Covering Problems for Brownian Motion on Spheres

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Summary

Bounds are given on the mean time taken by a strong Markov process to visit all of a finite collection of subsets of its state space. These are specialized to Brownian motion on the surface of the unit sphere Σ_p in R^p. This leads to bounds on the mean time taken by this Brownian motion to come within a distance ϵ of every point on the sphere and bounds on the mean time taken to come within a distance ϵ of every point or its opposite. The second calculation is related to the Grand Tour, a technique of multivariate data analysis that involves a search of low dimensional projections. In both cases, the bounds are asymptotically tight as $\epsilon \to 0$ on Σ_p for $p \geq 4$.

1. Introduction

The Grand Tour [Asimov (1985)] is a technique of data analysis that involves visual examination of a sequence of low, typically two, dimensional projections of a p–dimensional data set. Here a one dimensional Grand Tour, a sequence of one dimensional projections, is considered. One technique to construct such a Grand Tour is to generate a random walk on the surface of the unit sphere Σ_p in R^p and to look at the projections onto the lines spanned by the points visited by the random walk. If the random walk takes small steps, then the projections of the sequence will be close together, a desirable quality for visual inspection. Of interest is the time taken until such a sequence of projections has come within an angle ϵ of every possible projection. This is the time taken until the points visited by the random walk and their reflections in the origin are within a geodesic distance ϵ of every point on the sphere, or the time taken until caps of geodesic radius ϵ about these points cover Σ_p. Call this the two cap problem. There is also the one cap problem; the time taken by caps of geodesic radius ϵ about the points visited (and not their reflections) to cover Σ_p.

Now switch attention to Brownian motion on Σ_p. Covering times for Brownian motion are of interest in their own right and bounds on their expected covering times are asymptotically bounds on the expected covering times for random walks as the step sizes of the random walks shrink toward zero. Let $C_1(\epsilon,p)$ be the first time a Brownian path on Σ_p has come within a distance ϵ of all points of Σ_p, and define $C_2(\epsilon,p)$ analogously for the two cap problem. Since the sphere is separable both are measurable. The essence of this article is that $\mathrm{E}C_1(\epsilon,p)$ and $\mathrm{E}C_2(\epsilon,p)$ can be bounded above and below by quantities that involve only the expected hitting times of caps and the number of caps of various sizes needed to cover Σ_p. For Brownian motion with scale parameter λ on $\Sigma_p, p \geq 4$ the bounds are given herein and shown to be asymptotically tight as $\epsilon \to 0$. For the two cap problem the results are

$$\mathrm{E}C_2(\epsilon,p) = 2\frac{\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{\log(\epsilon^{-1})}{\epsilon^{p-3}}\left(1+0\left(\frac{\log\log(\epsilon^{-1})}{\log(\epsilon^{-1})}\right)\right) \text{ for } p \geq 4.$$

For the one cap problem

$$\mathrm{E}C_1(\epsilon,p) = 4\frac{\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{\log(\epsilon^{-1})}{\epsilon^{p-3}}\left(1+0\left(\frac{\log\log(\epsilon^{-1})}{\log(\epsilon^{-1})}\right)\right) \text{ for } p \geq 4.$$

For $p = 3$, for the two and one cap problems, respectively,

$$2 \leq \liminf \frac{\lambda \mathrm{E}C_2(\epsilon,p)}{\log^2(\epsilon^{-1})} \leq \limsup \frac{\lambda \mathrm{E}C_2(\epsilon,p)}{\log^2(\epsilon^{-1})} \leq 8$$

$$4 \leq \liminf \frac{\lambda \mathrm{E}C_1(\epsilon,p)}{\log^2(\epsilon^{-1})} \leq \limsup \frac{\lambda \mathrm{E}C_1(\epsilon,p)}{\log^2(\epsilon^{-1})} \leq 16$$

all as $\epsilon \to 0$. The bounds in the one cap problem are always twice those in the two cap problem. For brevity only the computations in the two cap problem will be presented.

To obtain these bounds, it is necessary to be able to calculate the expected hitting times for small caps on Σ_p. This is why Brownian motion, rather than random walks, is considered. If expected hitting times could be calculated for random walks, the methods of this paper would apply equally well to covering problems for random walks. The techniques used here would also apply in the case of two dimensional projections, a more interesting situation. In the two dimensional case interest would be in a random walk or Brownian motion on a Grassmann manifold. Again what is lacking is a way to calculate expected hitting times, even for Brownian motion. A partial differential equation rather than ordinary one must be solved. In a practical sense, the answer is already known. The space of possible projections is so big that even the most efficient Grand Tour would take a very long time See Huber (1985) for discussion.

The plan for the remainder is as follows. First some general bounds on the expected time taken by a Markov process to visit a fitite collection of subsets of its state space are given. Caps on spheres and Brownian motion on spheres are then discussed briefly and the expected hitting times for Brownian motion are calculated. These are all put together to give the results stated above.

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2. General Bounds on Times to Hit Collections of Subsets

This section gives upper and lower bounds on the mean time taken by a strong Markov process to visit all of a finite collection of subsets of its state space. Let $\{X(t)\}$, t≥0 be a time–homogeneous strong Markov process with state space **A**. Let $\{A_1, A_2, \ldots, A_N\}$ be a set of N closed subsets of **A**. Further let X(0) be the initial position of the process and let **T** stand for the first time, starting from X(0), that the process has visited all of $\{A_1, A_2, \ldots, A_N\}$. Let $T(a, A)$ denote the random time taken by $\{X(t)\}$ to hit A∈ $\{A_1, A_2, \ldots, A_N\}$ from a ∈ **A**. Define

(2.1) $$\hat{A}_i = X(0) \cup \bigcup_{j \neq i} A_j.$$

Define

$$\mu_- = \min_{i=1,2,\ldots,N} \inf_{a \in \hat{A}_i} \mathrm{E}T(a, A_i) \tag{2.2a}$$

and

$$\mu_+ = \max_{i=1,2,\ldots,N} \sup_{a \in \hat{A}_i} \mathrm{E}T(a, A_i). \tag{2.2b}$$

Thus μ_- and μ_+ are the minimum and maximum expected times taken to hit one of $\{A_1, A_2, \ldots, A_N\}$ from another or $X(0)$.

For a lower bound on **ET** the following assumption will be needed.

Assumption 2.3. $T(X(0), A_1), \ldots, T(X(0), A_N)$ are distinct and nonzero with probability one.

For Brownian motion on spheres, this assumption is implied by

$$A_i \cap A_j = \emptyset \quad \text{for} \quad i \neq j$$

and

$$X(0) \notin \bigcup_{j=1,2,\ldots,N} A_j.$$

The route to get bounds on **ET** is the introduction of auxilliary randomization. See Aldous (1983b) for a different approach . Let (Ω, F, P) be the probability space on which $\{X(t)\}$ is defined. Let Σ be the set of all N! permutations of $(1, 2, \ldots, N)$, G the set of all subsets of Σ, and U the uniform distribution on Σ. Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ be a random permutation from (Σ, G, U). Form the product space $(\Omega \times \Sigma, F \times G, P \times U)$. Let F_t be the sub–sigma field of $\Omega \times \Sigma$ generated by σ and $\{X(s), 0 \leq s \leq t\}$. Define S_1 to be the first time $\{X(t)\}$ visits A_{σ_1}. Let S_2 be the first time $\{X(t)\}$ has visited both A_{σ_1} and A_{σ_2}. In general let S_i be the first time $\{X(t)\}$ has visited each of $A_{\sigma_1}, \ldots, A_{\sigma_i}$. For notational convenience let $S_0 = 0$ and $R_i = S_i - S_{i-1}$. R_i is the additional time taken to visit A_{σ_i} after time S_{i-1}. Clearly $S_N = \mathbf{T}$. The following propositions are easy consequences of these definitions.

Proposition 2.4.

$$P(R_i \neq 0) \leq \frac{1}{i}$$

and assuming (2.3)

$$P(R_i \neq 0) = \frac{1}{i}.$$

Proof. For the second assertion, let $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ index the permutations of $(1, 2, \ldots, N)$. For each π, let Π denote the event

$$0 < T(X(0), A_{\pi_1}) < T(X(0), A_{\pi_2}) < \ldots < T(X(0), A_{\pi_N}).$$

By Assumption 2.3 the union over all permutations of these events is the entire sample space. Conditional on Π, the probability that $R_i \neq 0$ is the probability that σ_i occurs further to the right in π than $\sigma_1, \ldots, \sigma_{i-1}$. This has conditional probability $1/i$, hence unconditional probability $1/i$ as well. An identical but slightly messier argument proves the first assertion. ∎

Proposition 2.5. The event $\{S_i \leq t\}$ is in F_t for all t, i.e., S_i is a stopping time for $i = 1, 2, \ldots, N$ with respect to the family of sigma fields $\{F_t\}$.

Proof. Straightforward. ∎

Let F^i be the σ–field generated by the permutation σ and $\{X(t); 0 \leq t \leq S_i\}$.

Proposition 2.6. $\{R_i \neq 0\} \in F^{i-1}$.

Proof.

$$\{R_i = 0\} = \bigcup_{j=1}^{N} [\{A_{\sigma_i} = A_j\} \cap \{\{X(t)\} \text{ visits } A_j \text{ before time } S_i\}].$$

Each of the events above is in F^{i-1}. ∎

Now the theorem of this section can be stated.

Theorem 2.7.

$$E\mathbf{T} \leq \mu_+ \sum_{i=1}^{N} \frac{1}{i}$$

and assuming (2.3)

$$E\mathbf{T} \geq \mu_- \sum_{i=1}^{N} \frac{1}{i}.$$

Proof. If either μ_- or μ_+ is infinite, it is easy to see that the corresponding bound holds. Therefore assume both are finite. Write

$$\mathrm{E}\mathbf{T} = \sum_{i=1}^{N} \mathrm{E}(R_i) = \sum_{i=1}^{N} \mathrm{E}(\mathrm{E}(R_i | F^{i-1})) .$$

$$\mathrm{E}(\mathrm{E}(R_i|F^{i-1})) = \mathrm{E}(T(X(S_{i-1}), A_{\sigma_i}) I_{\{R_i \neq 0\}}) \tag{2.8}$$

by the strong Markov property, time homogeneity and Proposition (2.6). On the set $\{R_i \neq 0\}$, $X(S_{i-1}) \in \hat{A}_{\sigma_i}$. Thus by the definitions of μ_+ and μ_-,

$$I_{\{R_i \neq 0\}} \mu_- \leq E(R_i | F^{i-1}) \leq I_{\{R_i \neq 0\}} \mu_+. \tag{2.9}$$

So by Proposition 2.4, taking expectation throughout this inequality and summing over i from 1 to N yields the result. ∎

Similar arguments can be used to get bounds on the moment generating function of $\mathbf{T}$. See Matthews (1985) for an example of this in the context of random walks on finite groups. Better bounds can be obtained with more work. This involves considering the distance between A_{σ_i} and $X(S_{i-1})$. See Matthews (1985) for an example of this. It is possible that this kind of consideration could lead to tight bounds for Brownian motion on R^3.

3. Caps on Spheres

Here the number of caps of radius ϵ needed to cover a sphere and the number of disjoint caps of radius ϵ that can be packed on a sphere are considered. Recall that a cap of radius ϵ on Σ_p about a particular point x is the set of all point of Σ_p within a geodesic distance ϵ of x. The first problem is related to coding theory, and there is a substantial literature on the subject. See Sloane (1982) for results and references. The first problem has not been investigated as much; Rogers (1963) is one reference. Here only give crude answers to these problems will be offered. As is apparent from the last section, the mean time taken to visit a set of caps will depend on the number of caps only through its logarithm. This makes the crude answers to the sphere covering and packing problems given here sufficient to obtain tight results for the problem of interest here, the mean time taken by Brownian motion to become dense on the surface of a sphere. The following results are all that is needed.

Proposition 3.1. There exist positive constants (depending only on p) U and L such that there are N(ρ) caps of radius ρ that cover Σ_p and M(ρ, θ) disjoint caps of radius ρ on Σ_p with points in different caps a distance at least 2θ apart such that

$$N(\rho) \leq U\rho^{1-p} \ \text{ and } \ L(\rho+\theta)^{1-p} \leq M(\rho,\theta).$$

Proof. With the area of the surface of Σ_p normalized to one, the volume of a cap of radius ρ is

$$\frac{\int_0^\rho \sin^{p-2}(\theta)d\theta}{\int_0^\pi \sin^{p-2}(\theta)d\theta}.$$

This is $O(\rho^{p-1})$ as $\rho \to 0$. Thus at most $O((\frac{\rho}{2})^{1-p})$ disjoint caps of radius $\frac{\rho}{2}$ can be placed on Σ_p. With any placement of caps of this radius such that there is no room for any more, concentric caps of radius ρ will cover Σ_p. Therefore Σ_p can be covered by $O(\rho^{1-p})$ caps of radius ρ. Similarly, place as many disjoint caps of radius $\rho+\theta$ as possible on Σ_p. Since concentric caps of twice the radius will cover Σ_p, there must be at least $\Theta(2(\rho+\theta)^{1-p})$ of these caps, or else their total volume will be less than 1. Taking concentric caps of radius ρ gives the same number of caps with the property that no two points in distinct caps are closer than a distance 2θ apart. ∎

An identical result is needed for the two cap problem. For future convenience define a cappair of radius ρ to be a pair of caps of radius ρ whose centers lie on a line through the origin. If opposite points of the sphere are identified, then a cappair is a ball of radius ρ in the Grassman manifold of one–dimensional subspaces of R^p. Instead of using cappairs, Brownian motion on this manifold could be considered. However, cappairs seem intuitively simpler and therefore will be used here. Consider the problems of covering and packing Σ_p with cappairs.

Proposition 3.2. There exist positive constants (depending only on p) u and l such that there are N(ρ) cappairs of radius ρ that cover Σ_p and M(ρ, θ) disjoint cappairs of radius ρ with points in different cappairs a distance at least 2θ apart on Σ_p such that

$$N(\rho) \leq u\rho^{1-p} \ \text{ and } \ l(\rho+\theta)^{1-p} \leq M(\rho,\theta).$$

Proof. The proof is exactly the same as the proof of Proposition (3.1). Note that the constants in this proposition will be exactly half as large as those of the previous proposition. ∎

4. Brownian Motion on Spheres

Brownian motion on a Σ_p, as a limit of random walks, was studied by Roberts and Ursell (1960). See Watson (1983) for a more modern description. Here the full diffusion is not considered; only the cosine of distance of the process from a particular point, say the North Pole, must be considered. This itself is a diffusion on [-1,1], with drift

$$\mu(x) = \frac{-\lambda x}{2} \tag{4.1}$$

and infinitesimal variance

$$\sigma^2(x) = \frac{\lambda(1-x^2)}{p-1}. \tag{4.2}$$

Suppressing the dependence on p, call this diffusion $W_\lambda(t)$. Karlin and Taylor (1981, p.338) discusses this diffusion briefly.

λ is an arbitrary parameter like the infinitesimal variance of Brownian motion on the line. Consider a symmetric random walk on the surface of a sphere and let ϕ denote a random variable whose distribution is the same as the step lengths of the random walk. If time and space are rescaled as usual in convergence of random walks to Brownian motion, the limiting diffusion obtained will be Brownian motion with scale parameter $\lambda = \mathrm{E}\phi^2$. Thus, intuitively, for small λ, $W_\lambda(t)$ moves about as fast as a random walk taking steps of size $\sqrt{\lambda}$.

Given $\mu(x)$ and $\sigma^2(x)$, it is an elementary exercise to calculate the expected time taken by Brownian motion to hit a cap or a cappair. Let $T(x,r)$ be the first time $W_\lambda(t)$, started from x, leaves the interval $(-r,r)$ for $|x| < |r| < 1$. Then $T(x,r)$ has the same distribution as the first time Brownian motion on Σ_p starting from a point a distance $\cos^{-1}(x)$ from a chosen point hits a cappair of radius $\cos^{-1}(r)$ with one cap centered at the chosen point. A similar procedure or a limiting argument equates a hitting time for $W_\lambda(t)$ and the time taken by Brownian motion on Σ_p to hit a single cap.

Following Karlin and Taylor, $\mathrm{E}T(x,r)$, $f(x,r)$ for short, satisfies

$$-1 = \frac{-\lambda x}{2} f'(x,r) + \frac{\lambda(1-x^2)}{2(p-1)} f''(x,r) \tag{4.3}$$

subject to $f(-r,r) = f(r,r) = 0$ with $'$ denoting differentiation with respect to x. For the two cap problem, the values of $f(0,r)$ and $f(x(r),r)$, where $x(r)$ is slightly smaller than r, will be needed. These are the maximal and minimal hitting times to be used in Theorem 2.7. Solving (4.3) is a standard exercise; see Karlin and Taylor, p.191, for example. The derivative of the scale function, $s(x)$ of (3.5) of Karlin and Taylor, p.194 is

$$s(x) = (1-x^2)^{-(\frac{p-1}{2})}.$$

The density $m(x)$ of the speed measure is

$$m(x) = \frac{(p-1)(1-x^2)^{\frac{p-3}{2}}}{\lambda}.$$

Lengthy but elementary calculations yield the following, valid as $r \to 1$.

$$(4.4) \qquad f(0,r) = \begin{cases} \frac{2\sqrt{\pi}}{\lambda(p-3)} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \left(\frac{1}{1-r^2}\right)^{\frac{p-3}{2}} (1+O(1-r^2)^{\frac{1}{2}}) & p \geq 4 \\ \frac{4}{\lambda} \log\left(\frac{1}{1-r^2}\right) + O(1) & p = 3. \end{cases}$$

For $p \geq 4$

$$(4.5a) \qquad f(x(r),r) = \frac{2\sqrt{\pi}}{\lambda(p-3)} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \left(\frac{1}{1-r^2}\right)^{\frac{p-3}{2}} \left(1 + O\left(\frac{1}{\log(1-r^2)}\right)\right)$$

$$\text{where } x(r) = \sqrt{1-(1-r^2)\log^2\sqrt{1-r^2}}.$$

For $p = 3$

$$(4.5b) \qquad f(x(r),r) = \frac{4}{\lambda}(\log(a^2(r)))^{\frac{1}{2}} + O(1)$$

$$\text{where } x(r) = (1-(1-r^2)a^2(r))^{\frac{1}{2}},$$

$$\text{and } a^2(r) \text{ is arbitrary subject to } \frac{1}{1-r^2} > a^2(r) > 1.$$

An arbitrary $a^2(r)$ is included for $p = 3$ since the bounds on the mean time taken by Brownian motion to become dense in S_3 are not tight in this case. By examining all possible choices of $a^2(r)$ it can be seen that the bounds cannot be improved without a more detailed analysis.

For the one cap problem, expected times taken to hit one cap could also be calculated. Let $g(x,r)$ denote the time by $W_\lambda(t)$ to hit a cap of radius $\cos^{-1}(r)$ about a chosen point from a point a distance $\cos^{-1}(x)$ from the chosen point. The same calculations yield, as $r \to 1$,

$$g(-1,r) = \begin{cases} \frac{4\sqrt{\pi}}{\lambda(p-3)} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \left(\frac{1}{1-r^2}\right)^{\frac{p-3}{2}} (1+O((1-r^2)^{\frac{1}{2}})) & p \geq 4 \\ \frac{8}{\lambda} \log\left(\frac{1}{1-r^2}\right) + O(1) & p = 3. \end{cases}$$

For $p \geq 4$

$$g(x(r),r) = \frac{4\sqrt{\pi}}{\lambda(p-3)} \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \left(\frac{1}{1-r^2}\right)^{\frac{p-3}{2}} \left(1 + O\left(\frac{1}{\log(1-r^2)}\right)\right)$$

$$\text{where } x(r) = \sqrt{1-(1-r^2)\log^2\sqrt{1-r^2}}$$

and for $p = 3$

$$g(x(r), r) = \frac{8}{\lambda}\log(a^2(r))^{\frac{1}{2}} + O(1)$$

$$\text{where } x(r) = (1-(1-r^2)a^2(r))^{\frac{1}{2}},$$

$$\text{and again } \frac{1}{1-r^2} > a^2(r) > 1.$$

5. Expected Covering Times

In this section upper and lower bounds on the mean time taken by $W_\lambda(t)$ to come within a geodesic distance ϵ of all points of Σ_p, with opposite points identified (the two cap problem), are calculated. Result will only be stated for the one cap problem and can be obtained in the same manner. The bounds are quite messy and therefore will be given only as $\epsilon \to 0$. Since the bounds will be tight asymptotically as $\epsilon \to 0$ for $p \geq 4$ and not tight for $p = 3$, the two cases will be considered separately.

Recall that $\mathrm{E}C_2(\epsilon, p)$ is the expected time taken by Brownian motion to come within ϵ of all points of Σ_p in the two cap problem.

Theorem 5.1.

$$\mathrm{E}C_2(\epsilon, p) = 2\frac{\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{\log(\epsilon^{-1})}{\epsilon^{p-3}}\left(1 + 0\left(\frac{\log\log(\epsilon^{-1})}{\log(\epsilon^{-1})}\right)\right) \quad \text{for } p \geq 4.$$

Similarly, $\mathrm{E}C_1(\epsilon, p)$ is the expected time taken by Brownian motion to come within ϵ of all points of Σ_p in the one cap problem.

Theorem 5.2.

$$\mathrm{E}C_1(\epsilon, p) = 4\frac{\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{\log(\epsilon^{-1})}{\epsilon^{p-3}}\left(1 + 0\left(\frac{\log\log(\epsilon^{-1})}{\log(\epsilon^{-1})}\right)\right) \quad \text{for } p \geq 4.$$

Proof of Theorem 5.1. First consider a lower bound on $\mathrm{E}C_2(\epsilon, p)$. By Proposition 3.2 there is a set of at least $l(\epsilon + \epsilon\log(\epsilon^{-1}))^{-(p-1)}$ disjoint cappairs of radius ϵ such that two points in different cappairs are a distance at least $2\epsilon\log(\epsilon^{-1})$ apart. Choose X(0) without loss of generality to be at the center of one of the caps, and remove this cappair from the set under consideration. The expected time taken to hit the remaining cappairs is a lower bound on $\mathrm{E}C_2(\epsilon, p)$, since if $W_\lambda(t)$ has not visited one of the remaining pairs, then it has not been within a distance ϵ of either of the two centers, thus it has not been within ϵ of all points or their opposites. Now μ_-, the minimum expected time to hit a cappair from $X(0)$ or from inside another chosen cappair, is given by

$$f(\cos(\epsilon(1 + 2\log(\epsilon^{-1})), \cos(\epsilon)).$$

Let $x(r) = \left(1-(1-r^2)\log^2\sqrt{1-r^2}\right)^{\frac{1}{2}}$. For ϵ reasonably small,

$$\cos(\epsilon(1 + 2\log(\epsilon^{-1}) < x(\cos(\epsilon)),$$

so

$$f(\cos(\epsilon(1+2\log(\epsilon^{-1})),\cos(\epsilon)) > f(x(\cos(\epsilon)),\cos(\epsilon)).$$

$f(x(\cos(\epsilon)),\cos(\epsilon))$ is given by (4.5a) as

$$\frac{2\sqrt{\pi}}{\lambda(p-3)}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\left(\frac{1}{1-\cos^2(\epsilon)}\right)^{\frac{p-3}{2}}\left(1+O\left(\frac{1}{\log(1-\cos^2(\epsilon))}\right)\right) \quad \text{as } \epsilon\to 0,$$

$$\text{which is } \frac{2\sqrt{\pi}}{\lambda(p-3)}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{1}{\epsilon^{p-3}}\left(1+O\left(\frac{1}{\log(\epsilon^{-1})}\right)\right).$$

Everything needed for Theorem 2.7 is now available. The time taken by $W_\lambda(t)$ to come within ϵ of all points of Σ_p or their opposites is larger than the time taken to visit the

$$l(\epsilon+\epsilon\log(\epsilon^{-1}))^{-(p-1)}$$

chosen cappairs. Using the elementary fact that $\Sigma_{i=1}^{N}\frac{1}{i}=\log(N)+O(1)$, Theorem 2.7 says

$$\mathrm{E}C_2(\epsilon,p) > \frac{2\sqrt{\pi}}{\lambda(p-3)}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{1}{\epsilon^{p-3}}\left(1+O\left(\frac{1}{\log(\epsilon^{-1})}\right)\right)\log\left(L\left(\epsilon+\epsilon\log(\epsilon^{-1})\right)^{-(p-1)}\right)$$

which is

$$2\frac{\sqrt{\pi}}{\lambda}\frac{p-1}{p-3}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{\log(\epsilon^{-1})}{\epsilon^{p-3}}\left(1+0\left(\frac{\log\log(\epsilon^{-1})}{\log(\epsilon^{-1})}\right)\right) \quad \text{for } p\geq 4$$

as asserted.

Next consider an upper bound on $\mathrm{E}C_2(\epsilon,p)$. By Proposition 3.2 there is a set of at most

$$u\left(\frac{\epsilon}{\log(\epsilon^{-1})}\right)^{-(p-1)}$$

cappairs of radius $\epsilon/\log(\epsilon^{-1})$ that cover Σ_p. Place concentric cappairs of radius

$$\delta=\epsilon\left(1-\frac{1}{\log(\epsilon^{-1})}\right)$$

about the center of each of these cappairs. If $W_\lambda(t)$ visits one of these large cappairs, then it simultaneously comes within ϵ of every point, or its opposite, in the smaller concentric cappair. Thus if $W_\lambda(t)$ has visited each of the cappairs, then it has been within ϵ of all points of Σ_p or their opposites. To use Theorem 2.7 μ_+ is needed. The maximum expected time taken to hit a cappair of radius δ is $f(0,\cos(\delta))$, which is found from (4.4) to be

$$\frac{2\sqrt{\pi}}{\lambda(p-3)}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\left(\frac{1}{1-\cos^2(\delta)}\right)^{\frac{p-3}{2}}\left(1+O\left(1-\cos^2(\delta)\right)^{\frac{1}{2}}\right).$$

Thus $\mathrm{E}C_2(\epsilon, p)$ is less than

$$\frac{2\sqrt{\pi}}{\lambda(p-3)}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\left(\frac{1}{1-\cos^2(\delta)}\right)^{\frac{p-3}{2}}\left(1+O\left(1-\cos^2(\delta)\right)^{\frac{1}{2}}\right)(\log(\delta)+O(1)).$$

This is

$$\frac{2\sqrt{\pi}}{\lambda(p-3)}\frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})}\frac{1}{\epsilon^{p-3}}(p-1)\log(\epsilon^{-1})\left(1+O\left(\frac{\log\log(\epsilon^{-1})}{\log(\epsilon^{-1})}\right)\right)$$

as required. ∎

Theorem 5.2 is proven the same way.

Now consider the case $p = 3$.

Theorem 5.3. On Σ_3,

$$2 \le \liminf \frac{\lambda \mathrm{E}C_2(\epsilon,p)}{\log^2(\epsilon^{-1})} \le \limsup \frac{\lambda \mathrm{E}C_2(\epsilon,p)}{\log^2(\epsilon^{-1})} \le 8.$$

Theorem 5.4. On Σ_3,

$$4 \le \liminf \frac{\lambda \mathrm{E}C_1(\epsilon,p)}{\log^2(\epsilon^{-1})} \le \limsup \frac{\lambda \mathrm{E}C_1(\epsilon,p)}{\log^2(\epsilon^{-1})} \le 16.$$

Proof of Theorem 5.3. For an upper bound the same argument as the case $p \ge 4$ is used. Concentric cappairs of radii $\epsilon/\log(\epsilon^{-1})$ and $\epsilon - \epsilon/\log(\epsilon^{-1})$, Theorem 2.7, Proposition 3.2, and (4.4) give the result.

The best lower bound is obtained by choosing disjoint cappairs of radius $\epsilon + \sqrt{\epsilon}/2$ and concentric cappairs of radius ϵ. In (4.4) let $r = \cos(\epsilon)$ and $x(r) = \cos(\epsilon + \sqrt{\epsilon})$, so $a(r) = \sin(\epsilon - \sqrt{\epsilon})/\sin(\epsilon)$. Thus the lower bound can be shown to be

$$\frac{4}{\lambda}\log\left(\frac{1}{\sqrt{\epsilon}}\right)\log\left(\frac{U}{(\epsilon+\sqrt{\epsilon})^2}\right),$$

which is the assertion of the theorem. ∎

6. Discussion

An obvious question is: when will the bounds obtained by this method be tight? Intuitively, they will be tight when μ_- and μ_+ are close together; when the expected time taken by the process to hit a small cap is about the same whether the process starts quite near the cap or far from the cap. A process is rapidly mixing, in the sense of Aldous (1983a), if its distribution is close to its stationary distribution (assuming one exists) in a short time. For Markov chains, a short time is a number of transitions that is small compared to the size of the state space. For a continuous process, the analogous idea is that the process is rapidly mixing if it is close to its stationary distribution before if has come close to a non–negligible portion of its state space. Intuitively, the bounds given in Section 2 will be tighter for more rapidly mixing processes. Processes in high dimensions naturally tend to be rapidly mixing. Brownian motion on Σ_p, for $p \ge 4$, is unlikely to hit a small cap in a short period of time, even if it starts fairly near the cap. There are too many directions for it to wander off in. On Σ_3, Brownian motion is less rapidly mixing, leading to bounds that are not asymptotically tight.

It may be possible to give tight bounds for the expected time taken by Brownian motion to become dense on Σ_3, with extra effort. This paragraph discusses what the correct answer probably is. Returning attention to (2.8) and (2.9), it is clear that better bounds could be obtained by considering the distance between $X(S_{i-1})$ and A_{σ_i}. This is not an easy problem, but the technique was successful [Matthews (1985)] in getting the asymptotic distribution of the time taken by certain random walks on Z_2^n to visit every element of the group. The problem on S_3 boils down to the following: In the proof of Theorem 2.3 consider the expected time taken by Brownian motion to visit a set of M small disjoint cappairs. For small δ, how are the last δM cappairs situated on S_3? If they are in a few clumps, then the lower bound given would be hard to improve. However, if the are fairly spread out on the sphere, the with high probability $X(S_{i-1})$ and A_{σ_i} will be fairly far apart, allowing μ_- to be replaced by a larger number. Intuitively, this is probably the case. However a proof is hard to come by. Thus, there is a good suspicion that in the case of Brownian motion on S_3, the upper bound is the correct asymptotic expected covering time, and it is the lower bound that needs to be improved.

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