REVERSING GAUSSIAN SEMIMARTINGALES WITHOUT GAUSS

by

P. Protter+
Purdue University
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Department of Statistics
Purdue University

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ABSTRACT. Representation and time reversal results of Lindquist and Picci on Gaussian semimartingales with stationary increments are shown to hold without the Gaussian and stationarity assumptions, the key property being that the martingale term is a Wiener process. The methods of proof also yield a Girsanov-type formula for a process of the form $X_t = W_t - \int_0^t h_s ds$, where h is not necessarily adapted.

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1. Introduction

In an interesting paper Lindquist and Picci [3] investigate Gaussian processes with stationary increments. They characterize when such a process Y has a representation of the form:

$$Y_t - Y_s = \int_s^t Z_u du + M_t - M_s \qquad (-\infty < s < t < \infty) \tag{1.1}$$

where M is a martingale, and thus Y is a semimartingale. Recently time reversal of stochastic differential equations and more generally reversal of semimartingales have been of interest, and Lindquist and Picci further find sufficient conditions under which Y can be reversed, yet still have a (backwards) representation of the form (1.1).

In this article we put the results of Lindquist and Picci in a more traditional semimartingale framework and we show that the Gaussian hypothesis — fundamental to their approach — as well as the stationarity, are actually unnecessary. Instead we use Girsanov's theorem to show the reversibility. This idea is extended in paragraph five to show that if $X_t = W_t - \int_0^t h_s ds$, where h is not necessarily adapted to the filtration of W, then X might still have a law equivalent to that of W; that is, one can find an equivalent probability Q such that X is a Wiener process. This last topic is not considered in [3].

Throughout this article we will only consider one dimensional results; finite dimensional analogs, however, clearly hold. Also, since we are dropping the hypothesis of stationary increments we will assume our processes are indexed by $[0,1]$, rather than all of $\mathbb{R}$.

In [3] one begins with a process Y having stationary, Gaussian increments. In a representation such as (1.1), the processes $\int_0^t Z_u du$ and M_t will inherit these properties. Lindquist and Picci further assume M has orthogonal increments; thus it is a standard Wiener process. As we shall see it is this feature that is the key to the reversibility.

For all facts about semimartingales as well as any unexplained notation we refer the reader to Dellacherie and Meyer [1].

2. The Lindquist-Picci Semimartingale Representation Theorem.

In [3] Lindquist and Picci introduce the following hypothesis on a process Y:

$$|E\{Y_{t+h} - Y_t | \mathcal{F}_t\}| \leq Kh \quad \text{a.s.,} \tag{2.1}$$

for all $t \geq 0$ and all $h > 0$. The constant K can be random and is finite a.s. Lindquist and Picci assume further that Y is Gaussian with stationary increments and obtain necessary and sufficient conditions for (2.1) to hold.

(2.2) THEOREM. Let Y be an adapted process on $[0,1]$ with cadlag paths, with $Y_t \in L^1(dP)$, $0 \leq t \leq 1$.

(i) If there exists a r.v. $K \in L^1(dP)$ such that (2.1) holds, then there is a representation of the form

$$Y_t = \int_0^t Z_u du + M_t \tag{2.3}$$

where M is a martingale and $\int_0^1 E(|Z_u|)du < \infty$.

(ii) If for some $\epsilon > 0$, $E\left[\sup_{0 \leq u \leq 1} |Z_u|\right]^{1+\epsilon} < \infty$ in a representation of the form (2.3), then (2.1) holds with $K \in L^1$.

Proof. (ii) Let $C = \sup_{0 \leq u \leq 1} |Z_u|$. Then

$$|E\{Y_{t+h} - Y_t | \mathcal{F}_t\}| = |E\{\int_t^{t+h} Z_u du | \mathcal{F}_t\}|$$

$$\leq E\{\int_t^{t+h} |Z_u| du | \mathcal{F}_t\}$$

$$\leq E\{Ch | \mathcal{F}_t\} = hE\{C | \mathcal{F}_t\}.$$

Let $N_t = E\{C | \mathcal{F}_t\}$, and set $N^* = \sup_{0 \leq t \leq 1} |N_t|$. Since $C \in L^{1+\epsilon}$, we have $N^* \in L^1$, and we can take $K = N^*$ (cf. [1]).

Proof of (i): Let $\pi : 0 = t_0 < t_1 < \cdots < t_r = 1$ be a partition of $[0,1]$. Each $Y_{t_i} \in L^1$ and

$$E\{\sum_{t_i \in \pi} |E\{Y_{t_i} - Y_{t_{i-1}} | \mathcal{F}_{t_{i-1}}\}| \leq E\{K\Sigma(t_i - t_{i-1})\}$$

$$= E\{K\} < \infty,$$

and so Y is a quasimartingale. Therefore Y is a special semimartingale and has a canonical decomposition $Y = A + M$ where M is a martingale and A is a predictable process with paths of finite variation. Moreover letting $\int_0^1 |dA_s|$ denote the total variation of A, we know $E\{\int_0^1 |dA_s|\} < \infty$. We want to show A has absolutely continuous paths. Fix $s < t$ and let

$$\Lambda_{s,t} = \{\omega : E\{\int_s^t |dA_u| \ |\mathcal{F}_s\}(\omega) > K(\omega)(t-s)\}.$$

It suffices to show that $P(\Lambda_{s,t}) = 0$ for arbitrary s and t. Let $H_t = \sum_{t_i \in \pi} H_i 1_{]t_i, t_{i+1}]}$, with π being a partition of $]s,t]$, and with $|H_i| \leq 1$ a.s., and $H_i \in \mathcal{F}_{t_i}$. Then, writing Λ for $\Lambda_{s,t}$:

$$E\{1_\Lambda \int_s^t H_u dA_u\} = E\{1_\Lambda \sum_i H_i(A_{t_{i+1}} - A_{t_i})\}$$

$$= E\{1_\Lambda \sum_i H_i E\{A_{t_{i+1}} - A_{t_i} | \mathcal{F}_{t_i}\}\}$$

$$\leq E\{1_\Lambda \sum_i | E\{A_{t_{i+1}} - A_{t_i} | \mathcal{F}_{t_i}\}|\}$$

$$\leq E\{1_\Lambda K(t-s)\},$$

where H as above was arbitrary. On the other hand,

$$\sup_H E\{1_\Lambda \int_s^t H_u dA_u\} = E\{1_\Lambda \int_s^t |dA_u|\}$$

$$= E\{1_\Lambda E\{\int_s^t |dA_u| \ |\mathcal{F}_s\}\}$$

$$> E\{1_\Lambda K(t-s)\}, \quad \text{if} \quad P(\Lambda) > 0,$$

where the supremum is taken over all such H (cf. [1]). This yields a contradiction. Since then A is absolutely continuous, one can find an adapted process Z such that $A_t = \int_0^t Z_u du$. Moreover $\int_0^1 |dA_u| \in L^1$ implies that $E\left(\int_0^1 |dA_u|\right) = E\left(\int_0^1 |Z_u| du\right)$ $= \int_0^1 E(|Z_u|)du < \infty$, by Fubini's theorem. □

3. Backwards Processes and Backwards Stochastic Integration.

Let $(W_t)_{0\le t\le 1}$ be a standard Wiener process on a complete space $(\Omega,\mathcal{F},P)$ with filtration $(\mathcal{F}_t)_{0\le t\le 1}$ satisfying the "usual hypotheses": $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, and $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, each t. Let $\tilde{W}^t = W_t - W_1$, for $1 \ge t \ge 0$. Then as t descends to 0, the process $\tilde{W}^t$ has Gaussian, independent, stationary increments; we call it a backwards Wiener process. For $0 \le s < t \le 1$, note that $\tilde{W}^s - \tilde{W}^t = W_s - W_t$, and so the increments of $\tilde{W}$ are equal to those of W. We let $\mathcal{F}_0^t = \sigma\{\tilde{W}^u;\ u \ge t\}$, and $\mathcal{F}^t$ denote the completed version of $\mathcal{F}_0^t$. Processes that are adapted to $\mathcal{F}^t$ as t descends from 1 to 0 will be indexed by superscripts.

Suppose a backwards process $(J^t)_{1\ge t\ge 0}$ has a representation $J^t = \sum_{t_i\in\pi} J^i 1_{[t_i,t_{i+1}[}$, where $\pi : 0 = t_0 < t_1 < \cdots < t_n = 1$ is a partition of $[0,1]$, and where $J^i \in \mathcal{F}^{t_{i+1}}$. We can define a backwards Itô integral of J with respect to W by:

$$\int_0^1 J^t dW^t = \sum_{t_i\in\pi} J^i(W_{t_{i+1}} - W_{t_i}). \tag{3.1}$$

It is important to note that a more natural definition of the backwards integral would be $\sum J^i(\tilde{W}_{t_i} - \tilde{W}_{t_{i+1}}) = -\sum J^i(W_{t_{i+1}} - W_{t_i})$, but the definition (3.1) is that of Kunita [2] which has gained acceptance, so we use it here. One can now develop an entire theory of the backwards integral which is analogous to that of the customary (forwards) Itô integral. In particular one has a backwards change of variables formula (all Lebesgue integrals are meant in the forward — that is, traditional — sense):

(3.2) THEOREM. *Let* $(W_t)_{0\leq t\leq 1}$ *be a standard Wiener process,* $\tilde{W}^t = W_t - W_1$, *and let* f *be* $\mathscr{C}^2$. *Then for* $0 \leq s < t \leq 1$ *one has*

$$f(\tilde{W}^s) - f(\tilde{W}^t) = \int_s^t f'(\tilde{W}^s)dW^s + \frac{1}{2}\int_s^t f''(\tilde{W}^s)ds.$$

A proof of Theorem (3.2) can be found, e.g., in [6] as a special case of Theorem 6.1. In any event it is elementary. One also has a backwards Grisanov formula. We give the proof because it does not seem yet to exist in the literature.

(3.3) THEOREM. *Let* $(J^t)_{1\geq t\geq 0}$ *be jointly measurable and adapted to* $\mathscr{F}^t$ *such that* $P\{\int_0^1 (J^u)^2 du < \infty\} = 1$. *Let* $L^t = \exp\{\int_t^1 J^u d\tilde{W}^u - \frac{1}{2}\int_t^1 (J^u)^2 du\}$. *Suppose* $E(L^0) = 1$ *and set* $dQ = L^0 dP$. *Then under* Q *the process* $X^t = \tilde{W}^t - \int_t^1 J^u du$ *is a backwards Wiener process.*

Proof. Theorem (3.2) applied to $f(x) = e^x$ yields $L^t = 1 + \int_t^1 L^u J^u d\tilde{W}^u$, so L^t is a backwards martingale. Next consider $V_u^t = \exp\{u(\tilde{W}^t - \int_t^1 J^s ds) - \frac{u^2}{2}(t-1)\}$. It will suffice to show that $(V_u^t, \mathcal{F}^t, Q)$ is a backwards local martingale for all U. To see this, we need only show that $V_u^t L^t$ is an $(\mathcal{F}^t, P)$ backwards local martingale. But

$$V_u^t L^t = \exp\{\int_t^1 J^s d\tilde{W}^s - \frac{1}{2}\int_t^1 (J^s)ds + u\tilde{W}^t - \int_t^1 uJ^s ds\}$$

$$= \exp\{\int_t^1 (J^s+u)d\tilde{W}^s - \frac{1}{2}\int_t^1 (J^s+u)^2 ds\},$$

which again is a backwards local martingale by Theorem (3.2). Thus X^t is a backwards Wiener process. □

(3.4) COROLLARY. With the same hypotheses and notation as Theorem (3.3), the process

$$X^t - X^0 = W_t + \int_0^t J^s ds.$$

is a forwards Wiener process under Q.

(3.5) COMMENT. A sufficient condition to have $E(L^0) = 1$ is, of course, that of Novikov: that $E\{\exp(\frac{1}{2}\int_0^1 (J^u)^2 du)\} < \infty$. Also, the statement that $X^t - X^0$ is a forwards Wiener process is meant with respect to its natural filtration.

4. Reversibility of Some Semimartingales.

In this paragraph we investigate the reversibility of semimartingales Y of the form (2.3) under the additional restriction that the martingale M is a Wiener process. That is, we will assume our process Y has a (forward) representation:

$$Y_t = \int_0^t H_u du + W_t \qquad (0 \le t \le 1). \tag{4.1}$$

This extends the results of Lindquist and Picci [3] since we do not assume either that Y is Gaussian or that Y has stationary increments.

Let W be a standard Wiener process on $[0,1]$ defined on a filtered complete space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ satisfying the "usual hypotheses". Let H be a jointly measurable, adapted process such that $P\{\int_0^1 (H_s)^2 ds < \infty\} = 1$ and moreover that $E\{\exp(\int_0^1 H_s dW_s - \frac{1}{2}\int_0^1 H_s^2 ds)\} = 1$. We set:

(4.2) $$Y_t = \int_0^t H_s ds + W_t \qquad (0 \leq t \leq 1).$$

Let $\mathcal{G}_0^t = \sigma\{Y_u - Y_1;\ t \leq u \leq 1\}$, and $\mathcal{G}^t = \mathcal{G}_0^t \vee \mathcal{N}$, where $\mathcal{N}$ are the P-null sets of $\mathcal{F}$. Observe that Y as given in (4.2) can be written as:

(4.3) $$Y_t - Y_s = \int_s^t H_u du + W_t - W_s \qquad (0 \leq s < t \leq 1).$$

(4.4) THEOREM. _Let_ W,H _and_ Y _be as given above. Then_ Y _is reversible in the sense that there exists a backwards Wiener process_ M _and a_ $\mathcal{G}^t$_-adapted measurable process_ J _such that_

$$\tilde{Y}^t = Y_t - Y_1 = M^t + \int_t^1 J^u du \qquad (0 \leq t \leq 1)$$

or equivalently, for $0 \leq s < t \leq 1$:

$$Y_s - Y_t = M^s - M^t + \int_s^t J^u du.$$

Proof. Let Q be a measure equivalent to P given by:

$$dQ = \exp\left[\int_0^1 - H_s dW_s - \frac{1}{2}\int_0^1 H_s^2 ds\right] dP.$$

Then by Girsanov's theorem (e.g. [4], p. 232) we know that $Y_t = \int_0^t H_s ds + W_t$ is a Q-Wiener process. Therefore $\tilde{Y}^t = Y_t - Y_1$ is a backwards Wiener process under Q. Since Q and P are equivalent, we know there exists a process J such that

$$M^t = \tilde{Y}^t - \int_t^1 J^u du$$

is a backwards P-Wiener process, where J is adapted to $(\mathcal{G}^u)_{1 \geq u \geq 0}$. □

5. An Anticipating Girsanov Formula

In this paragraph we consider the question: under what hypotheses on a process h can the law of

$$X_t = W_t - \int_0^t h_s ds \qquad (0 \leq t \leq 1)$$

be transformed, by a change to an equivalent probability measure, to that of a Wiener process? $W = (W_t)_{0 \leq t \leq 1}$ is of course assumed to be an $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq 1}, P)$-Wiener process. The traditional assumptions are that h be adapted and satisfy an integrability condition. Here we relax the requirement that h be adapted.

Let H be adapted, jointly measurable, and suppose $E\{\exp(\frac{1}{2}\int_0^1 H_s^2 ds)\} < \infty$. Set

$$Y_t = W_t - \int_0^1 H_s ds.$$

Let $\mathscr{F}^t$ and $\mathscr{G}^t$ be the completions respectively of $\sigma\{W_u - W_1;\ t \le u \le 1\}$ and $\sigma\{Y_u - Y_1;\ t \le u \le 1\}$. Note that in general one has neither $\mathscr{G}^t \subseteq \mathscr{F}^t$ nor $\mathscr{F}^t \subseteq \mathscr{G}^t$. Define a new probability Q by

$$\text{(5.1)} \qquad dQ = \exp(\int_0^1 H_s dW_s - \frac{1}{2}\int_0^1 H_s^2 ds)dP.$$

Let $(J^u)_{1 \ge u \ge 0}$ be $\mathscr{G}^u$-adapted and suppose further that $E_Q\{\exp(\frac{1}{2}\int_0^1 (J^u)^2 du)\} < \infty$, or simply that $E_P\{\exp(\frac{1}{2}\int_0^1 H_s^2 + (J^s)^2 ds)\} < \infty$.

(5.2) THEOREM. _Let_ H _and_ J _be as given above. Then there exists a probability law_ R _equivalent to_ P _such that_ $U_t = W_t + \int_0^t (J^s - H_s)ds$ _is a Wiener process under_ R.

Proof. If $Y_t = W_t - \int_0^t H_s ds$, under Q as defined in (5.1) we have that Y is a Wiener process. Therefore $\tilde{Y}^t = Y_t - Y_1$ is a backwards Wiener process under Q. Next define

$$Z^t = \tilde{Y}^t - \int_t^1 J^u du$$

and define R by

$$dR = \exp\left[\int_0^1 J^u d\tilde{Y}^u - \frac{1}{2}\int_0^1 (J^u)^2 du\right] dQ.$$

Then Z^t is a backwards Wiener process under R, and hence

$$Z^t - Z^0 = \tilde{Y}^t - \tilde{Y}^0 - \int_t^1 J^u du + \int_0^1 J^u du$$

$$= W_t - \int_0^t H_s ds + \int_0^t J^u du$$

$$= W_t - \int_0^t (H_s - J^s) ds \qquad (0 \le t \le 1)$$

is a forwards Wiener process under R. Since R is equivalent to Q, and Q is equivalent to P, we have that R is equivalent to P. □

Let $\tilde{W}^t = W_t - W_1$. Then $\tilde{W}^t$ is a backwards Wiener process under P, but since $(J^u)_{1\geq u\geq 0}$ need not be adapted to $\mathcal{F}^u$, we cannot give a usual meaning to $\int_t^1 J^u d\tilde{W}^u$. However since $\tilde{W}^t - \tilde{W}^s = \tilde{Y}^t - \tilde{Y}^s + \int_s^t H_u du$, we can <u>define</u> the integral as follows:

(5.3) $$\int_t^1 J^u d\tilde{W}^u \equiv \int_t^1 J^u d\tilde{Y}^u + \int_t^1 J^u H_u du.$$

With the above convention, we have the following:

(5.4) COROLLARY. <u>With the assumptions and notation of Theorem</u> (5.2), <u>and</u> (5.3), <u>we have</u>:

$$\frac{dR}{dP} = \exp\left[\int_0^1 J^u d\tilde{W}^u + \int_0^1 H_u dW_u - \frac{1}{2}\int_0^1 (J^u + H_u)^2 du\right].$$

<u>Proof</u>. Since $\frac{dR}{dP} = \frac{dR}{dQ}\frac{dQ}{dP}$, we have

$$\frac{dR}{dP} = \exp\left[\int_0^1 J^u d\tilde{Y}^u - \frac{1}{2}\int_0^1 (J^u)^2 du\right]\exp\left[\int_0^1 H_s dW_s - \frac{1}{2}\int_0^1 (H_s)^2 ds\right]$$

$$= \exp\left[\int_0^1 J^u d\tilde{W}^u - \int_0^1 J^u H_u du + \int_0^1 H_u dW_u - \frac{1}{2}\int_0^1 (J^u)^2 + (H_u)^2 du\right]$$

$$= \exp\left[\int_0^1 J^u d\tilde{W}^u + \int_0^1 H_u dW_u - \frac{1}{2}\int_0^1 (J^u)^2 + 2H_u J^u + (H_u)^2 du\right].$$

□

Note that if one writes

$$\int_0^1 (J^u+H_u)dW(u) \equiv \int_0^1 J^u d\tilde{W}^u + \int_0^1 H_u dW_u ,$$

one has as a replacement for Corollary (5.4) the pretty formula:

(5.5) $$\frac{dR}{dP} = \exp\left[\int_0^1 (J^u+H_u)dW(u) - \frac{1}{2}\int_0^1 (J^u+H_u)^2 du\right].$$

(5.6) COMMENTS. The chief advantage of Theorem (5.2) and Corollary (5.4) is their simplicity and their reliance on well known theorems. Other approaches to developing a Girsanov type formula might be that of the expansion of filtrations, or that of using a stochastic integral that integrates anticipating processes. Nualart and Zakai [5], for example, have used this latter approach with the Skorohod integral, and they have achieved some success.

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