ESTIMATING NORMAL TAIL PROBABILITIES*

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Abstract

The estimation problem of normal tail probabilities is considered. The form of generalized Bayes estimators is derived and the asymptotic behavior of the mean square errors is studied. This study shows that the best unbiased estimator, a sample for which is given, is superior to the maximum likelihood estimator or to a class of generalized Bayes procedures for large parametric values, but can be significantly improved for moderate values of the parameter.

Key words and phrases: Normal tail probability, mean square error, best unbiased estimator, maximum likelihood estimator, generalized Bayes estimator, asymptotic behavior of the risk.

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1. Introduction

Let $x_1,\ldots,x_n$, $n \geq 2$, be a normal random sample with both parameters μ and σ unknown. In many practical situations arising in reliability theory, quality control, insurance problems, etc., it is important to estimate tail probability $\theta = P(x_1 < a) = \Phi((a-\mu)/\sigma)$ where a is a given constant. By considering a shifted sample $x_i - a$, $i = 1,\ldots,n$ we may and will assume that $a = 0$. Two statistical procedures are traditionally used to estimate θ: the uniformly minimum variance unbiased (UMVU) estimator δ_U and the maximum likelihood (ML) estimator $\hat{\delta}$. The former estimator has been given by Kolmogorov (1950), Bowker and Goode (1952) and Lieberman and Resnikoff (1955) (see also Vajda (1955), Barton (1961), Basu (1964)). Applying Rao-Blackwell theorem this estimator can be derived in the following form:

Let $X = \sum_1^n x_j/n$ and $S^2 = \sum_1^n (x_j-X)^2$. Then $\delta_U(X,S) = P(x_1<0/X,S)$

$= P(X-x_1)/S>X/S)$. For $n \geq 3$ the latter probability has the form

$\delta_U(X,S) = I_w(n/2-1, n/2-1)$

where $I_w(p,q)$ denotes the incomplete beta function

$$I_w(p,q) = \int_0^w t^{p-1}(1-t)^{q-1}dt/B(p,q), \qquad (1.1)$$

and for $0 \leq w \leq 1$

$w = 1/2 - n^{\frac{1}{2}} X/[2(n-1)^{\frac{1}{2}} S]$.

Also $\delta_U(X,S) = 0$ if $w < 0$, and $\delta_0(X,S) = 1$ if $w > 1$. Thus δ_U is not range-preserving since it takes extremal values 0 and 1 with positive probability. However exactly this fact makes the UMVU estimator very efficient for large parametric values.

An alternative form of this estimator in terms of t-distribution function has been obtained by Folks, Pierce and Stewart (1965).

Guenther (1971) discussed the relationship between these two forms and their relative advantages for numerical evaluation of the UMVU estimator.

From the mathematical point of view the equivalence of these forms is based on the following easily verified identity. For any positive p and $0 < u < 1$

$$I_{(1-(1-u)^{\frac{1}{2}})/2}(p,p) = I_u(p,\tfrac{1}{2})/2 = \int_{(u^{-1}-1)^{\frac{1}{2}}}^{\infty} (1+t^2)^{-p-\frac{1}{2}}\,dt/B(p,\tfrac{1}{2}) \,. \qquad (1.2)$$

For instance,

$$\delta_U(X,S) = 1/2 - \int_0^v (1+t^2)^{-(n-1)/2}\,dt/B(n/2-1,1/2).$$

$$v = n^{\frac{1}{2}} X/[(n-1)S^2 - nX^2]^{\frac{1}{2}}\,, \quad |X|/S < [(n-1)/n]^{\frac{1}{2}}\,.$$

Apparently it was unnoticed that because of (1.2) δ_U can be expressed in terms of elementary functions.

Indeed if $n = 2m + 2$ with a positive integer m

$$\delta_U(X,S) = 1/2 - \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} (2k+1)^{-1}\, u^{2k+1}/B(m,1/2). \qquad (1.3)$$

with $u = n^{\frac{1}{2}} X/[(n-1)^{\frac{1}{2}} S]$, and if $n = 2m+1$

$$\delta_U(X,S) = 1/2 - \operatorname{arctg} v/\pi$$

$$-(2m-1)^{-1} \sum_{k=1}^{m-1} \frac{(2m-1)\cdots(2m-2k+1)}{2^{2k}(m-1)\cdots(m-k)} \; \frac{v}{(1+v^2)^{n-k}} \,/B(m-1/2,1/2) \qquad (1.4)$$

Thus the UMVU estimator admits simple representations (1.3) and (1.4) which facilitate its numerical evaluation for small sample sizes and also the evaluation of its mean square error.

For instance, if n = 5

$\delta_U(X,S) = 1/2 - [v/(1+v^2) + \text{arctg} v]/\pi$

$v = \sqrt{5}\, X/[4S^2 - 5X^2]^{\frac{1}{2}}, \quad |X|/S < 2/\sqrt{5}.$

In the numerical example considered by Folks, Pierce and Stewart (1965) and Guenther (1971), X = -4, $S^2 = 40$, and the value of $\delta_U = .90915$ can easily be obtained from the formula above without using any tables and interpolation.

The ML estimator $\hat{\delta}$ is obtained from the formula of the parametric function by replacing the parameters by their ML estimators,

$\hat{\delta}(X,S) = \Phi(-n^{\frac{1}{2}} X/S).$

The mean square errors of these two estimators have been tabulated by Zacks and Milton (1971) and Brown and Rutemiller (1973). The latter authors show that the UMVU estimator is better than the ML estimator for very large or very small values of the ratio $|\mu|/\delta$. (Zacks and Milton erroneously claim that the ML estimator is more efficient for small values of θ). In this paper we give theoretical explanation for these findings by the study of asymptotical behavior of the risk functions for small and large parametrical values. To provide a larger class of alternative estimators we also develop in Section 2 the form of generalized Bayes rules for a family of prior densities. These rules have form similar to (1.3) and (1.4) and some of them substantially improve upon UMVU estimator for small values of the parameter.

2. Bayes Estimators

Let $\lambda(\mu,\sigma) = \lambda(\mu/\sigma)\sigma^{-\alpha}$ be a generalized prior density for parameters μ and σ. The corresponding Bayes estimator $\delta_B(X,S)$ for quadratic loss has the form

$$\delta_B(X,S) = \frac{\int_{-\infty}^{\infty}\int_0^{\infty} \Phi(-\mu/\sigma)\ \exp\{-[n(X-\mu)^2 + S^2]/(2\sigma^2)\}\ \sigma^{-n-\alpha}\lambda(\mu/\sigma)d\mu d\sigma}{\int_{-\infty}^{\infty}\int_0^{\infty} \exp\{-[n(X-\mu)^2+S^2]/(2\sigma^2)\}\sigma^{-n-\alpha}\lambda(\mu/\sigma)d\mu d\sigma}$$

$$= \frac{\int_{-\infty}^{\infty}\int_0^{\infty} \Phi(-\eta)\ \exp\{-[n(yz-\eta)^2 + y^2]/2\}\ \lambda(\eta)y^{n+\alpha-2}d\eta dy}{\int_{-\infty}^{\infty}\int_0^{\infty} \exp\{-[n(yz-\eta)^2 + y^2]/2\}\ \lambda(\eta)y^{n+\alpha-2}\ d\eta dy}$$

where $z = X/S$.

If for some τ

$$\lambda(\eta) = \exp\{-\eta^2/(2\tau^2)\} \tag{2.1}$$

then with $\nu^2 = n^{-1}$

$$\int_{-\infty}^{\infty} \exp\{-n(yz-\eta)^2/2\}\ \lambda(\eta)d\eta$$

$$= (2\pi)^{\frac{1}{2}}\tau\nu\ (\tau^2+\nu^2)^{-\frac{1}{2}}\exp\{-\ y^2z^2/(2(\tau^2+\nu^2))\}\ ,$$

and

$$\int_{-\infty}^{\infty} \Phi(-\eta)\ \exp\{-n(yz-\eta)^2/2\}\ \lambda(\eta)d\eta$$

$$= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\int_0^{\infty} \exp\{-\ [(t+\eta)^2 + n(yz-\eta)^2 + \eta^2/\tau^2]/2\}\ d\eta dt$$

$$= (1+\tau^{-2}+\nu^{-2})^{-\frac{1}{2}}$$

$$\int_0^{\infty} \exp\{-\ [t^2 + (yz/\nu)^2 - (t-yz\nu^{-2})^2/(1+\tau^{-2}+\nu^{-2})]/2\}dt\ .$$

Therefore for $n = \alpha > 1$

$$\delta_B(X,S) = \delta_B(z)$$

$$= (\tau^2+\nu^2)^{\frac{1}{2}}\ [2\pi(\nu^2\tau^2+\tau^2+\nu^2)]^{-\frac{1}{2}}$$

$$\frac{\int_0^\infty\int_0^\infty \exp\{-[y^2(z^2+z^2\tau^2+\nu^2\tau^2+\tau^2+\nu^2)+2tyz\tau^2+t^2(\tau^2+\nu^2)]/2(\nu^2\tau^2+\tau^2+\nu^2)\}\, y^{n+\alpha-2}dydt}{\int_0^\infty \exp\{-y^2[1+z^2/(\tau^2+\nu^2)]/2\}y^{n+\alpha-2}dy}$$

$$= \int_h^\infty (1+t^2)^{-(n+\alpha)/2}dt/B((n+\alpha-1)/2,\ 1/2) \tag{2.2}$$

where

$$h = z\tau^2[z^2+\tau^2+\nu^2)(\nu^2\tau^2+\tau^2+\nu^2)]^{-\frac{1}{2}}.$$

Because of (1.2)

$$\delta_B(z) = I_{(1+h)^{-1}}((n+\alpha-1)/2,\ 1/2)/2$$

$$= I_{(1-(h/(1+h))^{\frac{1}{2}})/2}((n+\alpha-1)/2,\ (n+\alpha-1)/2)$$

so that δ_B can be found from the tables of the incomplete beta function or from the tables of t-distribution.

If $n+\alpha$ is a positive integer, then the integral in (2.2) can be expressed in terms of elementary functions as in (1.3) and (1.4).

If $\tau\to\infty$, which corresponds to the uniform, non-informative prior for μ, one obtains

$$h \to h_0 = z(1+\nu^2)^{-\frac{1}{2}}.$$

Notice that h is an increasing function of τ, and

$$|\delta_B(X,S) - 1/2|$$

$$= \int_0^{|h|} (1+t^2)^{-(n+\alpha)/2}dt\ /\ B((n+\alpha-1)/2,\ 1/2).$$

Thus the risk of δ_B at $\mu = 0$, $\sigma = 1$, i.e. $\theta = 1/2$, is an increasing function of τ. This formula also shows that for a fixed τ this risk is an increasing function of α.

For the generalized Bayes estimators δ_B with $\tau < \infty$ one has

$$\delta_B(z) \to \delta_B(\infty)$$

$$= \int_{\tau^2(\nu^2\tau^2+\tau^2+\nu^2)^{-\frac{1}{2}}} (1+t^2)^{-(n+\alpha)/2} dt/B((n+\alpha-1)/2,\ 1/2),$$

so that as $\eta \to \infty$

$$E_\eta(\delta_B - \theta)^2 \to \delta_B^2(\infty),$$

and the mean square error of estimator (2.2) does not vanish at infinity if $\tau < \infty$.

Therefore we will assume $\tau = \infty$ and in the next section we study the asymptotic behavior of these estimators as well as this of δ_U and $\hat{\delta}$.

3. Asymptotical formulae for risk functions

All estimators δ considered in previous sections are functions of $z = X/S$ such that

$$\delta(-z) = 1 - \delta(z).$$

The quadratic risk of these estimators depends only on $\eta = \mu/\sigma$ and is symmetric:

$$R(\eta,\delta) = E_\eta(\delta(z) - \theta)^2 = R(-\eta,\delta). \tag{3.1}$$

If

$$\delta(z) = 1/2 + \psi(n^{\frac{1}{2}} z)$$

then an easy calculation shows that

$$R(0,\delta) = \int_{-\infty}^{\infty} \psi^2(z)(1+z^2)^{-n/2} dz/B((n-1)/2,\ 1/2).$$

An application of this formula for the estimators δ_U, $\hat{\delta}$ and δ_B with $\tau = \infty$ gives the following asymptotical formulas:

$$R(0,\delta_U) = [2\pi(n-3)]^{-1}[1 - 5/(2n) + o(n^{-1})]$$

$$R(0,\hat{\delta}) = [2\pi(n-3)]^{-1}[1 - 1/n + o(n^{-1})]$$

$$R(0,\delta_B) = [2\pi(n-3)]^{-1}[1 + (\alpha-1/2)/n + o(n^{-1})].$$

It follows from these formulas that for sufficiently large n ($n \geq 30$)

$R(0,\hat{\delta})/R(0,\delta_U) = 1 + 3/(2\eta) + O(n^{-1})$

which agrees with the results of Brown and Rutemiller (1973).

Also for large n and $\theta = 1/2$ δ_B is better than $\hat{\delta}$ if $\alpha < 5/2$, and if $\alpha < 1$, δ_B is better than δ_U. If $\alpha \leq -1$, then

$|\delta_B - 1/2| < |\delta_U - 1/2|$

and $R(0,\delta_B) < R(0,\delta_U)$ for all sample sizes.

Asymptotical study of the risk functions (of Appendix) shows that the efficiencies of these estimators are reversed if $|\eta| \to \infty$,

$R(\eta,\delta_U) < R(\eta,\hat{\delta}) < R(\eta,\delta_B)$.

Thus if small values of μ/σ are anticipated in practical situation negative α, $\alpha > -n+1$, can be chosen for the prior density $d\mu d\sigma/\sigma^{\alpha}$. For large values of μ/σ, δ_U is preferable.

4. Numerical Example

We have evaluated for n = 6 mean square errors and expectations of estimators δ_0, δ_1 and δ_2 given by (2.2) with $\tau = \infty$ which correspond to values $\alpha = -3, -1$ and 1 respectively. The corresponding characteristics for estimators δ_U and $\hat{\delta}$ were taken from the Table of Zacks and Milton (1971).

Notice that δ_2 is the generalized Bayes estimator which corresponds to uniform "non-informative" prior density $d\mu d\sigma/\sigma$. The results of this calculation are given in Tables 1 and 2.

It follows from Table 1 that δ_2 is slightly better than δ_U for all values of θ, $\theta > .05$. As is to be expected, estimators δ_0 and δ_1 substantially improve upon δ_u for moderate values of μ/σ, but δ_U is the best estimator for large values of this parameter.

Table 2 shows that δ_2 is the least biased estimator, and δ_0 is the most biased one.

Appendix

In this section we investigate the asymptotic behavior of $R(\eta,\delta)$ as $|\eta| \to \infty$. Because of (3.1) we can assume $\eta \to +\infty$.

We will use the notation $\xi = n^{\frac{1}{2}}\eta$ and

$C = 1/[(2\pi)^{\frac{1}{2}} 2^{(n-3)/2}\Gamma((n-1)/2)]$.

1. Let us start with maximum likelihood estimator $\hat{\delta}$. One has

$$E_\eta \Phi(-n^{\frac{1}{2}} z)$$

$$= C\int_{-\infty}^{\infty} \int_0^{\infty} \Phi(-x/S) \exp\{-[x - \xi)^2 + S^2]/2\} S^{n-2} dS dx$$

$$= C(2\pi)^{-\frac{1}{2}}$$

$$\times \int_0^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \exp\{-[(u + x/S)^2 + (x - \xi)^2 + S^2]/2\} S^{n-2} dS dx du$$

$$= C \int_0^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \exp\{-[S^2 + (\xi + uS)^2/(1 + S^2)]/2\} S^{n-1}(1 + S^2)^{-\frac{1}{2}} dS du.$$

Because of Laplace's asymptotical formula for integrals (see for example Erdelyi (1956) § 2.4) as $\xi \to \infty$

$$\int_0^{\infty} \exp\{-(2\xi uS + u^2S^2)/(2(1 + S^2))\} du$$

$$\sim (1 + S^2) / (\xi S)$$

and

$$E_\eta \Phi(-n^{\frac{1}{2}} z)$$

$$\sim C \int_0^{\infty} \exp\{-[S^2 + \xi^2/(1 + S^2)]/2\} S^{n-2}(1 + S^2)^{\frac{1}{2}} dS/\xi$$

$$= C \int_{\xi^{-1}}^{\infty} \exp\{-\xi (u + u^{-1})/2 + 1/2\}(\xi u - 1)^{(n-3)/2} u^{\frac{1}{2}} du\, \xi^{\frac{1}{2}} .$$

Since the minimum of the function $u + u^{-1}$ occurs when $u = 1$, the Laplace's method shows that

$$E_\eta \Phi(-n^{\frac{1}{2}} z) \sim C(2\pi)^{\frac{1}{2}} \xi^{(n-3)/2} \exp\{-\xi + 1/2\} .$$

Analogously

$$E_\eta \Phi^2(-n^{\frac{1}{2}} z) = C(2\pi)^{-1}$$

$$\int_0^\infty \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \exp\{-[(u_1 + x/S)^2 + (u_2 + x/S)^2 + (x - \xi)^2 + S^2]/2\} S^{n-2}\, dS\,dx\,du_1\,du_2$$

$$= C(2\pi)^{-\frac{1}{2}}$$

$$\int_0^\infty \int_0^\infty \int_0^\infty \exp\{-[(u_1^2 + u_2^2 + \xi^2) - ((u_1 + u_2)S - \xi)^2/(1 + 2/S^2) + S^2]/2\}$$

$$\times S^{n-2} (2 + S^2)^{-\frac{1}{2}} dS\,du_1\,du_2$$

$$\sim C(2\pi)^{-\frac{1}{2}}$$

$$\int_0^\infty \exp\{-[S^2 + 2\xi^2/(2 + S^2)]/2\} S^{n-4}(S^2 + 2)^{3/2} ds/\xi^2$$

$$\sim C\xi^{-(n-4)/2} \exp\{-2\xi + 1\} 2^{(n-3)/2} .$$

As $\eta \to \infty$

$$\theta \sim \exp\{-\eta^2/2\}(2\tau)^{-\frac{1}{2}} \eta^{-1}$$

$$= n^{\frac{1}{2}} \exp\{-\xi^2/(2n)\}(2\pi)^{-\frac{1}{2}} \xi^{-1} ,$$

so that

$$R(\eta,\hat{\delta}) \sim E_\eta \hat{\delta}^2(z)$$

$$\sim C\xi^{-(n-4)/2} \exp\{-2\xi - 1\} 2^{(n-3)/2}.$$

2. For UMVU estimator δ_U

$$R(\eta,\delta_U) = E_\eta \delta_U^2 - \theta^2$$

and

$$E_\eta \delta_U^2 = C \exp\{-\xi^2/2\}$$

$$\int_{-1}^{1} \tilde{\delta}^2(t(n(1 - t^2))^{-\frac{1}{2}})\,(1 - t^2)^{(n-3)/2}\,dt \int_0^\infty \exp\{\xi t S - S^2/2\} S^{n-1} dS.$$

Let

$$\tilde{\delta}(t) = \delta_U(t(n(1 - t^2))^{\frac{1}{2}})$$

$$(\text{i.e. } t = n^{\frac{1}{2}} z/(1 + n z^2)^{\frac{1}{2}},\ |t| < 1).$$

It follows from (1.1) that for $n \geq 3$

$$\tilde{\delta}(t) \sim n^{(n-2)/2}(n - 2)^{-1}(1 - nt^2/(n - 1))^{(n-2)/2}/B(1/2,(n-2)/2)$$

as $t \uparrow (1 - n^{-1})^{\frac{1}{2}}$, and

$$\tilde{\delta}(t) = 0 \text{ if } t > (1 - n^{-1})^{\frac{1}{2}}.$$

Thus

$$E_\eta \delta_U^2 \sim C(2\pi)^{\frac{1}{2}}(n - 2)^{-2} n^{n-2} \exp\{-\xi^2/(2n)\}/B^2(1/2,(n-2)/2)$$

$$\int_0^{(1-n^{-1})^{\frac{1}{2}}} (1 - nt^2/(n - 1))^{n-2}(1 - t^2)^{(n-3)/2} \exp\{-\xi^2(t^2 - 1 + n^{-1})/2\}dt$$

$$\sim \Gamma(n/2)(2/n)^{\frac{1}{2}}(n - 1)^{-n/2}/[\Gamma((n - 1)/2)B^2(1/2,(n-2)/2)] \exp\{-\xi^2/(2n)\}\xi^{-n}.$$

For $n = 2$ it is easy to see that δ_U has the form

$$\delta_U(x_1,x_2) = [I_{(-\infty,0)}(x_1) + I_{(-\infty,0)}(x_2)]/2,$$

so that in this case

$$E_\eta(\delta_U - \theta)^2 = \theta(1 - \theta)/2$$

$$\sim \exp\{-\xi^2/(2n)\}/(\xi(2\pi/n)^{\frac{1}{2}}).$$

3. For the generalized Bayes estimator δ_B with $\tau = \infty$

$$\delta_B(z) \sim (1+nz^2)^{-(n+\alpha-1)/2}(n+1)^{(n+\alpha-1)/2}(n+\alpha-1)^{-1}/B(1/2,(n+\alpha-1)/2)$$

as $z \to +\infty$.

To obtain asymptotical formula for the risk functions of these estimators observe that if

$$g(z) \sim \bar{g}(1 + nz^2)^{-p} + o(z^{-2p}) , \quad z \to \infty,$$

then

$$E_\eta g(z) \sim \bar{g}\, C$$

$$\int_0^\infty \int_{-\infty}^\infty \exp\{-[(x-\xi)^2 + S^2]/2\} S^{n+2p-2}(S^2 + x^2)^{-p}\, dS\, dx$$

$$\sim \bar{g}\, C(2\pi)^{\frac{1}{2}} \int_0^\infty \exp\{-S^2/2\} S^{n+2p-2}(S^2 + \xi^2)^{-p} dS$$

$$\sim \bar{g}\Gamma((n + 2p - 1)/2)2^p/[\xi^{2p}\Gamma((n-1)/2)] .$$

Therefore

$$E_\eta\delta_B \sim (n+1)^{(n+\alpha-1)/2}(n+\alpha-1)^{-1}\Gamma(n+(\alpha-2)/2)\, 2^{(n+\alpha-1)/2}/[\xi^{n+\alpha-1}\Gamma((n-1)/2)B(1/2,(n+\alpha-1)/2]$$

and

$$E_\eta\delta_B^2 \sim (n+1)^{(n+\alpha-1)}(n+\alpha-1)^{-2}\,\Gamma((3n+2\alpha-3)/2)\, 2^{n+\alpha-1}/[\xi^{2(n+\alpha-1)}\Gamma((n-1)/2)B^2(1/2,(n+\alpha-1)/2)].$$

Notice also that for $n \geq 2$

$$e^{-\eta^2}(1 + 2\eta^2)/n < R(\eta,\delta_U) \leq \theta(1-\theta)/n.$$

The first of these inequalities follows from Cramèr-Rao inequality, the second is a consequence of the comparison of the risks of UMVU estimator and unbiased estimator $\sum_1^n I_{(-\infty,0)}(x_j)/n$.

The formulas developed in this section incidentally show that for large values of η the bias of estimators $\hat{\delta}$ and δ_B is positive, which also contradicts to a statement of Zacks and Milton (1971) p. 591.

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Table 1 Mean square errors of generalized Bayes estimators δ_0, δ_1 and δ_2 and estimators δ_U and $\hat{\delta}$ for n = 6

θ	$R(\theta,\delta_0)$	$R(\theta,\delta_1)$	$R(\theta,\delta_2)$	$R(\theta,\delta_U)$	$R(\theta,\hat{\delta})$
.5	.00971	.01982	.02852	.02938	.03549
.4	.01138	.01932	.02680	.02831	.03320
.3	.01621	.01795	.02205	.02497	.02680
.2	.02333	.01595	.01542	.01899	.01759
.1	.02998	.01301	.00838	.01013	.00768
.05	.03007	.01026	.00495	.00481	.00334
.025	.02698	.00777	.00309	.00215	.00152

Table 2 Expected values of generalized Bayes estimators δ_0, δ_1 and δ_2 and maximum likelihood estimator $\hat{\delta}$ for n = 6

θ	$E\ \delta_0$	$E\ \delta_1$	$E\ \delta_2$	$E\ \hat{\delta}$
.4	.44266	.41783	.40137	.3900
.3	.38420	.33603	.30507	.2834
.2	.32259	.25416	.21278	.1833
.1	.25197	.16881	.12401	.0918
.05	.20615	.11997	.07842	.0490
.025	.17419	.08959	.05279	.0275