On the Arrival and Departure Processes Arising in Queues with Infinite Servers

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Abstract. We study the behavior of the departure processes associated with 'general' input infinite servers queues. In so doing, we establish certain characterizing relationships between the departure process and the corresponding input process (referred to here as an arrival process) for such a queue. These relationships between the two processes in turn help to study certain properties of one process by studying those of the other. In particular a kind of 'reversibility' result is also established between the Poisson departure and the arrival processes for the case of an infinite servers queue with *non*identically distributed service times. Finally, we also establish a new result (theorem 2(b)) about thinning of arrival processes where the thinning probabilities p_i's may be time-dependent.

Key Words: infinite servers queues; Poisson processes, departure processes, independence, compartment models, mixed Poisson processes, renewal processes, thinning of arrival processes.

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1. Introduction.

Consider N independent and identically conducted Bernoulli trials resulting in ξ_1 successes and $\xi_2 = N - \xi_1$ failures. The random variables ξ_1 and ξ_2 are completely dependent as long as N is nonrandom. On the other hand if N is random, depending upon its distribution, the random variables ξ_1 and ξ_2 may even become mutually independent. This happens to be the case when N has a Poisson distribution. In fact this is the only distribution enjoying such a property. The same characterizing property for Poisson distribution for N holds when N multinomial trials are conducted in an independent and identical manner resulting into $\xi_1, \xi_2, \ldots, \xi_k$ as the numbers of various types of 'successes', with $\xi_1 + \ldots + \xi_k = N$. In fact here the *mutual independence for any two* ξ_i*'s* forces N to have a Poisson distribution (see Moran (1952) and Patil and Seshadri (1964)). The continuous time analog of the above observation is given in theorem 1 below, where $\{A(t), t \geq 0\}$ with $A(0) = 0, A(t) < \infty, \forall\ t \geq 0$, a.s. and referred to as an arrival process, is a separable point process with right-continuous sample paths having successive unit steps at times $0 < \tau_1 < \tau_2 \ldots$. In particular $\{A(t); t \geq 0\}$ is a Poisson process whenever it has independent increments with each increment having a Poisson distribution and it has a finite nondecreasing mean function $E(A(t)) \equiv a(t)$, which is continuous for $t \geq 0$. Consider the

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situation where for such an arrival process, each arrival, independent of everything else, is allowed to depart instantly upon arrival to a compartment C_i with positive probability $p_i, i = 1, 2, \ldots, k, k \geq 2$, satisfying $p_1 + \ldots + p_k = 1$. Let $D_i(t)$ denote the number of 'ith type' departures i.e. to C_i during $(0, t]$. The following theorem is due to Fichtner (1975), which was stated originally in terms of thinnings of a process. Instead we have purposely preferred here to use the language of departure processes since we are intending to consider later on the case of departures after positive random service times (see section 3 onwards).

Theorem 1. *The arrival process $\{A(t), t \geq 0\}$ is Poisson if and only if the departure processes $\{D_i(t), t \geq 0\}, i = 1, 2, \ldots, k$, are mutually independent.*

Note that in the above case the processes $\{D_i(t), t \geq 0\}, i = 1, 2, \ldots k$, individually turn out also to be Poisson processes. *Furthermore the independence of any two of these will force the arrival process $\{A(t), t \geq 0\}$ to be a Poisson process.* A special case of the above theorem where the process $A(t)$ was assumed to be time homogeneous is due to Rényi (1964) (see also Srivastava (1971)). Also the reader may refer to Kimeldorf and Thall (1983) for a recent generalization of theorem 1. However their results do not cover the specific generalization in the direction where an allowance is made for the probabilities p_i's corresponding to a specific arrival time τ_j to depend upon this τ_j. We deal with such a generalization in section 2 (see theorem 2(b)). Here some conditions on the functions $p_i(\cdot)$'s are necessary in order that the result of theorem 1 holds. That the result fails to hold in general is illustrated through an example.

Again in the case covered by theorem 1, each arrival was allowed upon arrival to depart instantly to a compartment C_i with probability $p_i, i = 1, 2, \ldots, k$. We now consider the case where each arrival is served, independent of everything else, for a random length of time with a common cumulative distribution function (c.d.f.) $G(\cdot)$. At the end of this service period each arrival is allowed to depart to compartment C_i with the *constant* probability $p_i, i = 1, 2, \ldots, k$, generating this way again k-departure processes

$D_i(t), i = 1, 2, \ldots, k$, over $[0, \infty)$. Let $D(t) = \sum_{i=1}^{k} D_i(t)$, be the overall departure process. An interesting questions is what can one say about the arrival process $\{A(t), t \geq 0\}$, given that the processes $\{D_i(t), t \geq 0\}, i = 1, 2, \ldots, k$, are mutually independent. Note since $\{D_i(t), t \geq 0\}, i = 1, 2, \ldots, k$, are thinnings of the process $D(t)$, as follows from theorem 1 the mutual independence of these processes would be equivalent to $D(t)$ being a Poisson process. Thus the above question amounts to characterizing the arrival process $A(t)$, given that the departure process $D(t)$ is Poisson. For the case where $A(t)$ is assumed to be a renewal process we give a partial answer to this question in section 3 but under the weaker condition that $D(t)$ has a Poisson distribution for every $t \geq 0$. In section 4 (theorem 5) we have a rather interesting related result, namely given that the departure process $D(t)$ is Poisson and that for every $n \geq 1$, conditional on the first n departure epochs, the *corresponding* n service times are mutually independent, we show that the arrival process has to be a Poisson process. Moreover, in that case, the service times for the various *arrivals* have to be also mutually independent. A converse (theorem 4) of this result is also discussed. Finally the paper concludes with some remarks and touches some of the open questions that are still under investigation.

2. Case with time-dependent p_i's.

We return to the model considered in Theorem 1 and allow the probabilities $p_i(\cdot), i = 1, 2, \ldots, k$, to depend upon the arrival time τ of an arrival, with $\sum_{i=1}^{k} p_i(\tau) \equiv 1$. The following theorem is a generalization of Theorem 1.

Theorem 2. (a) *Let $\{A(t), t \geq 0\}$ be a Poisson process with $EA(t) \equiv a(t)$ and $p_i(\cdot)$'s be such that the Lebesgue-Stieltjes integrals $\int_0^t p_i(\tau) a(d\tau), i = 1, 2, \ldots, k$, exist for every $t > 0$. Then the departure processes $D_i(t)$, $i = 1, 2, \ldots, k$, are mutually independent Poisson processes with*

$$ED_i(t) = \int_0^t p_i(\tau) a(d\tau).$$

(b) *Conversely, let for two arbitrary nonempty disjoint subsets of $\{1, 2, \ldots k\}$, say K_1 and*

K_2, for every $\varepsilon > 0$ and $0 < t < \infty$,

$$(1) \qquad J_{\varepsilon,t} = \{\tau\colon\ \tau \le t,\ \sum_{i\in K_1} p_i(\tau) < \varepsilon \text{ or } \sum_{i\in K_2} p_i(\tau) < \varepsilon\}.$$

Let $A(J_{\varepsilon,t})$ denote the number of arrivals at time points in $J_{\varepsilon,t}$. We assume that the arrival process $A(t)$ and the probability functions $p_i(\tau)$ are such that for every small enough $\varepsilon > 0$ and for every $t > 0$, $A(J_{\varepsilon,t})$ is measurable and satisfies the condition

$$(2) \qquad A(J_{\varepsilon,t}) \xrightarrow{P} 0, \quad \text{as} \quad \varepsilon \downarrow 0.$$

Then the mutual independence of the departure processes

$$\sum_{i\in K_1} D_i(t) \qquad \text{and} \qquad \sum_{i\in K_2} D_i(t)$$

implies that $\{A(t), t \ge 0\}$ is a Poisson process.

Proof. Part (a) is well known and can be proved using the order-statistic property of Poisson processes (see Matis (1973), Puri (1973), Faddy (1979) and Harrison and Lemoine (1981)). We shall prove part (b) only for the case where $K_1 = \{1\}$ and $K_2 = \{2\}$. The general case follows along similar lines. Thus we have for $t > 0$,

$$(3) \qquad J_{\varepsilon,t} = \{\tau\colon\ \tau \le t, p_1(\tau) < \varepsilon \ \text{ or } \ p_2(\tau) < \varepsilon\},$$

and we are given that the departure processes $D_1(t)$ and $D_2(t)$ are mutually independent. Also let $\varepsilon \le \frac{1}{2}$ and

$$(4) \qquad J_\varepsilon = \bigcup_{t>0} J_{\varepsilon,t}.$$

We now construct two new point processes $\{B_i(t), t \ge 0\}, i = 1,2$, as follows. For each departure to compartment $C_i, i = 1,2$, if the corresponding arrival time $\tau \in J_\varepsilon$ we let this departure stay in C_i, otherwise we let it go upon arrival to two new compartments C_i' and C_i'' with probabilities $(1 - \varepsilon/p_i(\tau))$ and $(\varepsilon/p_i(\tau))$ respectively, for $i = 1,2$. Note that the four compartments C_i' and C_i'', $i = 1, 2$, are different from the original compartments $C_1, C_2, \ldots, C_k$. Let $B_1(t)$ and $B_2(t)$ denote the numbers of arrivals entering the compartments C_1'' and C_2'' respectively during $(0, t]$. Evidently from the mutual independence of the processes $D_1(t)$ and $D_2(t)$ follows the mutual independence of the processes $\{B_i(t), t \geq 0\}, i = 1, 2$. Again let

(5) $$H_\varepsilon(t) \equiv A(t) - A(J_{\varepsilon,t}), \quad t \geq 0,$$

which is also an arrival process. From the construction it follows that for each arrival of the process $\{H_\varepsilon(t), t \geq 0\}$, there is a probability ε of entering each of the compartments C_i'', $i = 1, 2$, generating the two processes $\{B_j(t), t \geq 0\}$, $i = 1, 2$, and the remaining probability $1 - 2\varepsilon$ of not entering any of these two compartments C_1'' and C_2''. In view of Theorem 1 and the remarks following it, the independence of these two processes in turn implies that the process $\{H_\varepsilon(t), t \geq 0\}$ is a Poisson process with expectation, say

(6) $$EH_\varepsilon(t) = a_\varepsilon(t), \qquad t \geq 0.$$

Also

(7) $$a_0(t) \equiv \lim_{\varepsilon \downarrow 0} a_\varepsilon(t) < \infty, \qquad t \geq 0,$$

for otherwise if $a_0(t) = \infty$, we would have

(8) $$P(A(t) > n) \geq P(H_\varepsilon(t) > n) \longrightarrow 1, \quad \forall\, n,$$

as $\varepsilon \downarrow 0$, contradicting thereby the finiteness of $A(t)$. Again the process defined in (5) being monotone in ε, converges a.s., as $\varepsilon \downarrow 0$, to another arrival process, say $\{H(t), t \geq 0\}$. Consequently using (7), being the limit of Poisson processes, $H(t)$ must itself be a Poisson process with

(9) $$EH(t) = a_0(t), \quad t \geq 0,$$

and hence in view of (2), the process $\{A(t), t \geq 0\}$ must also be a Poisson process with $EA(t) = a_0(t), t \geq 0$. □

The following example illustrates the need of condition such as (2) in order that the result (b) of the above theorem holds.

Example. Let $\{A_1(t), t \geq 0\}$ be a homogeneous Poisson process with parameter $\lambda > 0$ and $\{A_2(t), t \geq 0\}$ with $A_2(0) = 0$, be another arrival process with unit jumps occurring only at all integral values of t. Also let $A(t) \equiv A_1(t) + A_2(t), t \geq 0$. Choose two positive constants p_1 and p_2 with $p_1 + p_2 \leq 1$ and for $i = 1, 2$, and $t \geq 0$, let

$$(10) \qquad p_i(t) = \begin{cases} 0, & \text{if } t \text{ is a positive integer,} \\ p_i, & \text{otherwise.} \end{cases}$$

Then it is easy to see that while the corresponding departure process $\{D_i(t), t \geq 0\}, i = 1, 2$, are two mutually independent homogeneous Poisson processes with parameters $\lambda p_i, i = 1, 2$, the process $\{A(t), t \geq 0\}$ is not a Poisson process.

Consider now the case where each arrival of an arrival process $\{A(t), t \geq 0\}$ departs to a compartment C upon arrival with probability $p(\tau)$, independently of everything else, where τ is its arrival epoch. Let $D(t)$ denote the number of departures to C during $(0, t]$. For the special case where the function $p(u) \equiv p$ is a positive constant, it can be easily shown and must be known in literature that $\{A(t), t \geq 0\}$ is Poisson if and only if $\{D(t), t \geq 0\}$ is. For the case with time dependent $p(\cdot)$, the result that $\{A(t), t \geq 0\}$ is Poisson implies that $\{D(t), t \geq 0\}$ is also Poisson, has already been covered by Theorem 2(a). The following corollary to Theorem 2(b) covers its converse.

Corollary 1. *For the given arrival process $\{A(t), t \geq 0\}$ and the probability function $p(\cdot)$, define for every $\varepsilon > 0$* and $t > 0$,

$$(11) \qquad J'_{\varepsilon,t} = \{\tau : \ \tau \leq t, p(\tau) < \varepsilon\}.$$

Let $A(J'_{\varepsilon,t})$ denote the number of arrivals at the time points in $J'_{\varepsilon,t}$. We assume that for every small enough $\varepsilon > 0$ and for every $t > 0$, $A(J'_{\varepsilon,t})$ is measurable and satisfies

$$A(J'_{\varepsilon,t}) \xrightarrow{P} 0, \quad \text{as } \varepsilon \downarrow 0. \tag{12}$$

Finally let the corresponding departure process $\{D(t), t \geq 0\}$ be Poisson. Then the arrival process $\{A(t), t \geq 0\}$ must also be Poisson.

Proof. Let each departure of the process $\{D(t), t \geq 0\}$ be allowed to pass through compartments C' or C'' each with probability $\frac{1}{2}$. Let $D'(t)$ and $D''(t)$ denote the numbers passing through C' and C'' respectively during $(0, t]$. Since $\{D(t), t \geq 0\}$ is a Poisson process, it follows from Theorem 1 that the process $\{D'(t), t \geq 0\}$ and $\{D''(t), t \geq 0\}$ are mutually independent. Since these processes could have alternatively and yet equivalently been constructed directly by allowing each arrival of the process $\{A(t), t \geq 0\}$ to pass through either compartment C' or C'', each with probability $\frac{1}{2}p(\tau)$, the corollary easily follows from Theorem 2(b). □

3. The case of GI/G/∞ queues.

Consider the situation where the 'customers' arrive according to a renewal process $\{A(t), t \geq 0\}$. Upon arrival each customer is served immediately with the service times of various customers being i.i.d. r.v.'s with c.d.f. $G(\cdot)$, satisfying $G(0) < 1$. Also the service times are assumed to be independent of the arrival process $A(t)$. Let $D(t)$ and $N(t)$ denote the number of departures during $(0, t]$ and the number of customers being served at time t respectively. Then given that the random variable $D(t)$ has a Poisson distribution for every $t \geq 0$, we shall show (see theorem 3) under some minor conditions that $\{A(t), t \geq 0\}$ must be a homogeneous Poisson process. We shall need the following lemma to establish this assertion.

Lemma 1. *Let $\{A(t), t \geq 0\}$ be a renewal process with interarrival time c.d.f. $F(x) = P(X_k \leq x), k = 1, 2, \ldots$, where $\{X_k, k \geq 1\}$ are the times between successive arrivals.*

Then $\{A(t), t \geq 0\}$ *is a homogeneous Poisson process with parameter* $b > 0$, *i.e.* $F(\cdot)$ *is exponential with parameter* b, *if and only if either*

(i) $P(D(t) = 0) = \exp\{-bB(t)\}$, for $t \geq 0$, or

(ii) $P(N(t) = 0) = \exp\{-b(t - B(t))\}$, for $t \geq 0$,

where

$$B(t) = \int_0^t G(u)du, \quad t \geq 0. \tag{13}$$

Proof. The rest being either analogous or straightforward, we shall prove only that a renewal process with property (i) must be a Poisson process with parameter b. Let $L(t) \equiv P(D(t) = 0)$. Using a standard renewal argument it is easy to establish

$$L(t) = 1 - F(t) + \int_0^t (1 - G(t-x))\ L(t-x)\ dF(x). \tag{14}$$

In view of (i), substituting $L(t) = \exp(-bB(t))$ in (14) and taking Laplace transforms of both sides with respect to t, we obtain for $\theta > 0$, the relation

$$\xi(\theta) = \theta^{-1}(1 - F^*(\theta)) + F^*(\theta)[(1 + b^{-1}\theta)\ \xi(\theta) - b^{-1}], \tag{15}$$

where

$$F^*(\theta) = \int_0^\infty \exp(-\theta t)\ dF(t), \tag{16}$$

and

$$\xi(\theta) = \int_0^\infty \exp\{-\theta t - bB(t)\}\ dt. \tag{17}$$

Finally solving (15) for $F^*(\theta)$ yields

$$F^*(\theta) = b(b + \theta)^{-1}, \quad \theta > 0, \tag{18}$$

which in turn implies that F is exponential with parameter b. □

We may note that we did not impose the condition $F(0) = 0$ in the lemma to begin with, so that the process $\{A(t), t \geq 0\}$ was not strictly an arrival process since the jumps were not necessarily of unit sizes. And yet the conditions (i), (ii) forced $F(0) = 0$ to hold. The same remark applies to the following theorem.

Theorem 3. *Let $\{A(t), t \geq 0\}$ be a renewal process as in Lemma 1, and for every $t, D(t)$ have a Poisson distribution with $ED(t) \equiv \Lambda(t)$ and $\Lambda(0) = 0$. Furthermore let*

(*) *$\Lambda(t)$ be continuously differentiable $\forall\ t \geq 0$, with its derivative $\lambda(t) = 0, \forall\ 0 \leq t \leq a$ and $\lambda(t) > 0, \forall\ t > a$, for some constant $a \geq 0$. Then*

(i) *$G(\cdot)$ is continuous,*

(ii) *$\Lambda(t) \equiv bB(t)$, for some $b > 0$, where $B(t)$ is as defined in (13), and*

(iii) *$\{A(t), t \geq 0\}$ is a homogeneous Poisson process with parameter b.*

Proof. Note that condition (*) implies that $\Lambda(t) = 0\ \forall\ 0 \leq t \leq a$ and $\Lambda(t) > 0,\ \forall\ t > a$. Let

(19) $$R(s;t) = E(s^{D(t)}), \qquad 0 \leq s \leq 1,$$

then as given

(20) $$R(s;t) = \exp\{-\Lambda(t)(1-s)\}, \qquad 0 \leq s \leq 1, \ \ t \geq 0.$$

As in (14), using a renewal argument we obtain

(21) $$R(s;t) = 1 - F(t) + \int_0^t [1-(1-s)G(t-x)]R(s;t-x)dF(x).$$

Taking Laplace transforms of both sides with respect to t, we obtain for $\theta > 0$,

(22) $$\mathcal{L}(R(s;t)) = \theta^{-1}(1-F^*(\theta)) + F^*(\theta) \ \ \mathcal{L}([1-(1-s)G(t)]R(s;t)),$$

where $\mathcal{L}(\psi(t))$ denotes the Laplace transform of a function $\psi(t)$ and F^* is as defined in (16). Solving (22) for F^* we obtain

$$F^*(\theta) = \frac{\theta^{-1} - \mathcal{L}(R(s;t))}{\theta^{-1} - \mathcal{L}\left([1-(1-s)G(t)]R(s;t)\right)}. \tag{23}$$

Again since the left side of (23) is independent of s, the derivatives of all orders with respect to s of its right side are all equal to zero. This implies that for $0 \le s < 1$, and $i \ge 1$,

$$F^*(\theta) = \frac{\mathcal{L}(\Lambda^i(t)R(s;t))}{\mathcal{L}\left(R(s;t)[i\Lambda^{i-1}(t)G(t) + \Lambda^i(t)\{1-(1-s)G(t)\}]\right)}. \tag{24}$$

In particular, by letting $s \uparrow 1$ and using the fact that

$$\Lambda(t) = ED(t) \le EA(t) \le \delta_1 t + \delta_2, \ \forall\, t \ge 0,$$

for some constants δ_1 and δ_2 (see Feller (1971), page 359), so that the transforms such as $\mathcal{L}(\Lambda^i(t))$ exist, $\forall\, i \ge 1$, and $\theta > 0$, we have

$$\frac{\mathcal{L}(\Lambda(t))}{\mathcal{L}(G(t)) + \mathcal{L}(\Lambda(t))} = \frac{\mathcal{L}(\Lambda^i(t))}{\mathcal{L}(i\Lambda^{i-1}(t)G(t)) + \mathcal{L}(\Lambda^i(t))}, \tag{25}$$

for $i \ge 1$, or equivalently

$$\mathcal{L}(G(t))\mathcal{L}(\Lambda^i(t)) = \mathcal{L}(\Lambda(t)) \ \ \mathcal{L}(i\Lambda^{i-1}(t)G(t)); \quad i \ge 1, \tag{26}$$

which in turn is equivalent to

$$\int_0^t G(t-u)\Lambda^i(u)du \overset{t}{\equiv} \int_0^t i\Lambda^{i-1}(t-u)G(t-u)\Lambda(u)du; \quad i \ge 1. \tag{27}$$

Integrating the left side of (27) by parts while using the fact that $\Lambda(0) = 0$, yields the expression

$$i\int_0^t B(t-u)\lambda(u)\Lambda^{i-1}(u)du. \tag{28}$$

Thus we rewrite (27) as

$$\int_0^t [B(t-u)\lambda(u) - G(u)\Lambda(t-u)]\Lambda^i(u)du \stackrel{t}{\equiv} 0, \quad i \geq 0. \tag{29}$$

Again using the fact that $\Lambda(t) = 0$ for $0 \leq t \leq a$, we have from (27) with $i = 2$ and $t = 2a$,

$$\int_0^{2a} G(2a-u)\Lambda^2(u)du = 2\int_0^{2a} G(2a-u)\Lambda(2a-u)\Lambda(u)du = 0. \tag{30}$$

Here the second equality follows from the fact that $\Lambda(u)\Lambda(2a-u) = 0$ for $0 \leq u \leq 2a$. Furthermore, the first integral of (30) being zero and the fact that $\Lambda(u) > 0$ for $u > a$, imply that $G(u)$ and hence B(u) must be equal to zero $\forall\ 0 \leq u \leq a$. Consequently, $\forall\ t \geq 2a$, (29) can be rewritten as

$$\int_a^{t-a} [B(t-u)\lambda(u) - G(u)\Lambda(t-u)]\Lambda^i(u)du = 0, \quad i \geq 0. \tag{31}$$

Since $\Lambda(u)$ is strictly increasing for $u \geq a$, introducing a change of variable from u to x in (31), where

$$x = \Lambda(u)/\Lambda(t-a),$$

(31) can be shown to be of the form

$$\int_0^1 x^i \phi(x;t)\ dx = 0, \quad i \geq 0, \tag{32}$$

for an appropriate function $\phi(\cdot;t)$,which belongs to $L_1(0,1)$ for every $t \geq 2a$. Using Corollary 6.1b of Widder (1946, p. 61) in (32), it follows that $\phi(x;t) = 0$ for almost every $x \in (0,1)$ and for every fixed $t \geq 2a$. This in turn implies (see (31)) that for every fixed $t \geq 2a$,

$$B(t-u)\lambda(u) - G(u)\Lambda(t-u) = 0, \tag{33}$$

for almost every $u \in (a, t-a)$. However since the functions involved are all continuous from the right, equation (33) must hold $\forall\ u \in (a, t-a)$ and $t \geq 2a$. Finally taking $u = t/2$ in (33), it follows that

$$\frac{\lambda(t)}{\Lambda(t)} = \frac{G(t)}{B(t)}, \qquad \forall\ t > a, \tag{34}$$

which after integrating both sides yields the result (ii) keeping in mind that $\Lambda(t) = B(t) = 0$ for $0 \leq t \leq a$. The continuity of $G(\cdot)$ also follows from (34) and the assumption of continuity of $\lambda(\cdot)$. Result (iii) now follows from (ii) and Lemma 1. □

Remark. Note the condition (*) of the above theorem is neither restrictive nor unrealistic though it may look that way at a first glance. In particular the condition (*) does in fact imply that $\min(\tau_1+x_1, \tau_2+x_2, \ldots) > a$ holds with probability one, where τ_i's are the epochs of arrivals. Moreover our results (i) and (ii) of theorem 3 show, under the conditions of the theorem and in particular condition (*), that $\lambda(t) = bG(t)$ for $0 \leq t \leq a$ and hence $G(t) = 0, \forall\ 0 \leq t \leq a$, so that the service times X_i's must satisfy $P(X > a) = 1$. Thus the condition (*) covers the more general case than the one with $a = 0$ where $P(X > 0) = 1$. Finally the result of theorem 3 is interesting in that it only requires that $D(t)$ have a Poisson distribution for every t, a much weaker condition compared to requiring it to be a Poisson process. Indeed there are plenty of proccesses which are not Poisson but have Poisson distribution for every t.

4. Case of an infinite servers queue with nonidentically distributed service times.

Consider an infinite servers queueing system where the customers arrive at epochs $0 \leq \tau_1 < \tau_2 < \ldots$, of an arrival process $\{A(t), t \geq 0\}$. Each customer upon arrival is served for a (finite) random length V of time with c.d.f. $G(\cdot|\tau)$, which depends upon its arrival epoch τ. The various service times $V_1, V_2, \ldots$, given $\tau_1 < \tau_2 \ldots$, are otherwise mutually independent. The function $G(v|\tau)$ is assumed to be jointly Borel-measurable with respect to its two arguments. The customers depart at the end of their service periods. Let $D(t)$ denote the number of departures occurring during $(0,t]$. Thus looking from the 'arrival end' of the 'service box', the process is completely defined by the sequence $\{(\tau_i, V_i)\}$ with $0 \leq \tau_1 < \tau_2 < \ldots$. On the other hand looking from the 'departure end' of the service box, let $0 \leq U_1 \leq U_2 \leq \ldots$ denote the successive departure epochs for the departure process

$\{D(t), t \geq 0\}$. Also let W_i denote the length of service time that *corresponds* to the ith departure at calendar time $U_i, i = 1, 2, \ldots$.

The next theorem 4 deals with the distribution problem concerning the sequence $\{(U_i, W_i)\}$ at the departure end and describes it in terms of functions $G(\cdot|\tau)$ and the mean $M(t) \equiv EA(t)$ of the arrival process $\{A(t), t \geq 0\}$, which is assumed to be Poisson. Conversely in theorem 5, distributions concerning the sequence $\{\tau_i, V_i)\}$ at the arrival end are described in terms of $H(w|u)$, the c.d.f. of W given U and $ED(t)$ of the departure process $\{D(t), t \geq 0\}$, which is now assumed to be Poisson. Together these theorems exhibit a 'reversibility' flavor between the two processes $\{A(t), t \geq 0\}$ and $\{D(t), t \geq 0\}$.

Theorem 4. *Let $\{A(t), t \geq 0\}$ be a Poisson process with $EA(t) \equiv M(t)$, which is assumed to be absolutely continuous with respect to Lebesgue measure with density function $m(t)$, so that $M(t) = \int_0^t m(u)du, t \geq 0$. Let for every τ, the conditional service time c.d.f. $G(v|\tau)$ be absolutely continuous with probability density function (p.d.f.) $g(v|\tau)$, which is assumed to be jointly Borel-measurable with respect to the arguments v and τ. Let the integral*

$$I(w|u) \equiv \int_0^w m(u-v)g(v|u-v)dv, \tag{35}$$

be finite, $\forall\ 0 < w \leq u < \infty$. Then

(i) $\{D(t), t \geq 0\}$ *is a Poisson process with*

$$ED(t) = \int_0^t m(\tau)G(t-\tau|\tau)d\tau, \qquad t \geq 0. \tag{36}$$

(ii) *The random variables $\{U_i, W_i), i = 1, 2, \ldots, n\}$ admit a joint p.d.f. given by*

(37) $f_{U_1,\ldots U_n;W_1,\ldots W_n}(u_1,\ldots,u_n;w_1,\ldots,w_n)$

$$= \left\{ \prod_{k=1}^{n} m(u_k - w_k) g(w_k | u_k - w_k) \right\} \cdot \exp\left\{ - \int_0^{u_n} G(u_n - \tau | \tau) m(\tau) d\tau \right\},$$

for $0 < u_1 < u_2 < \ldots < u_n$ *and* $0 < w_k < u_k, k = 1,2,\ldots,n$, *and is equal to zero otherwise.*

(iii) *For any* $n \geq 1$ *and arbitrary* $0 < u_1 < \ldots < u_n < \infty$ with $\mathcal{I}(u_k|u_k) > 0, \forall\, k = 1,2,\ldots,n$,

$$P(W_i \leq w_i, i = 1,2,\ldots,n | U_i = u_i, \quad i = 1,2,\ldots,n) = \prod_{k=1}^{n} H(w_k|u_k), \tag{38}$$

where for $u > 0$,

$$H(w|u) = \begin{cases} 0, & w < 0, \\ \mathcal{I}(w|u)/\mathcal{I}(u|u), & 0 \leq w < u, \\ 1, & w \geq u. \end{cases} \tag{39}$$

Proof. Result (i) is well known and can be proved by using the order statistic property of Poisson process (see Marasol (1963), Kendall (1964) and Daley (1976)). We outline the proof of (ii) as follows. Note that for $n \geq 1$,

$$U_n \leq u \Rightarrow A(u) \geq n. \tag{40}$$

Thus for $0 \leq u_1 < u_2 < \ldots < u_n < \infty$, and $0 \leq w_i \leq u_i, i = 1,2,\ldots,n$, we may express

$$\begin{aligned} &P(U_i \leq u_i, W_i \leq w_i; i = 1,2,\ldots,n) \\ &= \sum_{\ell=n}^{\infty} P(U_i \leq u_i, W_i \leq w_i; i = 1,2,\ldots,n | A(u_n) = \ell) P(A(u_n) = \ell). \end{aligned} \tag{41}$$

The probability (41) is zero if $M(u_n) = 0$. Let $M(u_n) > 0$. Using the order statistic property of the Poisson arrival process $\{A(t), t \geq 0\}$, given $A(u_n) = \ell, \ell \geq n$, the joint distribution of the arrival epochs $0 \le \tau_1 < \tau_2 < \ldots < \tau_\ell$ is same as that of the order statistic based on a random sample $T_1^*, \ldots T_\ell^*$, of size ℓ from a common distribution with p.d.f.

$$r(u) = \begin{cases} m(u)/M(u_n), & \text{for } 0 < u < u_n, \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

Given $T_i^* = t_i^*, i = 1, 2, \ldots, \ell$, let $X_1^*, \ldots, X_\ell^*$ be mutually independent r.v.'s, with p.d.f of X_i^* being $g(x^*|t_i^*), i = 1, 2, \ldots, \ell$. Let $U_i^* = T_i^* + X_i^*, i = 1, 2, \ldots, \ell$. Note that (U_i^*, X_i^*) are i.i.d. pairs, for $i = 1, 2, \ldots, \ell$, with common joint p.d.f. given by

$$g(u^*, x^*) = \frac{m(u^* - x^*)}{M(u_n)} g(x^*|u^* - x^*), \tag{43}$$

for $0 \le x^* \le u^* \le u_n + x^*$. Let $U_{(1)}^* \le U_{(2)}^* \le \ldots \le U_{(\ell)}^*$ be the order statistic based on $U_i^*, i = 1, 2, \ldots, \ell$. Also let

$$W_i^* = X_j^*, \text{ if } U_{(i)}^* = U_j^*, \quad i = 1, 2, \ldots, \ell. \tag{44}$$

Thus in view of (40) and the above mentioned order statistic property of the process $\{A(t), t \ge 0\}$ (see also Karlin (1975), p. 128–131), it is easily seen that

$$\begin{aligned} &P(U_i \le u_i, W_i \le w_i; i = 1, 2, \ldots, n | A(u_n) = \ell) \\ &= P(U_{(i)}^* \le u_i, W_i^* \le w_i; i = 1, 2, \ldots, n) \end{aligned} \tag{45}$$

This last probability can be obtained using (43). In fact using (43) the joint p.d.f. of $(U_{(i)}^*, W_i^*, i = 1, 2, \ldots, n)$ can be easily shown to be

$$\begin{aligned} &\frac{\ell!}{(\ell - n)!} \left[\int_{u_{(n)}^*}^{\infty} \int_0^{u \wedge u_n} \frac{m(y) g(u - y|y)}{M(u_n)} dy du \right]^{\ell - n} \\ &\cdot \prod_{i=1}^{n} \left\{ \frac{m(u_{(i)}^* - w_i^*)}{M(u_n)} g(w_i^* | u_{(i)}^* - w_i^*) \right\} \end{aligned} \tag{46}$$

subject to $0 < u_{(1)}^* < u_{(2)}^* < \ldots < u_{(n)}^* < u_n$ and $0 < w_i^* < u_{(i)}^*, i = 1, 2, \ldots, n$. Finally this when used in (45) and the fact that $A(u_n)$ has a Poisson distribution with mean

$M(u_n)$, yield the expresion for (41) as a $2n$-fold integral of the function given by (37) over appropriate limits for the integrals, thereby establishing the validity of (ii). The unnecessary lengthy computational details are omitted. The result (iii) now follows from (ii). □

The next theorem is a converse to the above theorem.

Theorem 5. *Let the departure process* $\{D(t), t \geq 0\}$ *be a Poisson process with* $ED(t) \equiv \Lambda(t)$*, which is assumed to be absolutely continuous with respect to Lebesgue measure with density function* $\lambda(t)$*, so that* $\Lambda(t) = \int_0^t \lambda(u)du, t \geq 0$*. Furthermore, for any* $n \geq 1$ *and arbitrary* $0 < u_1 \leq \ldots \leq u_n < \infty$*, let the relation*

$$P(W_i \leq w_i, i = 1, 2, \ldots, n | U_i = u_i, i = 1, 2, \ldots, n) = \prod_{i=1}^{n} H(w_i | u_i) \tag{47}$$

hold (whenever the left side is defined), for some jointly Borel-measurable function $H(w|u)$*, which for every* $u > 0$ *is a c.d.f. over the interval* $[0, u]$*. In addition let for every* $u > 0$*, the c.d.f.* $H(w|u)$ *be absolutely continuous in* w *with p.d.f.* $h(w|u)$*. Also let*

$$\int_0^t \int_0^\infty \lambda(v+y) h(v|v+y) dv dy \tag{48}$$

be finite, $\forall\, t \geq 0$*. Finally let*

$$J(u|t) \equiv \int_0^u \lambda(v+t) h(v|v+t) dv, \quad u \geq 0, t > 0, \tag{49}$$

and

$$J(\infty|t) \equiv \lim_{u \to \infty} J(u|t). \tag{50}$$

Then

(i) $\{A(t), t \geq 0\}$ *is a Poisson process with*

$$EA(t) = \int_0^t J(\infty|y) dy, \quad t \geq 0. \tag{51}$$

(ii) *The random variables* $\{(\tau_i, V_i), i = 1, 2, \ldots, n\}$ *admit a joint p.d.f. given by*

(52)
$$f_{\tau_1,\ldots,\tau_n;V_1,\ldots V_n}(t_1,\ldots t_n; v_1,\ldots v_n) = \left\{ \prod_{k=1}^{n} \lambda(t_k + v_k) h(v_k | t_k + v_k) \right\} \exp\{-EA(t_n)\},$$

for $0 < t_1 < t_2 < \ldots < t_n$ *and* $v_k \geq 0, k = 1, 2, \ldots, n$, *and is equal to zero elsewhere.*

(iii) *For any* $n \geq 1$ *and arbitrary* $0 < t_1 < t_2 < \ldots < t_n$ *with* $J(\infty|t_i) > 0, \forall\, k = 1, 2, \ldots, n$,

(53)
$$P(V_i \leq v_i, i = 1, 2, \ldots, n | \tau_i = t_i,\ i = 1, 2, \ldots, n) = \prod_{i=1}^{n} G(v_i|t_i),$$

where

(54)
$$G(v|t) = \begin{cases} J(v|t)/J(\infty|t), & 0 \leq v < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

Proof. Once (ii) is proven, it can be easily shown that the joint p.d.f. of $(\tau_1, \ldots, \tau_n)$ is given by

(55)
$$f_{\tau_1,\ldots,\tau_n}(t_1, \ldots, t_n) = \left[\prod_{k=1}^{b} J(\infty|t_k) \right] \cdot \exp\left\{ -\int_0^{t_n} J(\infty|y)\, dy \right\},$$

for $0 < t_1 < \ldots < t_n$, and is zero elsewhere. From this and (52), (iii) easily follows. Also from (55) it can be shown that $\{A(t), t \geq 0\}$ is a Poisson process with mean given by (51). We omit these details here. Again to prove (ii) it is sufficient to show that the probability

(56)
$$P(\tau_i \leq t_i, V_i \leq v_i, \quad i = 1, 2, \ldots, n),$$

for arbitrary $0 \leq t_1 < t_2 < \ldots < t_n$ and $v_i \geq 0, i = 1, 2, \ldots, n$, is equal to the integral of the function given by (52). To compute (56) we need first to define the random vector

(57)
$$(T^*_{(1)}(s, \omega), \ldots, T^*_{(n)}(s, \omega); V^*_1(s, \omega) \ldots, V^*_n(s, \omega))$$

for every sample point ω belonging to the set $[\tau_n < \infty]$ and for every $s > 0$. Let $\omega \in [\tau_n < \infty]$. If $D(s,\omega)$, the number of departures during $(0,s]$ is less than n, we define the above vector as $\underset{\sim}{0}$. If $D(s,\omega) = \ell \geq n$, then let

$$0 \leq U_1^*(s,\omega) \leq U_2^*(s,\omega) \leq \ldots \leq U_\ell^*(s,\omega),$$

be the departure epochs of the ℓ departures during $(0,s]$, with $W_1^*(s,\omega),\ldots,W^*(s,\omega)$, as the corresponding service time lengths satisfying $0 \leq W_i^*(s,\omega) \leq U_i^*(s,\omega), i = 1,2,\ldots,\ell$. Let

$$T_i^*(s,\omega) = U_i^*(s,\omega) - W_i^*(s,\omega), i = 1,2,\ldots,\ell, \tag{58}$$

with

$$0 \leq T_{(1)}^*(s,\omega) \leq T_{(2)}^*(s,\omega) \leq \ldots \leq T_{(\ell)}^*(s,\omega), \tag{59}$$

as their ordered values. Finally let $V_i^*(s,\omega), i = 1,2,\ldots,\ell$, be the *respective* service time lengths *associated* with these ordered values, that is

$$V_i^*(s,\omega) = W_j^*(s,\omega), \text{ if } T_{(i)}^*(s,\omega) = T_j^*(s,\omega). \tag{60}$$

Then as $s \to \infty$, since $D(s,\omega) \nearrow D(\infty,\omega) \geq n$, a.s., it follows that as $s \to \infty$,

$$(T_{(i)}^*(s), V_i^*(s); i = 1,2,\ldots,n) I_{[\tau_n<\infty]} \to (\tau_i, V_i, i = 1,2,\ldots,n) I_{[\tau_n<\infty]}, \text{ a.s.} \tag{61}$$

From this it follows that for $0 \leq t_1 < t_2 < \ldots < t_n$ and $v_i \geq 0, i = 1,2,\ldots,n$,

$$\begin{aligned} &P(\tau_i \leq t_i, V_i \leq v_i, \quad i = 1,2,\ldots,n) \\ &= \lim_{s\to\infty} P(T_{(i)}^*(s) \leq t_i, \ V_i^*(s) \leq v_i, \quad i = 1,2,\ldots,n) \\ &= \lim_{s\to\infty} P(T_{(i)}^*(s) \leq t_i, V_i^*(s) \leq v_i, \quad i = 1,2,\ldots,n; D(s) \geq n). \end{aligned} \tag{62}$$

The last equality follows from the fact that for $s \geq \max(t_i + v_i, i = 1,2,\ldots,n)$,

$$[T_{(i)}^*(s) \leq t_i, V_i^*(s) \leq v_i; \ i = 1,2,\ldots,n] \subset [D(s) \geq n]. \tag{63}$$

Now to obtain (62), as in (41) and (45), we shall need to compute the conditional probabilities

(64) $$P(T^*_{(i)}(s) \le t_i, \;\; V_i^*(s) \le v_i, \quad i = 1,2,\ldots,n | D(s) = \ell)$$

for $\ell \ge n$, which are well defined for large enough s with $ED(s) = \Lambda(s) > 0$. from here on to compute (64) we can now follow the approach adopted for the proof of theorem 4, while using the order statistic property of the Poisson process $\{D(t), t \ge 0\}$. We leave the remaining algebraic computations leading to (ii) via (62) to the reader. □

5. Concluding remarks.

(a) In Theorem 4, if $\{A(t), t \ge 0\}$ is assumed to be a mixed Poisson process (see Puri (1982) for a definition) with $E\{A(t)|\mu\} = \mu M(t)$, where μ is a positive random variable with $E(\mu) = 1$, while other conditions remaining unchanged, it turns out that the departure process $\{D(t), t \ge 0\}$ is also a mixed Poisson process with

$$E\{D(t)|\mu\} = \mu \int_0^t m(\tau) G(t - \tau|\tau) d\tau,$$

and the relation (38) still holds. Conversely in Theorem 5, if $\{D(t), t \ge 0\}$ is a mixed Poisson process, a similar result also holds for the arrival process $\{A(t), t \ge 0\}$.

(b) In section 1, the question of characterizing the arrival process $\{A(t), t \ge 0\}$ was raised given that (i) the service times of various arrivals are i.i.d. with a common d.f. $G(\cdot)$ and are independent of everything else and that (ii) the departure process $\{D(t), t \ge 0\}$ is a Poisson process. While only assuming that $D(t)$ has a Poisson distribution for every t, this question was answered in Theorem 3 under the restriction that $\{A(t), t \ge 0\}$ is a renewal process. For the general question itself, in an attempt to prove the result namely that under the above conditions (i) and (ii), the arrival process itself must be a Poisson process, the authors unfortunately had to impose an additional rather not so natural looking condition on the arrival process $\{A(t), t \ge 0\}$. This and further work, which is in progress in this direction, will be reported elsewhere.

(c) Using the lines of proof of Theorem 5, one can construct examples where the departure process is still a Poisson process, while the condition (47) is somewhat relaxed in the sense that the conditional service time distributions *for the departures* not only depend upon the departure times but also on the order in which these departure times occur. In such a case the arrival process is no longer Poisson nor does the condition (i) given above for the *service times of various arrivals* holds any more (see Huang (1983) for one such example).

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