MOMENTS OF A RANK VECTOR WITH APPLICATIONS TO SELECTION AND RANKING

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ABSTRACT:

Let k populations π_i be given with continuous cumulative distribution function $F_i(x)$, $i = 1, 2,\dots,k$. Also let $X_1, X_2,\dots,X_k$ be a set of mutually independent observations from respective populations $\pi_1, \pi_2,\dots,\pi_k$. A rank vector $\underline{R} = (R_1, R_2,\dots,R_k)$ is defined for these observations as follows:

$$R_i = s \text{ if } X_i \text{ is the s-th smallest of the variables } X_1, X_2,\dots,X_k.$$

The purpose of this paper is to give the mean, variance and covariance of rank vector $\underline{R}$ under the population model stated above. Applications to the slippage configuration of distributions and the normal distribution are also given.

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1. INTRODUCTION

Suppose there exist k populations $\pi_i (i = 1, 2,...,k)$, whose cumulative distribution functions (c.d.f.'s) are continuous in x and have the form $F_i(x)$, $(i = 1, 2,...,k)$. (In later sections, we sometimes denote $F_i(x)$ by F_i briefly).

Let the corresponding random variables be $X_1, X_2,...,X_k$ and let $\underline{R} = (R_1, R_2,...R_k)$ be their respective ranks defined as follows:

$$R_i = s \text{ if } X_i \text{ is the s-th smallest of the variables } X_1, X_2,...,X_k.$$

Let n independent observations of the rank vector $\underline{R}$ be $(\underline{R}_1, \underline{R}_2,...,\underline{R}_n)$. A rank sum vector $\underline{T}$ is defined by $\underline{T} = \sum_{j=1}^{k} \underline{R}_j$.

Tests of hypotheses or ranking and selection procedures based on this statistic are considered elsewhere, see for example Lehmann [8], Gupta and Panchapakesan [5] and Dudewicz and Koo [2]. Among those works which treat this statistic, several authors, Gupta and McDonald [4], McDonald [10], Matsui [9], Lee and Dudewicz [7] pay attention to the moments of this statistic and use them under the slippage configuration defined below.

The purpose of this paper is to give, exactly, the mean, variance and covariance of the rank statistic $\underline{R}$ under the population model. The results will be useful for clarifying the relations between ranks and populations (or distributions) and for checking numerically the behavior of ranks.

In order to apply the results to the ranking and selection problems, it is often necessary to investigate them under a special parametric configuration, i.e., the slippage configuration.

So, the results obtained here are also applied to the slippage configuration of distributions, which is defined as follows.

DEFINITION

For given c.d.f.'s $F_1(x)$, $F_2(x),\dots,F_k(x)$, a slippage configuration of distributions is defined by

$$F_1(x) = \dots = F_{k-t}(x) \ (\equiv F_0(x)) \geqq F_{k-t+1}(x) = \dots = F_k(x)(\equiv F(x)) \qquad (1.1)$$

for all x, where t is any given integer such that $1 \leqq t \leqq k-1$.

In Section 2, mean, variance and covariance of the rank vector are given. The results are also applied to the slippage configuration of distributions. Applications to the normal distributions are given in Section 3.

2. MOMENTS OF THE RANK VECTOR

In this section we give mean, variance and covariance of the rank vector defined in previous section.

2.1. Means

Let us first consider the expectations of a rank R_1. A probability $\Pr(R_1 = s)$ is given by

$$\Pr(R_1 = s) = \sum_S \Pr(X_{i_1} <\dots< X_{i_{s-1}} < X_1 < X_{i_{s+1}} <\dots< X_{i_k}) \qquad (2.1)$$

where $\{i_1,\dots,i_{s-1},\ i_{s+1},\dots,i_k\}$ is the permutation of $\{2,3,\dots,k\}$, and summation $\sum_S$ is taken over all $(k-1)!$ permutations of $\{i_1,\dots,i_{s-1},\ i_{s+1},\dots,i_k\}$.

Thus we have

$$E(R_1) = \int \sum_{s=1}^{k} \sum_{S} s\, F_{i_1}(x)\cdots F_{i_{s-1}}(x)\{1-F_{i_{s+1}}(x)\}\cdots\{1-F_{i_k}(x)\}\, dF_1(x)$$

$$= \int \sum_{m=0}^{k-1} \sum_{Am} \sum_{s=1}^{m+1} s\ {}_mC_{s-1}(-1)^{m-s+1} F_{a_1}(x)\cdots F_{a_m}(x)\, dF_1(x) \tag{2.2}$$

where $\{a_1, a_2,\ldots,a_m\}$ is a combination of size m ($\leqq k-1$) from the set $\{2,3,\ldots,k\}$ and the summation $\sum\limits_{A_m}$ is taken over all ${}_{k-1}C_m$ combinations $\{a_1, a_2,\ldots,a_m\}$ out of $\{2,3,\ldots,k\}$. Since

$$\sum_{s=1}^{m+1} (-1)^{m-s+1} s\ {}_mC_{s-1} = \begin{cases} 0 & m \geqq 2, \\ 1 & m = 0 \text{ or } 1, \end{cases} \tag{2.3}$$

we have

$$E(R_1) = \int\{1+ \sum_{j=2}^{k} F_j(x)\}\, dF_1(x) \tag{2.4}$$

In a similar manner, we have the following result in general.

THEOREM 2.1.

$$E(R_i) = \int \{1 + \sum_{\substack{j=1\\ j\neq i}}^{k} F_j(x)\}\, dF_i(x)$$

$$= \frac{1}{2} + \sum_{j=1}^{k} \int F_j(x)\, dF_i(x), \quad (i = 1,2,\ldots,k) \tag{2.5}$$

For the slippage configuration of distributions defined in (2.1), we have the following.

COROLLARY 2.1.

$$E(R_i) = \begin{cases} (k+t+1)/2-t\int F_0(x)dF(x), & \text{for } 1 \leqq i \leqq k-t \\ (t+1)/2+(k-t)\int F_0(x)dF(x), & \text{for } k-t+1 \leqq i \leqq k \end{cases} \tag{2.6}$$

Note that by using the thoerem 2.1, we have the following property of ranks.

THEOREM 2.2.

$$E(R_i) \geqq E(R_j) \text{ if and only if } F_i \leqq F_j, \quad (i,j=1,2,\ldots,k).$$

This theorem may support the relationship between ranks and (location or scale) parameters of the distribution to which ranked data are originated.

2.2 Variances

In the same way as in the preceding section, we have

$$E(R_1^2) = \int \sum_{m=0}^{k-1} \sum_{A_m} \sum_{s=1}^{m+1} s^2 \; {}_mC_{s-1}(-1)^{m-s+1} \; F_{a_1}(x)\cdots F_{a_m}(x)dF_1(x) \tag{2.7}$$

Since

$$\sum_{s=1}^{m+1}(-1)^{m-s+1} \; s^2 \; {}_mC_{s-1} = \begin{cases} 0, & \text{if } m \geqq 3, \\ 2, & \text{if } m = 2, \\ 3, & \text{if } m = 1, \\ 1, & \text{if } m = 0, \end{cases} \tag{2.8}$$

we have, for $A_1 = \{a_1\}$, all k-1 values out of $\{2,3,\ldots,k\}$ and for $A_2 = \{a_1,a_2\}$, all ${}_{k-1}C_2$ combinations out of $\{2,3,\ldots,k\}$,

$$E(R_1^2) = \int\{1+3\sum_{A_1} F_{a_1}(x) + 2\sum_{A_2} F_{a_1}(x)F_{a_2}(x)\}\, dF_1(x) \tag{2.9}$$

Thus we have the following general form of variance.

THEOREM 2.3.

$$V(R_i) = \int\{1+3\sum_{\substack{j=1\\ j\neq i}}^{k} F_j(x) + \sum_{\substack{j,r=1\\ j,r\neq i\\ j\neq r}}^{k} F_j(x)\, F_r(x)\}\, dF_i(x)$$

$$- [\int\{1+\sum_{\substack{j=1\\ j\neq i}}^{k} F_j(x)\}\, dF_i(x)]^2, \quad (i = 1,2,\ldots,k) \tag{2.10}$$

This can also be rewritten as

$$V(R_i) = 2\sum_{j=1}^{k} \int F_j\, dF_i - 2\sum_{j=1}^{k} \int F_i F_j\, dF_i - \sum_{j=1}^{k} \int F_j^2\, dF_i$$

$$+ \sum_{j,r=1}^{k} \int F_j F_r\, dF_i - (\sum_{j=1}^{k} \int F_j\, dF_i)^2 - \frac{1}{12}\,. \tag{2.11}$$

Under the slippage configuration (2.1), we have the following.

COROLLARY 2.2.

$$V(R_i) = \begin{cases} \{(k-t)^2-1\}/12 + t(k+t-2)\int F_0 dF - 2t(t-1)\int F_0 F dF \\ \quad - t(k-t-1)\int F_0^2 dF - t^2(\int F_0 dF)^2, \quad \text{for } 1 \leqq i \leqq k-t, \\ (t^2-1)/12 - (k-t)(t-2)\int F_0 dF + 2(k-t)(t-1)\int F_0 F dF \\ \quad + (k-t)(k-t-1)\int F_0^2 dF - (k-t)^2(\int F_0 dF)^2, \quad \text{for } k-t+1 \leqq i \leqq k \end{cases} \tag{2.12}$$

2.3 Covariances

For given integers s and t such that $1 \leqq s < t \leqq k$, we have

$$Pr(R_1=s,\ R_2=t) = \sum_{S_1} Pr\{X_{i_1} < \cdots < X_{i_{s-1}} < X_1 < X_{i_{s+1}} < \cdots < X_{i_{t-1}} < X_2 < X_{i_{t+1}} < \cdots < X_k\}.$$

(2.13)

$$Pr(R_1=t,\ R_2=s) = \sum_{S_2} Pr\{X_{i_1} < \cdots < X_{i_{s-1}} < X_2 < X_{i_{s+1}} < \cdots < X_{i_{t-1}} < X_1 < X_{i_{t+1}} < \cdots < X_k\}.$$

Here, $(i_1,\ldots,i_{s-1},\ i_{s+1},\ldots,i_{t-1},\ i_{t+1},\ldots,i_k)$ is a permutation of $(3,4,\ldots,k)$ and the summation $\sum_{S_1}$, $\sum_{S_2}$ are taken overall (k-2)! permutations of $(i_1,\ldots,i_{s-1},$ $i_{s+1},\ldots,i_{t-1},\ i_{t+1},\ldots,i_k)$. Then we have

$$E(R_1R_2) = \int\int_{x<y} \sum_{s<t} \sum_{S_1} stF_{i_1}(x)\cdots F_{i_{s-1}}(x)\{F_{i_{s+1}}(x)-F_{i_{s+1}}(x)\}\cdots$$

$$\cdots\{F_{i_{t-1}}(y)-F_{i_{t-1}}(x)\}\{1-F_{i_{t+1}}(y)\}\cdots\{1-F_{i_k}(y)\}\{dF_1(x)dF_2(y)+dF_2(x)dF_1(y)\}$$

$$= \int\int_{x<y} \sum_{\substack{r,m\geq 0 \\ r+m\leq k-2}} \sum_{A_{rm}} \sum_{(s,t)\in E_{rm}} st \; {}_rC_{s-1} \; {}_mC_{t-r-s} \; (-1)^{m-s-t+1}$$

$$F_{a_1}(x)\cdots F_{a_r}(x)F_{b_1}(y)\cdots F_{b_m}(y)\{dF_1(x)dF_2(y)+dF_2(x)dF_1(y)\} \qquad (2.15)$$

where $(a_1,\ldots,a_r, b_1,\ldots,b_m)$ is a combination of size $(r+m)$, $(r+m\leq k-2)$ from the set $(3,4,\ldots,k)$, and the summation $\sum_{A_{rm}}$ is taken over all ${}_{k-2}C_{r+m}$ combinations $\{a_1,\ldots,a_r, b_1,\ldots,b_m\}$ out of $\{3,4,\ldots,k\}$ and $E_{rm} = \{(s,t);\ s<t,\ 0\leq r-s+1\leq t-s-1,\ 0\leq m-t+r+2\leq k-t\}$. The coefficient of $F_{a_1}(x)\cdots F_{a_r}(x)F_{b_1}(y)\cdots F_{b_m}(y)$ in expression (2.15) is given by

$$C_{rm} \equiv \sum_{t=r+2}^{m+r+2} \sum_{s=1}^{r+1} s\ t\ {}_rC_{s-1}\ {}_mC_{t-r-2}(-1)^{m-s-t+1}$$

$$= \sum_{t=r+2}^{m+r+2} (-1)^{m-t+1}\ t\ {}_mC_{t-r-2} \sum_{s=1}^{r+1} (-1)^{-s}\ s\ {}_rC_{s-1} \qquad (2.16)$$

By using (2.3), we find that the summation in C_{rm} vanishes except for r, m=0 and 1, and

$$C_{rm} = \begin{cases} 1, & \text{if } (r,m) = (1,1), \\ 1, & \text{if } (r,m) = (0,1), \\ 3, & \text{if } (r,m) = (1,0), \\ 2, & \text{if } (r,m) = (0,0). \end{cases} \tag{2.17}$$

We have thus

$$E(R_1R_2) = \iint_{x<y} \left\{ \sum_{\substack{i,j=3 \\ i \neq j}}^{k} F_i(x)F_j(y) + \sum_{i=3}^{k} F_i(y) + 3\sum_{i=3}^{k} F_i(x) + 2 \right\}$$

$$\{dF_1(x)dF_2(y) + dF_2(x)dF_1(y)\}.$$

Since it is shown that

$$\iint_{x<y} \sum_{\substack{i,j=3 \\ i \neq j}}^{k} F_i(x)F_j(y)\{dF_1(x)dF_2(y) + dF_2(x)dF_1(y)\}$$

$$= \sum_{\substack{i,j=3 \\ i \neq j}}^{k} \int F_i(x)dF_1(x)\int F_j(y)dF_2(y)$$

we have finally

$$E(R_1R_2) = \sum_{\substack{i,j=3 \\ i \neq j}}^{k} \int F_i(x)dF_1(x)\int F_j(x)dF_2(x) + 3\sum_{i=3}^{k} \int F_i(x)dF_1(x)$$

$$+ 3\sum_{i=3}^{k} \int F_i(x)dF_2(x) - 2\sum_{i=3}^{k}\int F_i(x)F_2(x)dF_1(x) - 2\sum_{i=3}^{k}\int F_i(x)F_1(x)dF_2(x) + 2 \tag{2.18}$$

Generalizing the result thus obtained, we have the following theorem.

THEOREM 2.4.

Covariance of the ranks Rm and Rs is given by

$$\mathrm{Cov}(Rm,Rs) = \sum_{\substack{i,j=1\\ i,j\neq m,s\\ i\neq j}}^{k} \int F_i(x)dF_m(x)\int F_j(x)dF_s(x) + 3\sum_{\substack{i=1\\ i\neq m,s}}^{k}\{\int F_i(x)dF_m(x)+\int F_i(x)dF_s(x)\}$$

$$-2\sum_{\substack{i=1\\ i\neq m,s}}^{k}\{\int F_i(x)F_m(x)dF_s(x) + \int F_i(x)F_s(x)dF_m(x)\}$$

$$-\int\{1+\sum_{\substack{i=1\\ i\neq m}}^{k}F_i(x)\}dF_m(x)\int\{1+\sum_{\substack{i=1\\ i\neq s}}^{k}F_i(x)\}dF_s(x) + 2 \qquad (2.19)$$

$$(1\leqq m,s\leqq k,\ m\neq s).$$

This can also be rewritten as

$$\mathrm{Cov}(Rm,Rs) = (2-\int F_s dF_m)\sum_{i=1}^{k}\int F_i dF_s + (2-\int F_m dF_s)\sum_{i=1}^{k}\int F_i dF_m$$

$$-\sum_{i=1}^{k}\int F_i dF_m \int F_i dF_s - 2\sum_{i=1}^{k}(\int F_i F_m dF_s + \int F_i F_s dF_m)$$

$$+\int F_s dF_m \int F_m dF_s - 2\int F_m F_s dF_m - 2\int F_m F_s dF_s + 1 \qquad (2.20)$$

The following corollary is for the slippage configuration (2.1).

COROLLARY 2.3.

$$\mathrm{Cov}(Rm,Rs) = \begin{cases} -(k-t+1)/12-t\int F_0dF+2t\int F_0^2dF-t(\int F_0dF)^2, \\ \qquad\qquad \text{for } 1\leq m,\ s\leq k-t, \\ -(t+1)/12+3(k-t)\int F_0dF-(k-t)(\int F_0dF)^2-4(k-t)\int F_0FdF, \\ \qquad\qquad \text{for } k-t+1\leq m,s\leq k, \\ (-2t+1)\int F_0dF+(k-1)(\int F_0dF)^2+(-k+t+1)\int F_0^2dF+2(t-1)\int F_0FdF, \\ \qquad\qquad \text{for } 1\leq m\leq k-t,\ k-t+1\leq s\leq k. \end{cases} \tag{2.21}$$

3. APPLICATIONS TO THE NORMAL DISTRIBUTION

Let us apply the results of previous section to the normal distribution. First, we give some prerequisites.

Let the p.d.f. and c.d.f. of the standard normal distribution N(0,1) be

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(x)dx, \tag{3.1}$$

respectively. Also let the upper probability of the bivariate normal be

$$L(k,h;\rho) = \int_h^\infty \int_k^\infty \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\}\, dxdy \tag{3.2}$$

L-function has the following properties (see Johnson and Kotz [6]).

$$\begin{aligned} &L(h,k;\rho) = L(k,h;\rho), \\ &L(-h,k;\rho) = 1-\Phi(k) - L(h,k;-\rho). \\ &L(0,0;\rho) = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1}\rho. \end{aligned} \tag{3.3}$$

Now, by the transformation of variables, we have the following relations (David [1], Gupta [3]).

LEMMA 3.1.

For any real a_i and w_i $(i = 1, 2,\ldots,k-1)$, k-fold integral

$$I_k = \iint_D \cdots \int \prod_{i=1}^{k} \varphi(x_i)dx_i \tag{3.4}$$

where $D = \{x_i \leqq a_i x_k + w_i,\ i = 1, 2,\ldots,k-1;\ -\infty < x_k < \infty\}$ is expressed as

$$I_k = \iint_{D_0} \cdots \int \frac{|\Sigma|^{-1/2}}{(2\pi)^{(n-1)/2}} \exp(-\frac{1}{2}\underline{u}'\Sigma^{-1}\underline{u})d\underline{u} \tag{3.5}$$

where $\underline{u} = (u_1,\ldots,u_{k-1})'$, $D_0 = \{u_i \leqq w_i,\ i = 1, 2,\ldots,k-1\}$ and $\Sigma = \{\sigma_{ij}\}$, $\sigma_{ii} = 1 + a_i^2$ $(i = 1, 2,\ldots,k-1)$, $\sigma_{ij} = a_i a_j$ $(i, j = 1, 2,\ldots,k-1, i \neq j)$.

By letting k=2 and 3 in the above lemma, we have the following.

$$I_2 = \Phi(c_1) \tag{3.6}$$

$$I_3 = L(-c_1,-c_2;\rho) \tag{3.7}$$

where $c_i = w_i/\sqrt{1+a_i^2}$, $i = 1,2$ and $\rho = a_1 a_2/\sqrt{1+a_1^2}\ \sqrt{1+a_2^2}$. Also we have the following relations for any real $a_1, a_2 \geqq 0$.

$$\int_0^\infty \int_0^{a_1 x} \varphi(x)\varphi(y)dydx = \frac{1}{2\pi}\sin^{-1}\rho_1, \tag{3.8}$$

$$\int_0^\infty \int_0^{a_2 x} \int_0^{a_1 x} \varphi(x)\varphi(y)\varphi(z)\ dzdydx = \frac{1}{4\pi}\sin^{-1}\rho_2, \tag{3.9}$$

where $\rho_1 = a_1/\sqrt{1+a^2}$, $\rho_2 = a_1a_2/\sqrt{1+a_1^2}\sqrt{1+a_2^2}$.

3.1. Location Parameter Case

Now, let us consider the location parameter case of k normal populations.

Let F_i be normal distribution with lcoation parameter θ_i and common variance σ^2, i.e., $F_i(x) = \Phi((x-\theta_i)/\sigma)$, $i = 1, 2,\ldots,k$. From (3.6) and (3.7), we have the following integral relations.

$$\int F_j(x)dF_i(x) = \Phi(\delta_{ij}), \tag{3.10}$$

$$\int F_j(x)F_r(x)dF_i(x) = L(-\delta_{ij}, -\delta_{ir}; 1/2) \tag{3.11}$$

where $\delta_{ij} = (\theta_i-\theta_j)/\sqrt{2}\sigma$.

Thus, by applying these to theorems 2.1, 2.3 and 2.4, we have the following relations to the ranks from normal distributions.

THEOREM 3.1.

$$E(R_i) = 1/2 + \sum_{j=1}^{k} \Phi(\delta_{ij}),\ i = 1, 2,\ldots,k. \tag{3.12}$$

$$V(R_i) = 2\sum_{j=1}^{k} \Phi(\delta_{ij}) - 2\sum_{j=1}^{k} L(\delta_{ji},0;1/2) - \sum_{j=1}^{k} L(\delta_{ji},\delta_{ji};1/2)$$
$$+ \sum_{r,j=1}^{k} L(\delta_{ji},\delta_{ri};1/2) - \left\{\sum_{j=1}^{k}\Phi(\delta_{ij})\right\}^2 - 1/12,\ i = 1, 2,\ldots,k. \tag{3.13}$$

$$\mathrm{Cov}(Rm,Rs) = \{2-\Phi(\delta_{ms})\}\sum_{j=1}^{k}\Phi(\delta_{sj})+\{2-\Phi(\delta_{sm})\}\sum_{j=1}^{k}\Phi(\delta_{mj})$$

$$-\sum_{j=1}^{k}\Phi(\delta_{mj})\Phi(\delta_{sj})-2\sum_{j=1}^{k}\{L(\delta_{js},\delta_{ms};1/2)+L(\delta_{jm},\delta_{sm};1/2)\}$$

$$+\Phi(\delta_{ms})\Phi(\delta_{sm}) - 2L(0,\delta_{sm};1/2)-2L(0,\delta_{ms};1/2) + 1, \qquad (3.14)$$

$$(1\leqq m,\ s\leqq k,\ m\neq s)$$

where $\delta_{ij} = (\theta_i-\theta_j)/\sqrt{2}\sigma$.

The slippage configuration (2.1) is equivalent to the following slippage configuration of location parameters. For $\delta \geqq 0$, we have

$$\theta_1 =\cdots=\theta_{k-t} = \theta-\delta,\ \theta_{k-t+1}=\cdots=\theta_k = \theta. \qquad (3.15)$$

In this case, above relations reduce to the following.

COROLLARY 3.1.

Under the slippage configuration of parameters (3.15), we have

$$E(R_i) = \begin{cases} (k+t+1)/2-t\Phi(\delta^*), & \text{for } 1\leqq i\leqq k-t, \\ (t+1)/2+(k-t)\Phi(\delta^*), & \text{for } k-t+1\leqq i\leqq k. \end{cases} \qquad (3.16)$$

$$V(R_i) = \begin{cases} \{(k-t)^2-1\}/12+t(k+t-s)\Phi(\delta^*)-2t(t-1)L(0,-\delta^*;1/2) \\ \quad -t(k-t-1)L(-\delta^*,-\delta^*;1/2)-t^2\{\Phi(\delta^*)\}^2, \quad \text{for } 1\leq i\leq k-t. \\ (t^2-1)/12-(k-t)(t-2)\Phi(\delta^*)+2(k-t)(t-1)L(0,-\delta^*;1/2) \\ \quad +(k-t)(k-t-1)L(-\delta^*,-\delta^*;1/2)-(k-t)^2\{\Phi(\delta^*)\}^2, \text{ for } k-t+1\leq i\leq k. \end{cases} \tag{3.17}$$

$$Cov(Rm,Rs) = \begin{cases} -(k-t+1)/12-t\Phi(\delta^*)+2tL(-\delta^*,-\delta^*;1/2)-t\{\Phi(\delta^*)\}^2, \\ \qquad \text{for } 1\leq m,\ s\leq k-t, \\ -(t+1)/12+3(k-t)\Phi(\delta^*)-(k-t)\{\Phi(\delta^*)\}^2-4(k-t)L(-\delta^*,0;1/2), \\ \qquad \text{for } k-t+1\leq m,\ s\leq k, \\ (-2t+1)\Phi(\delta^*)+(k-1)\{\Phi(\delta^*)\}^2+(-k+t+1)L(-\delta^*,-\delta^*;1/2)+2(t-1)L(-\delta^*,0;1/2), \\ \qquad \text{for } 1\leq m\leq k-t,\ k-t+1\leq s\leq k. \end{cases} \tag{3.18}$$

where $\delta^* = \delta/\sqrt{2}\sigma$.

3.2 Scale Parameter Case

Suppose $F_i(x)$ follows normal distribution with scale parameter θ_i and common location parameter (without loss of generality) zero, i.e., $F_i(x) = \Phi(x/\theta_i)$.

In case of scale parameter, ranking process should be carried out according to the absolute values of each observation $(X_1, X_2,\ldots,X_k)$. So, rank vector $\underline{R}$ defined in section 1 is given for $(|X_1|, |X_2|,\ldots,|X_k|)$.

By applying the relations (3.8) and (3.9), we have the following integral evaluations.

$$\int H_j(z)dH_i(z) = (2/\pi)\sin^{-1}\rho_{ij} \tag{3.19}$$

$$\int H_r(z)H_j(z)dH_i(z) = (2/\pi)\sin^{-1}\rho_{ij}\rho_{ir} \tag{3.20}$$

where $H_i(x)$ denotes the c.d.f. of the random variable $Z_i = |X_i|$, X_i follows normal $N(0,\theta_i^2)$, $i = 1, 2,\ldots,k$ and $\rho_{st} = (\theta_s/\theta_t)/\sqrt{1+(\theta_s/\theta_t)^2}$.

Thus by using the relations (3.19), (3.20), Theorems 2.1, 2.3 and 2.4, we have the following expressions of the moments of ranks for normal distribution.

THEOREM 3.2.

$$E(R_i) = 1/2 + (2/\pi)\sum_{j=1}^{k} \sin^{-1}\rho_{ij}, \quad i = 1, 2,\ldots,k \tag{3.21}$$

$$\begin{aligned} V(R_i) = {} & (4/\pi)\sum_{i=1}^{k}\sin^{-1}\rho_{ij}-(4/\pi)\sum_{j=1}^{k}\sin^{-1}\rho_{ij}/\sqrt{2} - (2/\pi)\sum_{j=1}^{k}\sin^{-1}\rho_{ij}^2 \\ & + (2/\pi)\sum_{r,j=1}^{k}\sin^{-1}\rho_{ij}\rho_{ir} - \{(2/\pi)\sum_{j=1}^{k}\sin^{-1}\rho_{ij}\}^2 - 1/12,\ i = 1,2,\ldots,k. \end{aligned} \tag{3.22}$$

$$\begin{aligned} Cov(Rm,Rs) = {} & (4/\pi)\{1-(1/\pi)\sin^{-1}\rho_{ms}\}\sum_{j=1}^{k}\sin^{-1}\rho_{sj}+(4/\pi)\{1-(1/\pi)\sin^{-1}\rho_{sm}\} \\ & \sum_{j=1}^{k}\sin^{-1}\rho_{mj} \\ & -(4/\pi^2)\sum_{j=1}^{k}\sin^{-1}\rho_{mj}\sin^{-1}\rho_{sj}-(4/\pi)\sum_{j=1}^{k}(\sin^{-1}\rho_{sj}\rho_{sm}+\sin^{-1}\rho_{mj}\rho_{ms}) \end{aligned}$$

$$+(4/\pi^2)\sin^{-1}\rho_{ms}\sin^{-1}\rho_{sm}-(4/\pi)\sin^{-1}\rho_{ms}/\sqrt{2} - (4/\pi)\sin^{-1}\rho_{sm}/\sqrt{2} + 1,$$

$$(1\leqq m,\ s\leqq k,\ m\neq s). \tag{3.23}$$

where $\rho_{ij} = (\theta_i/\theta_j)/\sqrt{1+(\theta_i/\theta_j)^2}$.

As a special case, let us consider the slippage configuration of distributions. Under the assumption of normality and parametric proposition given above, the configuration (2.1) is equivalent to

$$\theta_1 =\cdots=\theta_{k-t} = \theta/\rho,\ \theta_{k-t+1}=\cdots=\theta_k = \theta,\ \rho \geqq 1 \tag{3.24}$$

In this case, moments given in Theorem 3.2 reduces to the following respective forms.

COROLLARY 3.2.

Under the slippage configuration of parameter (3.24), we have

$$E(R_i) = \begin{cases} (k+t+1)/2 - (2t/\pi)\sin^{-1}\rho^*, & \text{for } 1\leqq i\leqq k-t, \\ (t+1)/2 + \{2(k-t)/\pi\}\sin^{-1}\rho^*, & \text{for } k-t+1\leqq i\leqq k. \end{cases} \tag{3.25}$$

$$V(R_i) = \begin{cases} \{(k-t)^2-1\}/12 + \{2t(k+t-2)/\pi\}\sin^{-1}\rho^* - \{4t(t-1)/\pi\}\sin^{-1}\rho^*/\sqrt{2} \\ \qquad -\{2t(k-t-1)/\pi\}\sin^{-1}\rho^{*2} - t^2\{(2/\pi)\sin^{-1}\rho^*\}^2, \\ \qquad\qquad \text{for } 1\leqq i\leqq k-t \\ (t^2-1)/12-\{2(k-t)(t-2)/\pi\}\sin^{-1}\rho^*+\{4(k-4)(t-1)/\pi\}\sin^{-1}\rho^*/\sqrt{2} \\ \qquad +\{2(k-t)(k-t-1)/\pi\}\sin^{-1}\rho^{*2}-(k-t)^2\{(2/\pi)\sin^{-1}\rho^*\}^2, \\ \qquad\qquad \text{for } k-t+1\leqq i\leqq k. \end{cases} \tag{3.26}$$

$$\mathrm{Cov}(Rm,Rs) = \begin{cases} -(k-t+1)/12-(2t/\pi)\sin^{-1}\rho^*+(4t/\pi)\sin^{-1}\rho^{*2}-t\{(2/\pi)\sin^{-1}\rho^*\}^2 \\ \qquad\qquad\qquad \text{for } 1\leqq m,\ s\leqq k-t \\ -(t+1)/12+\{6(k-t)/\pi\}\sin^{-1}\rho^*-(k-t)\{(2/\pi)\sin^{-1}\rho^*\}^2 \\ \qquad -\{8(k-t)/\pi\}\sin^{-1}\rho^*/\sqrt{2}, \quad \text{for } k-t+1\leqq m,\ s\leqq k, \\ \{2(-2t+1)/\pi\}\sin^{-1}\rho^* + (k-1)\{(2/\pi)\sin^{-1}\rho^*\}^2 +\{2(k-t+1)/\pi\}\sin^{-1}\rho^{*2} \\ \qquad + \{4(t-1)/\pi\}\sin^{-1}\rho^*/\sqrt{2}, \quad \text{for } 1\leqq m\leqq k-t,\ k-t+1\leqq s\leqq k. \end{cases} \tag{3.27}$$

where $\rho^* = \rho/\sqrt{1+\rho^2}$.

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