Further Results on Testing a Poisson Hypothesis Against Compound Poisson Alternatives* **

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Further Results on Testing a Poisson Hypothesis
Against Compound Poisson Alternatives*

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The paper extends the author's previous results for testing a Poisson hypothesis against compound Poisson alternatives based on a sample $X_1, X_2, \ldots, X_n$ of i.i.d. random variables to the case where the X_i's, although mutually independent, are not necessarily identically distributed. In particular, we address ourselves to those situations where for each compound Poisson point process $\{X_i(t),\ \ t \geq 0\}$ all we are able to observe is the total number of events that have occurred by the end of a time period of length t_i, and are otherwise unable to monitor both the points and the sizes of various jumps of the process over time. The approach adopted here is that of Bartoo-Puri (1967); this is an extension of Neyman's (1959) theory of $C(\alpha)$-tests.

1. Introduction and Statement of the Problem.

This work is in continuation to the author's recently appeared paper Puri (1985). The problem considered there was to develop a suitable procedure for testing a Poisson hypothesis against compound Poisson alternatives based on a sample $X_1, X_2, \ldots, X_n$ of independent and identically distributed (i.i.d.) nonnegative integer-valued random variables (r.v.'s). The common distribution of these r.v.'s was taken to be compound Poisson with probability generating function (p.g.f.) given by

$$G(s) \equiv E(s^X) = \exp\{-\lambda(1 - h(s|\xi))\}, \quad |s| \leq 1, \tag{1}$$

where λ is a positive constant and for each $\xi \geq 0$, $h(s|\xi)$ is a p.g.f. given by

$$h(s|\xi) = \sum_{k=1}^{\infty} R(k|\xi)s^k, \qquad |s| \leq 1, \tag{2}$$

with

$$h(0|\xi) = 0, \qquad R(k|\xi) \geq 0, \qquad \sum_{k=1}^{\infty} R(k|\xi) = 1.$$

Here ξ is a nonnegative parameter and is such that for $k \geq 1$,

$$\lim_{\xi \to 0} R(k|\xi) = \delta_{1k}; \qquad \lim_{\xi \to 0} h(s|\xi) \equiv h(s|\xi = 0) \equiv s, \tag{3}$$

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where δ_{1k} is the Kronecker delta. With this the above mentioned hypothesis testing problem corresponds to testing $H_0: \ \xi = 0$ against the alternative $H_1: \ \xi > 0$ in the presence of λ as the nuisance parameter. The reader may refer to Puri (1985) for a detailed motivation to the above formulation along with some applications. In Puri (1985), an optimal $C(\alpha)$-test was developed for the Poisson hypothesis H_0 against H_1 based on Neyman's $C(\alpha)$-test theory (see Neyman (1959)).

Again in many practical situations the X_i's represent the number of events taking place over possibly different lengths t_i of time, based on independently conducted experiments. In particular here (as well as in Puri (1985)) we are addressing ourselves to those situations where we are unable to monitor both the jump points of a compound Poisson point process over time and the sizes of the various jumps. All we are able to observe is the total number of events that have occurred by the end of a time period of length t. The reader may refer to the author's joint work with Neyman in the area of radiation biology for precisely such a situation arising in the case of UV-radiation (see Neyman and Puri (1976, 1981) and in particular Puri (1982)).

Since typically different X_i's correspond to observations taken over different lengths of time, we need to reconsider the original problem of testing our hypothesis H_0 for the case where the X_i's are independent, but not necessarily identically distributed. For this we need to use the Bartoo-Puri generalization of Neyman's $C(\alpha)$-test theory which handles precisely such situations (see Bartoo and Puri (1967)). A brief summary of this generalization is outlined in the next section. In the light of the above remarks, we provide a reformulation of our problem where we allow not only the period of observation but also the p.g.f. h of (2) to vary from observation to observation. Furthermore, it is assumed that the p.g.f.'s h_i otherwise depend not only on the same parameter ξ under test but also on the same nuisance parameter λ. Thus we have $X_1, X_2, \ldots, X_n$, mutually independent nonnegative integer-valued r.v.'s with the p.g.f. G_i of X_i given by

$$(4) \qquad G_i(s) \equiv E(s^{X_i}) = \exp\{-\lambda t_i(1 - h_i(s|\xi,\lambda))\}, \quad |s| \le 1,$$

$i = 1, 2, \ldots, n$, with $\xi \ge 0, \ \lambda > 0$,

$$(5) \qquad h_i(s|\xi,\lambda) = \sum_{k=1}^{\infty} R_i(k|\xi,\lambda)s^k, \qquad |s| \le 1,$$

$$(6) \qquad R_i(k|\xi,\lambda) \ge 0; \qquad \sum_{k=1}^{\infty} R_i(k|\xi,\lambda) \equiv 1,$$

and for every $\lambda > 0, k \ge 1$ and $i = 1, 2, \ldots, n$,

$$(7) \qquad \lim_{\xi \to 0} R_i(k|\xi,\lambda) = \delta_{1k}, \ \lim_{\xi \to 0} h_i(s|\xi,\lambda) \equiv h_i(s|\xi = 0,\lambda) \equiv s,$$

where the t_i's are the known lengths of the corresponding time intervals. Our object is to develop an optimal $C(\alpha)$-test based on $X_1, X_2, \ldots, X_n$, for testing H_0: $\xi = 0$ against

H_1: $\xi > 0$. For this the needed generalization of Neyman's $C(\alpha)$-test theory is summarised in the next section.

2. The Bartoo-Puri generalization of Neyman's $C(\alpha)$-test theory.

In (1959), Neyman developed his $C(\alpha)$-test theory keeping in mind those situations where the distributions of the observable random variables turn out to be highly non-standard and often involve many nuisance parameters, $\theta_1, \ldots \theta_r$, besides the parameter ξ under test. Also, the estimators available for the nuisance parameters may not be too good and in particular may be biased. Under these circumstances, for testing the hypothesis, say H_0: $\xi = \xi_0$, against the alternative hypothesis, say, H_1: $\xi > \xi_0$ (in the presence of nuisance parameters) Neyman developed tests that are locally asymptotically most powerful in a class of so called $C(\alpha)$-tests and are based on a sample of i.i.d. r.v.'s, $X_1, X_2, \ldots, X_n$. Below we present a summary of the needed generalization of Neyman's theory where the observable random variables $\{X_{nk}, k = 1, 2, \ldots, n\}$ are independently but not necessarily identically distributed. The reader may find further details of these and other generalizations of Neyman's theory in Bartoo and Puri (1967), Bühler and Puri (1966).

Let $X_{nk}, k = 1, 2, \ldots, n; n = 1, 2, \ldots$, be a double sequence of independent r.v.'s with each X_{nk} having a probability density $p_{n,k}(x; \xi, \underset{\sim}{\theta})$ with respect to some σ-finite measure γ_{nk}, which is independent of the parameters ξ and $\underset{\sim}{\theta}$, where $\xi \epsilon [0, a)$ for some $a > 0$ and $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_r) \epsilon \Theta$, Θ being an open set in R^r. Also we assume that the support of the distribution of each X_{nk} is independent of ξ and $\underset{\sim}{\theta}$. For convenience we let the null hypothesis be H_0: $\xi = 0$, which is to be tested against H_1: $\xi > 0$, in the presence of the nuisance parameters $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_r) \epsilon \Theta$. We impose the conditions $(C_1) - (C_3)$ given below on the sequence $p_{n,k}(x; \xi, \underset{\sim}{\theta})$ of density functions.

(C_1) *The derivatives*

(8)
$$\varphi_{nk(j)}(x; \underset{\sim}{\theta}) = \left. \frac{\partial \ell n \; p_{n,k}(x; \xi, \underset{\sim}{\theta})}{\partial \theta_j} \right|_{\xi=0}, \qquad j = 1, 2, \ldots, r,$$

and

(9)
$$\varphi_{nk(\xi)}(x; \underset{\sim}{\theta}) = \left. \frac{\partial \ell n \; p_{n,k}(x; \xi, \underset{\sim}{\theta})}{\partial \xi} \right|_{\xi=0},$$

exist for arbitrary $\underset{\sim}{\theta} \epsilon \Theta$, *the derivative in* (9) *being the right hand derivative. If* $p_{nk} = 0$, *we define* $\varphi_{nk(\xi)} = \varphi_{nk(j)} = 0, j = 1, 2, \ldots, r$. *We assume that*

(10)
$$E_{0,\underset{\sim}{\theta}} \left[\varphi^2_{nk(\xi)}(X_{nk}; \underset{\sim}{\theta}) \right], \quad E_{0,\underset{\sim}{\theta}} \left[\varphi^2_{nk(j)}(X_{nk}; \underset{\sim}{\theta}) \right], \quad j = 1, 2, \ldots, r,$$

are finite for every n *and* k *and for all* $\underset{\sim}{\theta} \epsilon \Theta$, where the subscripts $0, \underset{\sim}{\theta}$ *under the expectation sign indicate that the expectations are obtained under* H_0.

(C_2) *Under* H_0, *whatever be* $\underset{\sim}{\theta}\epsilon\Theta$ *and* n *and* k, *the quantities* $\varphi_{nk(\xi)}(X_{nk};\underset{\sim}{\theta})$ *and* $\varphi_{nk(j)}(X_{nk};\underset{\sim}{\theta}), j=1,2,\ldots,r$, *are linearly independent with positive probability.*

Let $\underset{\sim}{a}_n^o(\underset{\sim}{\theta})=(a_{n1}^o(\underset{\sim}{\theta}),\ldots,a_{nr}^o(\underset{\sim}{\theta}))$ be a vector which minimizes the variance of

$$\sum_{k=1}^{n}\varphi_{nk(\xi)}\ (X_{nk};\underset{\sim}{\theta})-\sum_{j=1}^{r}a_{nj}(\underset{\sim}{\theta})\sum_{k=1}^{n}\varphi_{nk(j)}\ (X_{nk};\underset{\sim}{\theta}), \tag{11}$$

under H_0 for each n and fixed $\underset{\sim}{\theta}$. The symbol $S_n^{*2}(\underset{\sim}{\theta})$ will denote this minimum variance. Note that in view of the conditions $(C_1)-(C_2)$ the values $a_{nj}^o(\underset{\sim}{\theta})$ are always determined and $S_n^{*2}(\underset{\sim}{\theta})$ is always positive. We now add the condition (C_3).

(C_3) *We assume that* $\underset{\sim}{a}^o(\underset{\sim}{\theta})=(a_1^o,a_2^o,\ldots,a_r^o)=\lim_{n\to\infty}\underset{\sim}{a}_n^o(\underset{\sim}{\theta})$ *exists and that the sequence* $\{p_{nk}(x;\xi,\underset{\sim}{\theta})\}$ *is regular enough for the sequence*

$$f_{nk}^*(x,\underset{\sim}{\theta})=\varphi_{nk(\xi)}(x;\underset{\sim}{\theta})-\sum_{j=1}^{r}a_j^o(\underset{\sim}{\theta})\ \varphi_{nk(j)}(x;\underset{\sim}{\theta}),\qquad k=1,2,\ldots,n, \tag{12}$$

to form an expectation centered Cramér sequence. (See Neyman (1959) *and in particular Bartoo and Puri* (1967) *for their definition).*

It may be remarked here that the "regularity conditions" assumed for the Cramér sequences are similar to the ones imposed by Cramér (1946) in his treatment of consistency of maximum likelihood estimates. Consequently following Neyman (1959), Bartoo and Puri (1967) referred to these as Cramér sequences of functions. While avoiding the statement of these conditions in detail here in defining these sequences, we refer the reader to Bartoo and Puri (1967) for their definition.

Let $B(\alpha)$ be an arbitrary measurable set on the real line whose indicator function is continuous almost everywhere and is such that

$$(2\pi)^{-1/2}\int_{B(\alpha)}\exp(-x^2/2)\ dx=\alpha. \tag{13}$$

Let $\{f_{nk}(x,\underset{\sim}{\theta})\}$ be an expectation centered Cramér sequence such that for $j=1,2,\ldots,r$

$$\lim_{n\to\infty}\ n^{-1}\sum_{k=1}^{n}Cov_{0,\underset{\sim}{\theta}}\{f_{nk}(X_{nk},\underset{\sim}{\theta}),\ \varphi_{nk(j)}(X_{nk};\underset{\sim}{\theta})\}=0. \tag{14}$$

Here again the subscripts $0,\underset{\sim}{\theta}$ of Cov and of Var indicate that the covariances and the variances respectively are obtained under H_0. A typical member of Neyman's class of $C(\alpha)$-tests is now defined for each pair $\{f_{nk}(x,\underset{\sim}{\theta})\}$, an expectation centered Cramér sequence and a set $B(\alpha)$, by rejecting H_0 whenever

$$Z_n(\hat{\underset{\sim}{\theta}}_n)\epsilon B(\alpha), \tag{15}$$

where

$$Z_n(\hat{\theta}_n) = \left[S_n(\hat{\theta}_n)\right]^{-1} \sum_{k=1}^{n} f_{nk}(X_{nk}, \hat{\theta}_n), \tag{16}$$

$$S_n^2(\theta) = \sum_{k=1}^{n} Var_{0,\theta}\{f_{nk}(X_{nk}, \theta)\}, \tag{17}$$

and $\hat{\theta}_n = (\hat{\theta}_{n1}, \ldots, \hat{\theta}_{nr})$ is a locally $\sqrt{n}$ consistent estimator for θ defined by Neyman (1959) to be such that for every $j = 1, 2, \ldots, r$, and for some constant A_j, the quantities

$$|\hat{\theta}_{nj} - \theta_j - A_j \xi|\sqrt{n} \tag{18}$$

remain bounded in probability, as $n \to \infty$, for all (ξ, θ).

Using Neyman's (1959) local asymptotic optimality criterion based on a class Γ of local alternatives $\{\xi_n\}$ such that $\sqrt{n}\xi_n$ remains bounded as $n \to \infty$, a test which is optimal in the above class $C(\alpha)$ of tests corresponds to rejecting H_0: $\xi = 0$ in favour of the alternative hypothesis H_1: $\xi > 0$, whenever

$$Z_n^*(\hat{\theta}_n) = \left[S_n^*(\hat{\theta}_n)\right]^{-1} \sum_{k=1}^{n} f_{nk}^*(X_{nk}, \hat{\theta}_n) > z_{1-\alpha}. \tag{19}$$

Furthermore for the local alternatives Γ its asymptotic power is given by

$$1 - \Phi(z_{1-\alpha} - \xi_n S_n^*(\theta)), \tag{20}$$

where $z_{1-\alpha}$ satisfies $\Phi(z_{1-\alpha}) = 1 - \alpha$, and Φ is the cumulative distribution function of the standard normal distribution. It may be remarked here that an asymptotically equivalent optimal $C(\alpha)$ test results if the $a_j^o(\theta)$'s are replaced by $a_{nj}^o(\theta), j = 1, 2, \ldots, r$. The test function so obtained has the same asymptotic properties as those of $Z_n^*(\hat{\theta}_n)$.

3. An Optimal $C(\alpha)$-test for the Poisson Hypothesis.

We return to our problem introduced in section 1 and apply the $C(\alpha)$-test theory of the preceding section to arrive at a solution. For our random variables X_i of (4) with $i = 1, 2, \ldots, n$, let

$$p_i(m|\xi, \lambda) = P(X_i = m|\xi, \lambda), \quad m = 0, 1, 2, \ldots. \tag{21}$$

Then it can be easily seen (see Katti (1967)) that for every i, these probabilities are recursively related to each other and to the probabilities $R_i(k|\xi, \lambda)$ of (6) as follows:

$$\begin{cases} p_i(0|\xi, \lambda) = \exp(-\lambda t_i) \\ p_i(m+1|\xi, \lambda) = \frac{\lambda t_i}{m+1} \sum_{k=0}^{m} (m-k+1)\, p_i(k|\xi, \lambda)\, R_i(m-k+1|\xi, \lambda), \quad m \geq 0. \end{cases} \tag{22}$$

Also in view of (7), we have for $m \geq 0$

$$\lim_{\xi \to 0} p_i(m|\xi, \lambda) = p_i(m|0, \lambda) = \frac{(\lambda t_i)^m}{m!} \exp(-\lambda t_i). \tag{23}$$

We add further conditions $(A_1) - (A_3)$ on the probabilities $R_i(k|\xi, \lambda)$ and p_i's given below.

(A_1) For $i = 1, 2, \ldots, n$, and $k \geq 1$, the probabilities $R_i(k|\xi, \lambda)$ are twice differentiable with respect to ξ for $\xi \geq 0$ and also with respect to λ for $\lambda > 0$ (the derivatives with respect to ξ at $\xi = 0$ are to be considered as right hand derivatives), and these differentiations are valid under the summation sign of

$$\sum_{k=1}^{\infty} R_i(k|\xi, \lambda) \equiv 1. \tag{24}$$

We denote these derivatives by $R_{i\xi}(k|\xi, \lambda)$, $R_{i\lambda}(k|\xi, \lambda)$, etc. It is assumed that for $i = 1, 2, \ldots, n$ and for all $\lambda > 0$, $\sum_{j=1}^{\infty} j\ R_{i\xi}(j|0, \lambda)$ is finite, with $R_{i\xi}(1|0, \lambda) < 0$. Furthermore

$$R_{i\lambda}(j|0, \lambda) \equiv 0, \quad j = 1, 2, \ldots, \forall \lambda > 0, \tag{25}$$

where

$$R_{i\xi}(j|0, \lambda) = \left.\frac{\partial R_i(j|\xi, \lambda)}{\partial \xi}\right|_{\xi=0}, \quad R_{i\lambda}(j|0, \lambda) = \left.\frac{\partial R_i(j|\xi, \lambda)}{\partial \lambda}\right|_{\xi=0}. \tag{26}$$

Remark 1. Note that in view of (6) and (7) it is easily seen that $R_{i\xi}(1|0, \lambda) \leq 0$ and $R_{i\xi}(j|0, \lambda) \geq 0$, for $j \geq 2$. Moreover the condition (A_1) implies that

$$-R_{i\xi}(1|0, \lambda) = \sum_{j=2}^{\infty} R_{i\xi}(j|0, \lambda), \qquad i = 1, 2, \ldots, n, \tag{27}$$

holds with $R_{i\xi}(1|0, \lambda) < 0$. Again the condition (25) holds for most of the standard cases. For instance, in the case of the negative binomial distribution viewed as a compound Poisson distribution, the corresponding probabilities $R_i(k|\xi, \lambda)$ depend both on ξ as well as λ and in particular they satisfy (25) (see concluding remarks in Puri (1985)). Basically all that condition (25) requires is that for all $j \geq 1$ and $\lambda > 0$, in (28)

$$\lim_{\xi \to 0} \lim_{\delta \to 0} \frac{R_i(j|\xi, \lambda + \delta) - R_i(j|\xi, \lambda)}{\delta} \tag{28}$$

an interchange of the two limits be allowed. Finally it follows from (22) and the condition (A_1) that for every $i = 1, 2, \ldots, n$ and $m \geq 0$, the probabilities $p_i(m|\xi, \lambda)$ are twice differentiable with respect to $\xi \geq 0$ and also for $\lambda > 0$.

(A_2) *It is assumed that the above differentiations of $p_i(m|\xi,\lambda)$ are permitted under the summation sign of*

$$\sum_{m=0}^{\infty} p_i(m|\xi,\lambda) = 1. \tag{29}$$

We define for $i = 1, 2, \ldots, n$,

$$\varphi_{i\xi}(m|\lambda) = \left.\frac{\partial\ \ell n\ p_i(m|\xi,\lambda)}{\partial\xi}\right|_{\xi=0}, \qquad \varphi_{i\lambda}(m|\lambda) = \left.\frac{\partial\ \ell n\ p_i(m|\xi,\lambda)}{\partial\lambda}\right|_{\xi=0}. \tag{30}$$

(A_3) *We assume that the t_i's are bounded both from above as well as away from zero. In particular $n^{-1} \sum_{i=1}^{n} t_i$ and*

$$\frac{1}{n}\sum_{i=1}^{n} t_i \left[\sum_{j=1}^{\infty} j\ R_{i\xi}(j|0,\lambda)\right] \tag{31}$$

both converge as $n \to \infty$. Finally it is assumed that both $\{\varphi_{i\xi}(m|\lambda), i = 1, 2, \ldots\}$ and $\{\varphi_{i\lambda)}(m|\lambda), i = 1, 2, \ldots\}$ form Cramér sequences (see Bartoo and Puri (1967) *for their definition), which implies that the limit*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} E_{0,\lambda}\ [\varphi_{i\xi}(X_i|\lambda)]^2 \tag{32}$$

exists and is strictly positive for all $\lambda > 0$.

As in Puri (1985) standard calculations using (22) and (25) lead us to the following lemma.

Lemma 1. *For $i = 1, 2, \ldots$*

$$\begin{cases} \varphi_{i\xi}(m|\lambda) = \sum_{j=0}^{m} \frac{m!}{(m-j)!}\ (\lambda t_i)^{-(j-1)}\ R_{i\xi}(j|0,\lambda), & m \geq 1 \\ \varphi_{i\xi}(0|\lambda) = 0, \end{cases} \tag{33}$$

$$\varphi_{i\lambda}(m|\lambda) = \left(\frac{m}{\lambda} - t_i\right), \qquad m \geq 0, \tag{34}$$

where we have adopted the convention that

$$R_{i\xi}(0|0,\lambda) = R_{i\lambda}(0|0,\lambda) = 0.$$

Remark 2. As was shown in Puri (1985), subject to (7) and from the condition (A_1) it follows that for every i

$$E_{0,\lambda}\left[\varphi_{i\xi}(X_i|\lambda)\right]^2 > 0, \tag{35}$$

and that under the hypothesis H_0: $\xi = 0$, the random quantities $\varphi_{i\xi}(X_i|\lambda)$ and $\varphi_{i\lambda}(X_i|\lambda)$ are linearly independent with a positive probability for every $\lambda > 0$, a condition needed for (C_2) as part of the existence of an optimal $C(\alpha)$-test for the present case.

The next lemma gives further expressions that are needed for the construction of the optimal $C(\alpha)$-test. The proof is omitted as it involves lengthy but routine calculations using (33) and (34).

Lemma 2. For $i = 1, 2, \ldots,$ we have

$$Var_{0,\lambda}(\varphi_{i\lambda}(X_i|\lambda)) = t_i/\lambda; \tag{36}$$

$$Cov_{0,\lambda}(\varphi_{i\xi}(X_i|\lambda),\ \varphi_{i\lambda}(X_i|\lambda)) = t_i \sum_{j=0}^{\infty} j\ R_{i\xi}(j|0,\lambda); \tag{37}$$

$$a_n^o(\lambda) = \lambda\left(\sum_{i=1}^{n} t_i\right)^{-1} \sum_{i=1}^{n} t_i \left[\sum_{j=0}^{\infty} j\ R_{i\xi}(j|0,\lambda)\right]; \tag{38}$$

$$\mathrm{Var}_{0,\lambda}(\varphi_{i\xi}(X_i|\lambda)) = V_i(\lambda t_i); \tag{39}$$

where

$$V_i(\lambda t_i) = (\lambda t_i)^2 \sum_{k=1}^{\infty} R_{i\xi}\ (k|0,\lambda) \sum_{\ell=1}^{\infty} R_{i\xi}(\ell|0,\lambda) B_{\ell,k}(\lambda t_i), \tag{40}$$

and

$$B_{\ell,k}(\lambda t_i) = \sum_{r=0}^{\min(\ell,k)} \binom{k}{r} \binom{\ell}{r} (r!)\ (\lambda t_i)^{-r}. \tag{41}$$

On comparing the expression (40) with the corresponding one of Puri ((1985), equation (31)) we note that a factor λ^2 was inadvertently left out there. Finally, applying the $C(\alpha)$-test theory of section 2 (in particular (19) and (20) and lemmas 1 and 2), the following theorem gives the desired optimal $C(\alpha)$-test for our problem.

Theorem 1. *Subject to the assumptions* $(A_1) - (A_3)$, *an optimal* $C(\alpha)$*-test for testing* H_0: $\xi = 0$ *against* H_1: $\xi > 0$, *is to reject* H_0 *whenever*

$$\tilde{Z}_n(\hat{\lambda}) > z_{1-\alpha}, \tag{42}$$

where

$$\tilde{Z}_n(\hat{\lambda}) = \left[\tilde{S}_n(\hat{\lambda})\right]^{-1} \sum_{i=1}^{n} g_{ni}(X_i, \hat{\lambda}), \tag{43}$$

$$g_{ni}(X_i, \lambda) = \varphi_{i\xi}(X_i|\lambda) - a_n^o(\lambda)\varphi_{i\lambda}(X_i|\lambda), \tag{44}$$

$$\tilde{S}_n^2(\lambda) = \sum_{i=1}^{n} V_i(\lambda t_i) - \lambda \left(\sum_{i=1}^{n} t_i\right)^{-1} \left[\sum_{i=1}^{n} t_i \sum_{j=0}^{\infty} j\ R_{i\xi}(j|0,\lambda)\right]^2, \tag{45}$$

and $\hat{\lambda}$ *stands for a locally* $\sqrt{n}$ *consistent estimator of the nuisance parameter* λ.

Of particular importance is the special case when $R_{i\xi}(j|0,\lambda) = 0$ for $j \geq 3$ and for all $i = 1, 2, \ldots, n$. This condition is satisfied for several well known compound Poisson alternatives arising in practice, such as negative binomial distributions, and Neyman type A distributions, among others. Furthermore the test statistic (43) simplifies somewhat for this case. The next theorem deals with this special case.

Theorem 2. *In addition to the conditions of theorem 1, if* $\forall i = 1, 2, \ldots, R_{i\xi}(j|0,\lambda) = 0$ *for* $j \geq 3$, *also holds, the test statistic* (43) *reduces to*

(46)
$$\tilde{Z}_n(\hat{\lambda}) = \left[\tilde{S}_n(\hat{\lambda})\right]^{-1} \left\{\sum_{i=1}^{n} \left[\frac{X_i(X_i-1)}{\hat{\lambda} t_i} - X_i\right] R_{i\xi}(2|0,\hat{\lambda}) - a_n^o(\hat{\lambda}) \sum_{i=1}^{n} \left(\frac{X_i}{\hat{\lambda}} - t_i\right)\right\}$$

with the asymptotic power of the optimal $C(\alpha)$*-test, for the local alternatives* $\{\xi_n\}$, *given by*

$$1 - \Phi(z_{1-\alpha} - \xi_n \tilde{S}_n(\lambda)) \tag{47}$$

where

$$a_n^o(\lambda) = \lambda \left[\sum_{i=1}^{n} t_i\right]^{-1} \cdot \left[\sum_{i=1}^{n} t_i\ R_{i\xi}(2|0,\lambda)\right], \tag{48}$$

$$\begin{aligned}\tilde{S}_n^2(\lambda) = 2 \sum_{i=1}^{n} [R_{i\xi}(2|0,\lambda)]^2 + \lambda \sum_{i=1}^{n} t_i (R_{i\xi}(2|0,\lambda))^2 \\ - \lambda \left[\sum_{i=1}^{n} t_i\right]^{-1} \left[\sum_{i=1}^{n} t_i\ R_{i\xi}\ (2|0,\lambda)\right]^2.\end{aligned} \tag{49}$$

Finally the next theorem further specializes the test to the case where the $R_{i\xi}(2|0,\lambda)$ are all equal for $i \geq 1$.

Theorem 3. *If, subject to the conditions of theorem 2, the $R_{i\xi}(2|0,\lambda)$ are all equal for $i \geq 1$, the test statistic* (46) *further reduces to*

$$\tilde{Z}_n(\hat{\lambda}) = (2n)^{-1/2} \sum_{i=1}^{n} (\hat{\lambda} t_i)^{-1} \left\{ (X_i - \hat{\lambda} t_i)^2 - X_i \right\}. \tag{50}$$

Furthermore for the local alternatives $\{\xi_n\}$ the asymptotic power of the optimal $C(\alpha)$-test is given by (47) *with*

$$\tilde{S}_n^2(\lambda) = 2n \left[R_\xi(2|0,\lambda) \right]^2 . \tag{51}$$

4. Concluding Remarks

(a) For the case considered in theorem 3 it is interesting to note that the asymptotic power for the local alternatives (as is exhibited through (51)) does not depend upon the lengths t_i of the various observation periods. This implies, under the assumption of theorem 3, the following interesting observation regarding the design of our experiments. For instance, if the total length of the periods $\sum_{i=1}^{n} t_i$ is fixed, it would be advisable to take a larger n with smaller t_i's, for example by doubling the number n of observations and at the same time cutting down the average period length per observation to half that of the original one. Intuitively this sounds reasonable, since for small t_i's most of the time the positive X_i's would constitute the results of single jumps of the process, thereby enabling us to distinguish more effectively between the two situations, namely the one where the jump sizes are unity (Poisson process case) and the other where jump sizes are greater than one with a positive probability (compound Poisson process case).

(b) Again for the same case of theorem 3, it is interesting to note that the optimal test based on the test statistic (50) is independent of the common p.g.f. $h(s|\xi,\lambda)$. This property has been referred to as the 'robustness of optimality' property by Neyman (see Bartoo and Puri (1967) for more comments on this point).

(c) In the above tests one needs to have a locally $\sqrt{n}$ consistent estimator of the nuisance parameter λ. One possible way of constructing such an estimate for the most general case of theorem 1 is by using the fact that $P(X_i = 0) = \exp(-\lambda t_i)$, $i = 1, 2, \ldots, n$. Let $\eta_i = 1$ if $X_i = 0$ and $\eta_i = 0$ otherwise, $i = 1, 2, \ldots, n$. The likelihood function based on $\eta_1, \eta_2, \ldots, \eta_n$ is then given by

$$L(\lambda|\eta_1, \eta_2, \ldots, \eta_n) = \exp(-\lambda \sum_{i=1}^{n} t_i \, \eta_i) \prod_{i=1}^{n} \{1 - \exp(-\lambda t_i)\}^{1-\eta_i} . \tag{52}$$

The desired estimate of λ is now simply the maximum likelihood estimate based on (52) and is given by the unique solution for λ of the equation

$$\sum_{i=1}^{n} \eta_i \, t_i = \sum_{i=1}^{n} (1-\eta_i) \, t_i \, [\exp(\lambda t_i) - 1]^{-1} . \tag{53}$$

We need hardly mention that other more involved estimates of λ which use more information about the X_i's are also possible.

(d) Finally the approach adopted here for the testing of the Poisson hypothesis falls within the classical α-level hypothesis testing framework. The question of testing the Poisson hypothesis is currently being studied in the present context from a decision-theoretic as well as a Bayesian point of view. The results pertaining to these investigations will be reported elsewhere.

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