# ON AN EXTREMAL PROBLEM OF FÉJER

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The aim of this paper is to give new proofs to some theorems in Karlin and Studden's book (1966) and Balázs's paper (1979). We also obtained some new results of a similar nature.

The general problem is as follows. Let $-\infty \leq a < b \leq \infty$ and let $\mathcal{P}_{n+1}$ denote the class of n+1 tuples $(x_1,\ldots,x_{n+1})$ with $a \leq x_1 < \ldots < x_{n+1} \leq b$ ($x_1$ and $x_{n+1}$ are finite). Let $w(x)$ be a nonnegative function on $[a,b]$. Define

$$r_i(x) = \frac{\ell_i(x)}{w^{\frac{1}{2}}(x_i)} \qquad i = 1, 2, \ldots, n+1$$

where

$$\ell_i(x) = \frac{L_{n+1}(x)}{L'_{n+1}(x_i)\,(x-x_i)} \qquad \text{and } L_{n+1}(x) = \prod_{i=1}^{n+1} (x-x_i)$$

are the Lagrange interpolating polynomials. The extremal problem is to determine the value

(1) $$N = \inf_{\mathcal{P}_{n+1}} \sup_{a \leq x \leq b} w(x) \;\{r_1^2(x) + \ldots + r_{n+1}^2(x)\}$$

$$= \inf_{\mathcal{P}_{n+1}} \tilde{N}(x_1,\ldots, x_{n+1})$$

and the set of points $\{x_i\}_{i=1}^{n+1}$ which minimizes $\tilde{N}$ for a fixed $w(x)$.

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Fejer (1932) showed that for the case $w(x) = 1$ and $[a,b] = [-1,1]$, the quantity $N$ is minimized if $\{x_i\}_{i=1}^{n+1}$ are the roots of the equation $(1-x^2)P_n'(x) = 0$ where $P_n'(x)$ is the derivative of the n-th Legendre polynomial and that the minimum value $N = 1$. The cases that will be treated in this paper are listed in the following:

A. $[a,b] = [-1,1]$.

(i) $w(x) = 1$

(ii) $w(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}$ $(\alpha > -1,\ \beta > -1)$

(iii) $w(x) = (1-x)^{\alpha+1}$ $(\alpha > -1)$

(iv) $w(x) = (1+x)^{\beta+1}$ $(\beta > -1)$

(v) $w(x) = (1-x^2)^{\alpha+1} \quad |x|^{\gamma}$ $(\alpha > -1, \gamma > 0$, n is odd)

(vi) $w(x) = |x|^{\gamma}$ $(\gamma > 0$, n is odd)

B. $[a,b] = [0,\infty)$

(i) $w(x) = e^{-x}$

(ii) $w(x) = x^{\alpha+1}\ e^{-x}$ $(\alpha > -1)$

C. $[a,b] = (-\infty,\infty)$

(i) $w(x) = e^{-x^2}$

(ii) $w(x) = |x|^{\gamma}\ e^{-x^2}$ $(\gamma > 0$, n is odd)

Karlin and Studden (1966) gave proofs for cases A, (i), (ii); B, (i), (ii), and C, (i). Balázs (1979) considered in addition the cases A, (iii), (iv) and C, (ii). The cases A (v), (vi) are new and were suggested by Askey (1). The solutions to the above cases are listed in Theorem 1 below. The following notations, which are mainly from Szegö (1975), will be used:

$P_n'(x)$ denotes the derivative of the n-th Legendre polynomial

$P_n^{(\alpha,\beta)}(x)$ denotes the n-th Jacobi polynomial

$R_n^{(\alpha,\gamma)}(x)$ denotes the n-th orthogonal polynomial with respect to $(1-x^2)^\alpha\ |x|^\gamma$.

$L_n(x)$ denotes the n-th Laguerre polynomial

$L_n^\alpha(x)$ denotes the n-th Generalized Laguerre polynomial

$H_n(x)$ denotes the n-th Hermite polynomial

$H_n^\gamma(x)$ denotes the n-th Sonin-Markov polynomial (generalized Hermite polynomial) orthogonal with respect of $|x|^\gamma e^{-x^2}$.

Theorem 1. The solutions for the cases A, B, and C are the zeros of the following polynomials:

A. [a,b] = [-1,1].

(i) $(1-x^2)P_n'(x)$

(ii) $P_{n+1}^{(\alpha,\beta)}(x)$

(iii) $(1+x)P_n^{(\alpha,1)}(x)$

(iv) $(1-x)P_n^{(1,\beta)}(x)$

(v) $R_{n+1}^{(\alpha,\gamma)}(x)$ (n is odd)

(vi) $(1-x^2)R_{n-1}^{(1,\gamma)}(x)$ (n is odd)

B. [a,b] = [0,∞)

(i) $xL_n(x)$

(ii) $L_{n+1}^\alpha(x)$

C. $[a,b] = (-\infty,\infty)$

(i) $H_{n+1}(x)$

(ii) $H^{\gamma}_{n+1}(x)$ (n is odd)

The value of N in each case is N = 1.

In this paper we give a new proof using the coefficients in certain continued fraction expansions of Stieltjes transforms or equivalently, the coefficients in the three-term recursion formula for arbitrary orthogonal polynomials with leading coefficient one. Part of the proofs is modeled after the results in Karlin and Studden (1966) which use a theorem of Kiefer and Wolfowitz (1960) from statistical design theory. The problem of identifying the points $\{x_i\}_{i=1}^{n+1}$ minimizing $\tilde{N}$ in (1) is turned into one of identifying the polynomial coefficients maximizing certain determinants. The determinants in each case can be simply written down in terms of the coefficients from the continued fraction expansion and the maximization trivially carried out. The resulting coefficients are then identified with the solutions in Theorem 1. The solutions are greatly unified and all the cases for a given interval type $[-1,1]$, $[0,\infty)$ or $(-\infty,\infty)$ can be handled at the same time.

In the following we outline the proof in a number of steps. The details for the various steps are given later.

Step 1. Let $\xi$ denote a probability measure with mass $1/(n+1)$ on each point $x_i$, $i=1, 2,\ldots,n+1$. Write $f^t(x) = (1, x,\ldots,x^n)$ and $M(\xi) = \int f(x)\, f^t(x)\, w(x)\, d\xi(x)$. Then

(2) $$(n+1)\, w(x) \sum_{i=1}^{n+1} r_i^2(x) = w(x)\, f^t(x)\, M^{-1}(\xi)\, f(x) = v(x;\xi)$$

and hence

(3) $$N = \inf_{\xi} \sup_{x} v(x;\xi)$$

The infinum in (3) is over $\xi$ with equal masses on n+1 points. It turns out in our situation that the same value is obtained for N if $\xi$ is allowed to be an arbitrary probability measure.

Step 2. Let $\xi$ be an arbitrary probability measure.

Theorem 2 (Kiefer-Wolfowitz): The conditions

(i) $\xi^*$ minimizes sup $v(x;\xi)$

(ii) $\xi^*$ maximizes $|M(\xi)| \doteq \det M(\xi)$

(iii) $\sup_x v(x;\xi^*) = n+1$

are equivalent. The set B consisting of all $\xi^*$ fulfilling (i), (ii), (iii) is convex and closed and the matrix $M(\xi^*)$ is the same for all $\xi^*$ in B.

Step 3. If $\xi^*$ maximizes $|M(\xi)|$, where $w(x)$ is any one of the cases in A, B, or C, then $\xi^*$ is supported on n+1 points $x_1, x_2, \ldots x_{n+1}$ and $\xi^*(x_i) = 1/(n+1)$. For $\xi$ supported on n+1 points

$$(4) \quad |M(\xi)| = |M_o(\xi)| \prod_{i=1}^{n+1} w(x_i)$$

Here $M_o(\xi)$ denotes the matrix $M(\xi)$ corresponding to $w(x) \equiv 1$. Note that the weights for $\xi$ in (4) will still be thought of as arbitrary. If $w(x)$ is defined on $(-\infty,\infty)$ or $[-1,1]$ and is symmetric about o, then $\xi^*$ may be assumed symmetric also. Symmetry arguments are used only for Case C. Any symmetric situations in Case A result directly in a symmetric solution without a separate argument.

For the next step we introduce some parameters to characterize $\xi$. We then write down (4) in terms of these parameters, perform the maximization and identify the solutions in Theorem 1.

The following theorem gives us a set of parameters to characterize a probability measure on the various intervals.

Theorem 3 (A) The Stieltjes transform of every probability measure on $[-1,1]$ with n+1 support points has the continued fraction expansion

$$(5)\quad \int_{-1}^{1} \frac{d\xi(x)}{z-x} = \frac{1\,|}{|z+1-2\zeta_1} - \frac{4\zeta_1\zeta_2\,|}{|z+1-2\zeta_2-2\zeta_3} - \frac{4\zeta_3\zeta_4\,|}{|z+1-2\zeta_3-2\zeta_4} - \cdots - \frac{4\zeta_{2n-1}\zeta_{2n}\,|}{|z+1-2\zeta_{2n}-2\zeta_{2n+1}}$$

or equivalently

$$(5a)\quad \int_{-1}^{1} \frac{d\xi(x)}{z-x} = \frac{1\,|}{|z+1} - \frac{2\zeta_1\,|}{|1} - \frac{2\zeta_2\,|}{|z+1} - \cdots - \frac{2\zeta_{2n}\,|}{|z+1} - \frac{2\zeta_{2n+1}\,|}{|1}$$

where $\zeta_1=p_1$, $\zeta_i=q_{i-1}p_i$ for $i \geq 2$, $0 < p_i < 1$ for $i < 2n$ $0 \leq p_i \leq 1$ for $i=2n$, $2n+1$, $q_i=1-p_i$ for all i.

(B) Every probability measure $\xi$ on $[0,\infty)$ with n+1 support points has a Stieltjes transform expansion

$$(6)\quad \int_0^\infty \frac{d\xi(x)}{z-x} = \frac{1\,|}{|z} - \frac{d_1\,|}{|1} - \frac{d_2\,|}{|z} - \quad - \frac{d_{2n}\,|}{|z} - \frac{d_{2n+1}\,|}{|1}$$

where $d_i > 0$ for $i \leq 2n$ and $d_{2n+1} \geq 0$.

(C) Every probability measure $\xi$ on $(-\infty,\infty)$ with n+1 support points satisfies

$$(7)\quad \int_{-\infty}^{\infty} \frac{d\xi(x)}{z-x} = \frac{1\,|}{|z-b_1} - \frac{a_1\,|}{|z-b_2} - \frac{a_2\,|}{|z-b_3} - \cdots - \frac{a_n\,|}{|z-b_{n+1}}$$

where $a_i > 0$ for $i \leq n$ and $-\infty < b_i < \infty$ for $i \leq n+1$. For symmetry $\xi$ the $b_i$ are all zero.

Proof: For part A, see Wall (1948) or (1940). The form given in (5) is a contraction of 5(a). Parts B and (C) follow from Shohat and Tamarkin (1943) pages 47 and 32 respectively.

In order to calculate the determinant $|M(\xi)|$ in (4) we need both the determinant $|M_o(\xi)|$ and the product $\prod_{i+1}^{n+1} w(x_i)$.

Step 4. The determinants $|M_o(\xi)|$ for the three cases are given by

(A) $$|M_o(\xi)| = \prod_{i=1}^{n} (\zeta_{2i-1}\zeta_{2i})^{n-i+1}$$

(B) $$|M_o(\xi)| = \prod_{i+1}^{n} (d_{2i-1}d_{2i})^{n-i+1}$$

(C) $$|M_o(\xi)| = \prod_{i=1}^{n} a_i^{n-i+1}$$

Step 5. The products $\prod_{i=1}^{n+1} w(x_i)$ for the three cases are as follows:

(A) In this case let us write $w(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}(x)^{\gamma}$ with the understanding that $\gamma > 0$ iff $\alpha = \beta$. We have

$$\Pi w(x_i) = \text{const}\left(\prod_{i=1}^{2n+1} q_i\right)^{\alpha+1}\left(\prod_{i=0}^{n} \zeta_{2i+1}\right)^{\beta+1}\left(p_2q_4p_6q_8\cdots q_{2n-2}\ p_{2n}\right)^{\gamma}$$

(B) $$\Pi\, w(x_i) = \prod_{i=1}^{n+1} x_i^{\alpha+1} \exp\left(-\sum_{i=1}^{n+1} x_i\right)$$

$$= \left(\prod_{i=0}^{n} d_{2i+1}\right)^{\alpha+1} \exp\left(-\sum_{i=1}^{2n+1} d_i\right)$$

(C) Here we note that $\gamma > 0$ iff n is odd.

$$\Pi\, w(x_i) = \prod_{i=1}^{n+1} |x_i|^{\gamma} \exp\left(-\sum_{i=1}^{n+1} x_i^2\right)$$

$$= (a_1\; a_3\; \ldots\; a_n)^{\gamma} \exp\left(-2 \sum_{i=1}^{n} a_i\right)$$

<u>Step 6</u> If we now multiply the corresponding expressions for $|M(\xi_0)|$ and $\Pi w(x_i)$ from Steps 4 and 5 in each case, the resulting value for $|M(\xi)|$ can be maximized using simple calculus. The resulting parameters in each case are given by

(A) $p_{2i+1} = \dfrac{\beta+n-i+1}{\alpha+\beta+2(n-i+1)}$ $\qquad 0 \le i \le n$

$p_{2i} = \dfrac{\gamma+n-i+1}{\alpha+\beta+\gamma+2(n-i+1)+1}$ $\qquad$ i odd, $1 \le i \le n$

$p_{2i} = \dfrac{n-i+1}{\alpha+\beta+\gamma+2(n-i+1)+1}$ $\qquad$ i even, $1 \le i \le n$

(B) $d_{2i+1} = \alpha+n-i+1$ $\qquad 0 \le i \le n$

$d_{2i} = n-i+1$ $\qquad 1 \le i \le n$

(C) $a_{2i+1} = \frac{n-2i+\gamma}{2}$ $\quad 0 \le i \le \frac{n-1}{2}$

$a_{2i} = \frac{n-2i+1}{2}$ $\quad 1 \le i \le \frac{n-1}{2}$ $\quad$ } n odd, $\gamma > 0$

$a_i = \frac{n-i+1}{2}$ $\quad 1 \le i \le n$ $\quad \gamma = 0$

Step 7. Identify the parameters in Step 6 to obtain the results in Theorem 1.

Proofs of Steps 1-7

Step 1: Equation (2) follows by noting that $v(x;\xi)$ is invariant under basis change for the powers 1, x, $x^2$, ..., $x^n$ and we convert to the lagrange form $\ell_1(x)$, ..., $\ell_{n+1}(x)$.

Step 2: This is the Kiefer-Wolfowitz "equivalence theorem" from statistical design theory. The proof is actually fairly simple and uses the fact that $\ln|M(\xi)|$ is strictly concave in M. Thus a local maximum is a global maximum and the maximizing $\xi^*$ have the same M value. Let $\xi_\alpha = (1-\alpha)\xi^* + \xi_x$, $0 \le \alpha \le 1$ where $\xi_x$ denotes the point mass at x. Let $g(\alpha) = \ln|M(\xi_\alpha)|$. Then $\xi^*$ is a local maximum iff $g'(0) \le 0$ which is equivalent to $v(x,\xi^*) \le n+1$.

Step 3: The proof uses the fact that $v(x;\xi) = w(x)S_{2n}(x)$ where $S_{2n}(x)$ is a polynomial of degree 2n and $w(x)S_{2n}(x) \le n+1$ and touching on the support of $\xi^*$ forces the support of $\xi^*$ to be n+1 points.

If $\xi^*$ is supported by n+1 points then $|M(\xi^*)|$ can be written as

$$|M(\xi^*)| = \prod_{i=1}^{n+1} w(x_i) \prod_{i=1}^{n+1} \xi^*(x_i)\, F^2\,(x_1, \ldots, x_{n+1})$$

where $F(x_1, \ldots, x_{n+1})$ is the Vandermonde determinant involving $x_0, \ldots, x_{n+1}$. The maximization over the weights $\xi^*(x_i)$ can be done separately and they must be all equal. For later purposes we leave the $\Pi\, w(x_i)$ as it is and recombine the $\Pi\, \xi^*(x_i)$ with $F^2(x_i,\ldots,x_{n+1})$ to give the expression in equation (4).

The last sentence in Step 3 follows by considering the map $x \to -x$. The resulting measure $\xi_1$ then satisfies $|M(\xi)| = |M(\xi_1)|$ and the conclusion follow by considering $(\xi_1+\xi)/2$ and the concavity of $\ln|M(\xi)|$.

Step 4. The values given by the determinants are taken from Theorem 51·1 in Wall (1948).

Step 5. The right-hand sides of (5a), (6), are (7) are rational functions in z and it is easy to see that the support of $\xi$ in each case is given by the roots of the polynomial in the denominator. Let

$$K\begin{pmatrix} & u_1 & u_2 & \cdots & u_n & \\ v_1 & v_2 & v_3 & \cdots & & v_{n+1}\end{pmatrix} = \begin{vmatrix} v_1 & -1 & & & & \\ u_1 & v_2 & -1 & & \mathbf{0} & \\ & u_2 & v_3 & -1 & & \\ & & & \ddots & & \\ & \mathbf{0} & & & & -1 \\ & & & & u_n & v_{n+1} \end{vmatrix}$$

We can write down the denomiantors $D_{n+1}(z)$ in each case.

$$\text{(A)}\quad D_{n+1}(z) = K\begin{pmatrix} & -2\zeta_1 & & -2\zeta_2 & \cdots & -2\zeta_{2n+1} & \\ z+1 & & 1 & & z+1 & \cdots & 1\end{pmatrix}$$

$$\text{(B)}\quad D_{n+1}(z) = K\begin{pmatrix} & -d_1 & & -d_2 & \cdots & -d_{2n+1} & \\ z & & 1 & & z & \cdots & 1\end{pmatrix}$$

$$\text{(C)}\quad D_{n+1}(z) = K\begin{pmatrix} & -a_1 & & -a_2 & \cdots & -a_n & \\ z & & z & & z & & z\end{pmatrix}$$

By expanding the above determinants for cases (B) and (C), we obtain

$$\text{(B)}\quad z^{n+1} - \left(\sum_{i=1}^{2n+1} di\right) z^n + \dots + (-1)^{n+1} d_1 d_3 \dots d_{2n+1}$$

$$\text{(B)}\quad \begin{cases} z^{n+1} - \left(\sum_{i=1}^{n} a_i\right) z^n + \dots, & n \text{ is even} \\ z^{n+1} - \left(\sum_{i=1}^{n} a_i\right) z^n + \dots + (-1)^{\frac{n+1}{2}} a_1 a_3 \dots a_n & n \text{ is odd.} \end{cases}$$

The term $\prod_{i=1}^{n+1} w(x_i)$ can be expressed in terms of $d_i$'s and $a_i$'s for cases B and C respectively. More explicitly, we have

$$\text{(B)}\qquad e^{-\sum_{i=1}^{n+1} xi} \prod_{i=1}^{n+1} x_i^{\alpha+1} = e^{-\sum_{i=1}^{2n+1} di} \quad d_1 d_3 \dots d_{2n+1}{}^{\alpha+1}$$

$$\text{(C)}\qquad \begin{cases} e^{-\sum_{i=1}^{n+1} x_i^2} \prod_{i=1}^{n+1} |x_i|^{\gamma} = e^{-2\sum_{i=1}^{n} a_i} (a_1 a_2, \dots a_n)^{\gamma} & n \text{ is odd and } \gamma > 0. \\ e^{-\sum_{i=1}^{n+1} x_i^2} = e^{-2\sum_{i=1}^{n} a_i} & n \text{ arbitrary } \gamma = 0. \end{cases}$$

To compute $\prod_{i=1}^{n+1} w(x_i)$ for case A, we first show by induction,

$$D_{n+1}(1) = \prod_{i=1}^{n+1} (1-x_i) = \text{const} \prod_{i=1}^{2n+1} q_i$$

$$D_{n+1}(-1) = \prod_{i=1}^{n+1} (1+x_i) = \text{const} \prod_{i=0}^{n} \zeta_{2i+1}$$

Next, it can be seen that $1-2\zeta_1 = 0$ and $1-2\zeta_{2k}-2\zeta_{2k+1} = 0$ for $k \geq 1$ if $\xi$ is symmetric about 0. This is the case iff $P_i = \frac{1}{2}$ for all i. (This is the same as $b_i = 0$ in the expression (7)). Thus, in case $\xi$ is symmetric about 0 and n is odd, we have

$$D_{n+1}(0) = \prod_{i=1}^{n+1} x_i = \text{const. } p_2q_4p_6q_8\cdots q_{2n-2}p_{2n}.$$

Writing $w(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}|x|^{\alpha}$ with the understanding that $\alpha \neq 0$ iff $\alpha = \beta$. We have

$$\prod_{i=1}^{n+1} w(x_i) = \text{const } (\prod_{i=1}^{2n+1} q_i)^{\alpha+1}(\prod_{i=0}^{n} \zeta_{2i+1})^{\beta+1}(p_2q_4p_6q_8\cdots q_{2n-2}p_{2n})^{\gamma}$$

Step 6. This step is straight forward. We multiply the values for $\Pi\, w(x_i)$ and $|M(\xi_0)|$, for cases A, B, and C separately, to give $|M(\xi)|$. The maximization in each case is relatively easy.

Step 7. The identification of the appropriate roots in Theorem 1 from the parameters in Step 6 revolves around an interesting symmetry property of the parameters defined generally in Theorem 3. This property is stated in the following Theorem. The proof uses an induction argument and will be omitted. Full details can be found in Lau (1983).

Theorem 4 If $0 \leq p_i < 1$ for $i = 1, 2, \ldots, m$ then

(a) the probability measures corresponding to the sequences $(p_1,p_2,\ldots,p_m,0)$ and $(p_m,p_{m-1},\ldots,p_1,0)$ have the same support. Further the probability measures corresponding to $(p_1,p_2,\ldots,p_m,1)$ and $(q_m,q_{m-1},\ldots,q_1,1)$ have the same support.

Similarly in cases (B) and (C), if $d_i > 0$, $a_i > 0$ for $i = 1,\ldots,m$ then

(b) $(d_1,\ldots,d_m,0)$ and $(d_m,\ldots,d_1,0)$ have the same support and

(c) $(a_1,\ldots,a_m,0)$ and $(a_m,\ldots,a_1,0)$ have the same support.

With the aid of Theorem 4 the results given in Theorem 1 are more or less immediate. The required Stieltjes expansions for the identification for case A, B and C are taken from Van Rossum (1953). Case A on pages 51 and 56, Case B on page 41 and Case C on page 45. Some of these are also given in Wall (1948) formulas (89.16) and (92.4).

To identify parts (i) -(iv) in Case A we make use of the fact that the Stieltjes Transform of the Jacobi weight function $(1-x)^{\alpha+1}(1+x)^{\beta+1}$ has an infinite expansion as in Case A in Theorem 3 with parameters given by

$$p_{2k} = \frac{k}{\alpha+\beta+2k+3} \qquad k \geq 1$$

(8)

$$p_{2k+1} = \frac{\beta+k+2}{\alpha+\beta+2k+4} \qquad k \geq 0$$

Special cases of interest correspond to $\alpha=\beta=-\frac{3}{2}$ and $\alpha=\beta=-1$. The situation $\alpha=\beta=-\frac{3}{2}$ is associated with the Chebychev polynomials of the 1st kind and we have $p_i = \frac{1}{2}$, for all i. For the Lebesque or uniform measure with $\alpha=\beta=-1$ the resulting parameters are $p_i = \frac{1}{2}$ for i odd and $p_{2i} = i/(2i+1)$.

Consider part (i) of Case A. The parameters $p_i$ maximizing $|M(\xi)|$ are given from Step 6 as

$$p_{2i+1} = \tfrac{1}{2} \qquad i = 0, 1,\ldots,n-1$$

(9)
$$p_{2i} = \frac{n-i+1}{2(n-i+1)+1} \qquad i = 1, 2,\ldots,n-1$$

$$p_{2n} = 1$$

On comparing this sequence with the sequence for the uniform measure we note they both have $p_i = \frac{1}{2}$ for i odd. The even indexed parameters for the uniform measure are 1/3, 2/5, 3/7,..., etc. while the even parameters, maximizing $|M(\xi)|$ are given, in reverse order, by 2/3, 3/5, 4/7,....

Part (a) of Theorem 4 implies that the measure maximizing $|M(\xi)|$ for part (i), Case A has the same support as the sequence

$$p_{2i+1} = \tfrac{1}{2} \qquad i = 0, 1,\ldots,n-1$$

(10)
$$p_{2i} = i/(2i+1) \qquad i = 1, 2,\ldots,n-1$$

$$p_{2n} = 1$$

This sequence is obtained by "truncating" the sequence from the uniform measure by setting $p_{2n} = 1$. The resulting support in this case is on ±1 and the zeros of the $(n-1)^{st}$ polynomial orthogonal with respect to $(1-x^2)dx$ which is precisely the polynomial $P_n'(x)$ given in part (i) Case A of Theorem 1.

It is interesting to note that the finitely supported measure corresponding to (10) is associated with a classical Gauss-type quadrature formula using the end points ±1. The reversed sequence in (1) has precisely the same support and uniform weights.

The same phenomenon occurs in all the other cases including Cases B and C. To illustrate further, consider part (ii) of Case A. Here the maximizing sequence from Step 6 is given by

$$p_{2i+1} = \frac{\beta+n+1-i}{\alpha+\beta+2n-2i+2} \qquad 0 \le i \le n$$

(11)
$$p_{2i} = \frac{n+1-i}{\alpha+\beta+2n-2i+3} \qquad 1 \le i \le n$$

$$p_{2n+2} = 0$$

If we take the sequence from (8) with the weight $(1-x)^{\alpha}(1+x)^{\beta}$, truncate the sequence with $p_{2n+2} = 0$ and reverse the 1st 2n+1 parameters we obtain (11). Using part (a) of Theorem 4 the support is on the zeros of Jacobi polynomial as stated in Theorem 1.

The remaining cases are similarly and are omitted.

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