On the Barlow-Yor Inequalities for Local Time

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Summary. An idea of Burkholder is used to give a simple proof of the Barlow-Yor martingale local time inequalities. Related inequalities are proved for some stable processes.

Let $L_t^a, -\infty < a < \infty,\ t \geq 0$, be jointly continuous local time for the standard brownian motion $B = B_t,\ t \geq 0$, and put $L_t^* = \sup_a L_t^a$. In [2], (see also [3]), M.T. Barlow and M. Yor show the existence of absolute constants c_p and C_p such that, if τ is a stopping time for B,

$$c_p E\tau^{p/2} \leq EL_\tau^{*p} \leq C_p E\tau^{p/2}, p > 0. \tag{1}$$

Brownian motion is the normalized symmetric stable process of index 2, and Trotter [6] proved it has a jointly continuous local time. The symmetric stable processes of index $\alpha \in (1,2)$, as well as some other stable processes, also have a jointly continuous local time (see [1]). We prove the following theorem.

Theorem 1. Let $Z = Z_t,\ t \geq 0$, be a stable process of index α with jointly continuous local time L_t^a, and put $L_t^* = \sup_a L_t^a$. There exist positive constants k_p and K_p, depending only on Z, such that if τ is a stopping time for Z,

$$k_p E\tau^{p/\alpha} \leq EL_\tau^{*p} \leq K_p \tau^{p/\alpha},\ p > 0. \tag{2}$$

Our proof of Theorem 1 uses scaling to prove good-bad lambda inequalities and should be thought of as an adaptation of a similar argument used by D.L. Burkholder ([4]) in the context of maximal functions for n dimensional Brownian motion. The Barlow-Yor proofs also involved good-bad lambda inequalities and thus both proofs give a generalization of (1) (and in our case (2)) to functions other than x^p which satisfy a growth condition. See [5], p. 154, (3). Also, (1) may be rephrased as a result about continuous martingales. See [2]. Theorem 1 is the first extension we know of (1) to discontinuous processes, a question mentioned in [3].

Now (1) is proved. The proof immediately generalizes to a proof of Theorem 1. It will be shown that there are functions $\alpha(t)$ and $\beta(t)$ on $(0,\infty)$ which approach zero as t approaches zero and such that for any stopping time τ and any δ, λ both exceeding 0,

$$P(\tau^{1/2} > 2\lambda,\ L_\tau^* \leq \delta\lambda) \leq \alpha(\delta)P(\tau^{1/2} > \lambda), \tag{3}$$

and

$$P(L_\tau^* > 2\lambda,\ \tau^{1/2} \leq \delta\lambda) \leq \beta(\delta)P(L_\tau^* > \lambda). \tag{4}$$

These are the Burkholder-Gundy good-bad lambda inequalities. They quickly, essentially upon integration, give (1). We have written (3) and (4) in such a form that readers unfamiliar with this may follow, line for line, the presentation in [5], p.154, with δ^2 there replaced by $\alpha(\delta)$ and $\beta(\delta)$.

The functions α and β are defined by $\alpha(\delta) = P(L_1^* \le \delta/\sqrt{3})$ and $\beta(\delta) = P(v_1 \le \delta^2)$, where $v_a = \inf\{t : L_t^* = a\}$. To show that both $\alpha(\delta)$ and $\beta(\delta)$ approach zero as $\delta \to 0$ we must show $P(L_1^* = 0) = 0$ and $P(v_1 = 0) = 0$. The first of these equalities is immediate, for example, from the facts that $L_1^* \ge L_1^0$ and $P(L_1^0 = 0) = 0$, or in several other ways. That $P(v_1 = 0) = 0$ follows from the joint continuity of L_t^a in t and a, and the fact that $L_t^a = 0$ if $| a |> \sup_{0\le s\le t} | B_s |= \Phi(t)$. Since $\Phi(t) \to 0$ as $t \to 0$, on $\{v_1 = 0\}$, $L_t^a \ge 1$ for (a, t) arbitrarily close to $(0,0)$ which, since $L_0^0 = 0$, contradicts joint continuity.

Now if $\gamma > 0$, the process $\gamma^{-1/2}B_{\gamma t}$, $t \ge 0$, is standard Brownian motion, so if $a_1, \ldots, a_m$ are any numbers and $t_1, \ldots, t_m$ are nonnegative numbers the distributions of the two random vectors $(L_{t_i}^{a_j})_{1\le j\le m,\ 1\le i\le n}$ and $(\gamma^{-1/2}L_{\gamma t_i}^{\sqrt{\gamma}a_j})_{1\le j\le m,\ 1\le i\le n}$ are the same. Together with the joint continuity of L_t^a this yields

$$L_t^* \overset{\text{dist.}}{=} \sqrt{t}L_1^*, \tag{5}$$

and

$$v_{\sqrt{\gamma}} \overset{\text{dist.}}{=} \gamma v_1. \tag{6}$$

Let $L_{[c,d]}^* = \sup_a(L_d^a - L_c^a)$. The third of the following inequalities follows from the first two.

$$L_{[x,y]}^* + L_{[y,z]}^* \ge L_{[x,z]}^*,\ 0 \le x \le y \le z. \tag{7}$$

$$L_{[x,y]}^* \overset{\text{dist.}}{=} L_{y-x}^*,\ 0 \le x \le y. \tag{8}$$

$$P(v_b - v_a \le \theta) \ \le\ P(v_{b-a} \le \theta) \ \text{ if } \ 0 \le a \le b,\ \theta \ge 0. \tag{9}$$

Next we prove (3). Assume $P(\tau^{1/2} > \lambda) > 0$. Then

$$\begin{aligned} P(\tau^{1/2} > 2\lambda,\ L_\tau^* \le \delta\lambda \mid \tau^{1/2} > \lambda) &\le P(L_{4\lambda^2}^* \le \delta\lambda \mid \tau^{1/2} > \lambda) \\ &\le P(L_{[\lambda^2,4\lambda^2]}^* \le \delta\lambda \mid \tau^{1/2} > \lambda) \\ &= P(L_{3\lambda^2}^* \le \delta\lambda) \\ &= P(L_1^* \le \delta/\sqrt{3}), \end{aligned}$$

using the Strong Markov Property and (5) for the last two inequalities. The proof of (4) is similar. Assume $P(L_\tau^* > \lambda) > 0$. Then

$$\begin{aligned} &P(L_\tau^* > 2\lambda,\ \tau^{1/2} \le \delta\lambda \mid L_\tau^* > \lambda) \\ &\qquad = P(v_{2\lambda} < \tau,\ \tau^{1/2} \le \delta\lambda \mid v_\lambda < \tau) \\ &\qquad \le P(v_{2\lambda} < \tau,\ (v_{2\lambda} - v_\lambda)^{1/2} \le \delta\lambda \mid v_\lambda < \tau) \\ &\qquad \le P((v_{2\lambda} - v_\lambda)^{1/2} \le \delta\lambda \mid v_\lambda < \tau) \\ &\qquad = P((v_{2\lambda} - v_\lambda)^{1/2} \le \delta\lambda) \\ &\qquad \le P(v_\lambda^{1/2} \le \delta\lambda) = P(v_1 \le \delta^2), \end{aligned}$$

using (9) and (6) for the last two steps.

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