also $H \subset F \cup T$. Since F and K are disjoint, $H \subset D \cup T \cup Z$. Clearly $n(D) < \sum n(E_i) + \epsilon$, so $n(H) < \sum n(E_i) + 3\epsilon$, qed.

Trivially, if almost compactness holds with arbitrary covers, the measure n is almost Lindelöf.

An immediate consequence of this theorem is the result that a countably additive almost Lindelöf regular measure can be extended from the field generated by a lattice base to the σ-field generated by all open sets. Another consequence is that a net of measures on the field generated by a lattice base has the property that every subnet has a subnet converging weakly to a countably additive almost Lindelöf measure (and correspondingly weaker results with weaker conditions) if it is asymptotically almost compact; this is the appropriate condition to replace tightness.

A problem which appears to be totally different also yields to the foregoing analysis. Let P_α be a net of probability measures on the field generated by $\mathcal{A}$. Let us define $m(U) = \liminf P_\alpha(U)$ for U "open" and $m(F) = \limsup P_\alpha(F)$ for F "closed", and suppose that whenever $F \subset U$ we have $m(F) \leq m(U)$. We need a compactness condition; we say that we have *k-asymptotic regular almost compactness* if for every open covering and $\epsilon > 0$ there is a finite closed refinement which comes within ϵ for α sufficiently large. We have

THEOREM 2. *Under the above conditions, the "natural extension" of m to a measure n on B satisfies the conclusions of Theorem 1.*

Unfortunately, I have not found a proof that the consructed measure n has its properties without using non-constructive means. Define n on closed sets as $n(F) = \inf_{F \subset U} m(U)$. We need only show that this is modular. Now let Q_β be any subnet of P_α which converges for all elements of $\mathcal{A}$, and let m' be the corresponding limit. Since the conditions of Theorem 1 are satisfied, m' has a regular extension n'. To simplify arguments let us assume $\mathcal{A} = \mathcal{B}$, as otherwise the proof can be made in two stages.

Clearly, for all open U, $m(U) \leq m'(U)$. Hence for all closed F, $n(F) \leq n'(F)$. However, since $m(F) \leq m'(F) \leq n'(F)$, we must have for all $U \supset F$ that $m(U) \geq n'(U)$ since n' is regular. Again using the regularity of n', we see that $n(F) \geq n'(F)$. Thus for all closed F, we have $n(F) = n'(F)$, n extends in the natural manner to the same regular measure as does n', q.e.d.

There should be a result just based on the properties of m, and a direct proof similar to that of Theorem 1. Unfortunately, the proof of Theorem 1 uses the fact that we start out with a finitely additive measure, and I am unaware of the precise requirements of the "measure" m in Theorem 2.

If the initial lattice base is countable, and we consider "random strategies", that is, functions from a measure space to (probability) measures, we obtain a weak convergence result because the extension is the infimum of a countable number of measurable functions on closed sets, and is otherwise formed by explicit finite and countable operations. Thus, if a net of random strategies converges in L_∞ to a countable-almost compact measure on the field generated by a countable lattice base, the net converges weakly to a countably additive random strategy.