# A New Approach to Integration

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ABSTRACT

We give an approach to integration which is easy to use, conveys the intuitive concept, and covers many of the Riemann and Lebesgue type integrals, as well as some others. We present a one-sentence definition of the integral which has these properties, and point out how the definition extends to more general cases. The author believes that the teaching of integration by this approach would alleviate the lack of understanding typically found.

Among the criteria of a good definition of a mathematical concept are conciseness and the conveyance of an understanding of the concept. The usual treatment of the Riemann integral comes close to meeting these requirements. However, there is no need to confuse things by requiring a good approximation to hold for all partitions into sufficiently small intervals instead of for some partition into intervals. The usual treatments of Lebesgue type integration frequently fail to meet either of these criteria. In fact, most textbooks do not give a formal definitions of the Lebesgue or Lebesgue-Stieltjes integral. We claim that the definition given here satisfies these criteria, and we also illustrate the ease of direct generalization of the definition.

We now give an approach to integration which can be started at an elementary level and extends with little change to include many of the integrals in mathematics.

We start with the discrete integral. Let $\mathcal{M}$ be a collection of "measurable" sets. These do not have to form a field. For example, the collection of intervals on the line or rectangles in the plane forms a suitable collection. It is sufficient that for two "permitted" partitions of the space into disjoint measurable sets, there is a common permitted refinement partition. It is not necessary that all partitions are permitted. Partitions may be finite or countable, and may be required to be ordered in a "natural" order.

The elementary notion is that if $f$ is a function constant on cells of a partition $S_i, i \in I$ then

$$\int f \, d\mu = \sum f(x)\mu(S_i),$$

where for each $i$, $x \in S_i$.

This can even be exemplified by the evaluation of the total of a grocery bill, where $\mu(S_i)$ is the quantity of $i$ and the corresponding value $f(x)$ is the price of $i$, or the computation of a student's total number of grade points, where $\mu(S_i)$ is the number of credits in course $i$ and $f(x)$ is the numerical value of the grade. Additional complications can be introduced in the case of order; the sum need not be absolutely convergent, and in the case of a non-abelian group, addition need not be commutative. It is not necessary that the index set be the integers; a sum which has several points at which it is necessary to take a limit is certainly possible. A somewhat complicated example of this is the set

$$I = \{m/n \ : \ \text{for some } i, j, \ 0 \leq i < j < 5, \ |mj - ni| < 3\}.$$

and we require that if $m/n < p/q$, all elements of $S_{m/n}$ precede all elements of $S_{p/q}$.

The reader may not be used to seeing integration presented in this manner. Unfortunately, most students and many mathematicians are even ignorant that the above is integration. One may object that the usual integral is not subsumed in the above definition. That is correct and will now be rectified. We will call a function integrable if it can be adequately approximated by such a function above. Formally, we can define in the commutative case:

$a = \int f d\mu$ if for every $\epsilon > 0$ there exist:

a partition $\{S_i\colon \ i \ \in \ I\}$ of measurable sets,

a sequence $\{v_i\colon \ i \ \in \ I\}$ of values, and

a sequence $\{e_i\colon \ i \ \in \ I\}$ of errors, such that

(1) $|f(x) - v_i| \leq e_i$ for all $x \in S_i$,

$$(2)\quad |a - \Sigma v_i \mu(S_i)| < \epsilon,$$

$$(3)\quad \Sigma e_i \mu(S_i) < \epsilon.$$

The error sums are in the sense of unconditional convergence and $0 \cdot \infty = 0$ wherever it occurs.

If we take I to be the integers and require that the sums are in the sense of absolute convergence, this gives the usual Lebesgue integral. If we require I to be finite and require the $S_i$ to be intervals, we get the Riemann integral. A very useful integral is obtained if we require the $S_i$ to be intervals, and allow $I$ to be countable.

If $\mu$ is a signed measure, we can obtain the usual integral by replacing $\mu$ by $|\mu|$ in (3). Furthermore, we do not need $f$ or $\mu$ to be real-valued. If $f$ takes values in the linear space $F$, $\mu$ takes values in the linear space $G$, and we have suitable magnitude functions, which need not necessarily be real valued, the modifications of (1), (2), and (3) are immediate. We can obtain the various integrals of Banach-space valued functions in this way, and the spectral theorem in Hilbert space is easily obtained in this form, even becoming a Riemann type integral for bounded Hermitian operators.

By requiring the index set I to be ordered, we can certainly treat the case of integrals which require limits as certain points are approached. The extent of this approach is limited by the ability to define the sum of a countable ordered collection. It does not extend to Cauchy principal values, stochastic integrals which are not themselves of the Riemann- or Lebesgue-Stieltjes type, Denjoy integrals, etc.

The non-commutative case is somewhat more difficult, and we do not have a satisfactory general definition. However, the idea that the integral is an entity which can be approximated by discrete sums remains, and the additional work needed is to obtain the appropriate sufficient, and if possible necessary, conditions for passage to the limit.

Since this approach is natural and simple, one may ask why it was not expounded much earlier. I believe that, while the notion of the integral, and even the symbol, were derived from the notion of sum, mathematicians did not have available set notation and the formal concept of set. This is evident in many of the papers on probability by great mathematicians of the nineteenth and early twentieth centuries.