INFERENCE ABOUT COMPARATIVE PRECISION IN LINEAR STRUCTURAL RELATIONSHIPS*

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Abstract: We study point and confidence region estimators of the precisions of instruments, and of the ratios of such precisions to the precision of a standard instrument, in situations where a nonreplicated linear structural model describes the measurements simultaneously obtained from several instruments. Some drawbacks of maximum likelihood estimators of the precision ratios are noted. A union-intersection test for the null hypothesis H_0 that the standard instrument is at least as precise as the other instruments is proposed for situations where the relative precision of the standard instrument is known.

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1. Introduction

The problem of comparing the precisions of several instruments (or methods of measurement) arises frequently in scientific and technological contexts. For example, Miller (1980) compares two methods for measuring Kanamycin levels in premature babies, while Barnett (1969) compares four instrument-operator combinations for measuring human lung function. Grubbs (1973) discusses comparison of three velocity chronographs. Numerous other examples can be found in the literature of the social, environmental, agricultural and physical sciences (see Thompson, 1963; Cochran, 1968; Jansen, 1980).

Here, precision refers to the repeatability of measurements (how close they tend to be to each other), as opposed to accuracy, which refers to how close the measurements are to the true value measured (lack of bias). In the experimental model adopted in this paper (see (1.1) below), bias can potentially be eliminated by a linear transformation (calibration) of the instrument reading; the precision of an instrument is then defined to be the inverse of the measurement error variance of the calibrated instrument.

Of the p instruments to be compared, we assign to one (instrument 0) the special role of the standard (or control) instrument. In many applications, the standard instrument is the one currently in use, and the other p-1 instruments are studied in the hope of finding a better (more precise) instrument (Jansen, 1980). Because the standard instrument has been used previously, information may be available concerning its properties.

1.1 The Model and Design.

To compare the p instruments, the following experimental design and accompanying model is frequently adopted (Cochran, 1968; Barnett, 1969;

Theobald and Mallison, 1978). It is assumed that n experimental units (persons, physical objects, etc.) are randomly selected from a population of such units. Let u_j be the true value of the quantitative characteristic to be measured for the jth unit. Each unit is measured once by all p instruments. (Note: If measurement is destructive, it is assumed that each unit can be divided into p homogeneous parts, each having true value u_j.) Let y_{ij} be the reading on the ith instrument for the jth unit. We assume that the mean of y_{ij} is linearly related to the true value u_j; that is, instrument i measures u_j, but not necessarily in the correct scale. Thus, our model is

$$y_{ij} = \alpha_i + \beta_i u_j + e_{ij}, \quad i = 0,1,\ldots,p-1, \; j = 1,\ldots,n. \qquad (1.1)$$

The errors of measurement e_{ij} are assumed to be mutually statistically independent, and also independent of the (random) true values $u_1,\ldots,u_n$. Further, for each i, the random variables $e_{i1},\ldots,e_{in}$ are assumed to have a common normal distribution with mean 0 and variance σ_i^2, $i = 0,1,\ldots,p-1$. Finally, we assume that $u_1, u_2,\ldots,u_n$ are a random sample (i.i.d.) from a normal distribution with mean μ and variance σ_u^2.

Model (1.1) is recognizable as a linear structural errors-in-variables model (Kendall and Stuart, 1979, Chapter 29), and also (Theobald and Mallinson, 1978) as a congeneric single factor model. It is easily seen that the parameters of this model are not identifiable. Indeed, the model (1.1) retains the same form if α_i is replaced by $\alpha_i - \beta_i d$, β_i by $c\beta_i$, and u_j by $c^{-1}(u_j+d)$, $i = 0,\ldots,p-1$, $j = 1,\ldots,n$, where $c, d \neq 0$. Consequently, restrictions must be placed on the parameters to identify them.

Theobald and Mallinson (1978) adopt the usual factor analytic constraints $\mu = 0$, $\sigma_u^2 = 1$. Using such constraints is equivalent to changing the definition of the quantity to be measured from u_j to the standardized value $u_j^* = \sigma_u^{-1}(u_j - \mu)$.

Alternatively, recall our earlier discussion about the standard instrument (instrument 0). If enough experience has been obtained with use of this instrument to calibrate it, then we can assume that $E(y_{0j}) = u_j$, all j, and hence that

$$\alpha_0 = 0, \quad \beta_0 = 1. \tag{1.2}$$

Note that even when assumption (1.2) is not appropriate, adopting it simply means that u_j is redefined to be $E(y_{0j})$.

The restraints $\mu = 0$, $\sigma_u^2 = 1$ and the restraints (1.2) define two of a large class of equivalent identifiable parameterizations for the model (1.1). For purposes of comparing precisions, either parameterization yields the same conclusion. It is worth noting that the slopes β_0, $\beta_1,\ldots,\beta_{p-1}$ in Theobald and Mallinson's factor analysis parametrization correspond to σ_u, $\sigma_u\beta_1,\ldots,\sigma_u\beta_{p-1}$ in the model (1.1) under the restraints (1.2) used in this paper.

1.2 Definition of Precision

If we know the parameters α_i and β_i, then the observations y_{ij}, $j = 1,\ldots,n$, on instrument i can potentially be calibrated (rescaled) to provide readings

$$y^*_{ij} = \frac{1}{\beta_i}(y_{ij} - \alpha_i) = u_j + \frac{1}{\beta_j} e_{ij} \equiv u_j + e^*_{ij} ,$$

which are unbiased for u_j, $j = 1,\ldots,n$. The measurement errors e^*_{ij} of y^*_{ij} have variance (σ_i^2/β_i^2). Since after calibration all instruments measure u_j without bias, a natural definition for the precision of instrument i is

$$\pi_i = \frac{1}{\mathrm{var}(e^*_{ij})} = \frac{\beta_i^2}{\sigma_i^2}, \quad i = 0,1,\ldots,p-1. \tag{1.3}$$

This definition of precision is widely used (Cochran, 1968; Thompson, 1963). However, other authors (e.g., Theobald and Mallinson, 1978) use the relative precision:

$$\tau_i = \pi_i \sigma_u^2, \quad i = 0, 1,\ldots,p-1. \tag{1.4}$$

The advantage of using τ_i is that τ_i is a dimensionless quantity. However, τ_i has the disadvantage of depending upon the variance σ_u^2 of u in the population of units used for the experiment; this population need not be the same as the population of units to be used in future applications of the instruments. For this reason, we prefer to use (1.3) as our measure of precision.

However, if we are comparing the precisions of instruments $1,2,\ldots,p-1$ to the precision of the standard instrument, it does not matter which definition of precision is used, since the ratios

$$\psi_i = \frac{\pi_i}{\pi_0} = \frac{\tau_i}{\tau_0}, \qquad i = 1,\ldots,p-1 \tag{1.5}$$

have the same value using either definition of precision. Notice that since we assume (see (1.2)) that $\beta_0 = 1$,

$$\pi_0 = \frac{1}{\sigma_0^2}, \ \tau_0 = \frac{\sigma_u^2}{\sigma_0^2}, \ \tau_i = \frac{\beta_i^2\sigma_u^2}{\sigma_i^2}, \quad i = 1,2,\ldots,p-1. \tag{1.6}$$

1.3 Summary of the Paper

The questions that are treated in this paper are:

(I) How to estimate $\pi = (\pi_0,\pi_1,\ldots,\pi_{p-1})'$ and $\psi = (\psi_1,\ldots,\psi_{p-1})'$;

(II) How to test the null hypothesis

$$H_0: \ \psi_i \leq 1, \quad i = 1,2,\ldots,p-1 \tag{1.7}$$

that the standard instrument is at least as precise as any other instrument, versus alternatives

$$H_1: \ \psi_i > 1, \text{ some } i.$$

If H_0 is rejected, we may also want to follow up by determining the instrument or instruments which are more precise than the standard instrument. In consequence, we are interested in finding tests of H_0 which lend themselves naturally to such follow-up procedures.

When $p = 2$ (only one instrument other than the standard instrument is included in the experiment), the model (1.1) remains nonidentifiable even after imposing the constraints (1.2). A further parametric constraint

is required. Given our earlier characterization of the standard instrument, it is reasonable to assume that prior experience with this instrument on the given population of units has yielded an accurate estimate of the relative precision $\tau_0 = \sigma_0^{-2}\sigma_u^2$ of the standard instrument (see the Appendix). Thus, we assume in Section 2 that τ_0 is known, and show how to find maximum likelihood estimators of the parameters (and also π_0, π_1 and ψ_1) under this assumption. We also construct a $100(1-\gamma)\%$ confidence interval for ψ_1, and show that H_0: $\psi_1 \leq 1$ can be tested using a test for correlation. The power function and other properties of this test are determined.

When $p \geq 3$, the model (1.1), (1.2) is identifiable. Maximum likelihood estimators $\hat{\pi}$, $\hat{\psi}$ for $\pi = (\pi_0,\pi_1,\ldots,\pi_{p-1})'$ and $\psi = (\psi_1,\ldots,\psi_{p-1})'$, respectively, can be obtained, but for $p > 3$, require use of computer algorithms. In Section 3, we use the recent results of Fuller, Pantula, and Amemiya (1982) to obtain the asymptotic ($n\to\infty$) distributions of $\hat{\pi}$ and of $\hat{\psi}$, and to construct approximate $100(1-\gamma)\%$ confidence regions for π and for ψ.

Our results in Section 3 reveal that the variance of the asymptotic marginal distribution of $\hat{\psi}_i$ increases as any of the ratios ψ_j, $j \neq i$, are allowed to become small, and converges to infinity when $\max_{j\neq 1} \psi_j$ goes to 0. Consequently, care must be used in selecting instruments to be used in comparative precision experiments. If we cannot rule out the possibility that the ψ_j, $j \neq 1$, are all small, then the maximum likelihood estimator of ψ_i can be less efficient than alternative estimators based on data taken only from instrument i and the standard instrument (provided that τ_0 is known). Similarly, the likelihood ratio test of H_0 v.s. H_1 may be less powerful

than alternative tests when imprecise instruments are included in the experiment.

Thus, in Section 4 we propose a union-intersection test (Roy, 1953) for H_0 v.s. H_1 based on simultaneously testing the subhypotheses H_{0i}: $\psi_i \leq 1$, $i = 1,\ldots,p-1$, using the test for $p = 2$ instruments obtained in Section 2. Although this test requires us to know τ_0, it has two important advantages over the likelihood ratio test: (1) it lends itself more naturally to follow up searches for more precise instruments when H_0 is rejected, and (2) its power is not affected by the inclusion of imprecise instruments in the experiment.

2. The case of two instruments

When $p = 2$, the model (1.1), (1.2) becomes

$$y_j \equiv \begin{pmatrix} y_{0j} \\ y_{1j} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_1 \end{pmatrix} + \begin{pmatrix} 1 \\ \beta_1 \end{pmatrix} u_j + \begin{pmatrix} e_{0j} \\ e_{1j} \end{pmatrix}, \quad j = 1,2,\ldots,n, \qquad (2.1)$$

where $z_j = (u_j, e_{0j}, e_{1j})'$ are i.i.d. with a trivariate normal distribution with mean vector $(\mu,0,0)'$ and covariance matrix $\mathrm{diag}(\sigma_u^2,\sigma_0^2,\sigma_1^2)$. That is,

$$z_j \sim \mathrm{MVN}_3((\mu,0,0)', \mathrm{diag}(\sigma_u^2,\sigma_0^2,\sigma_1^2)),$$

$j = 1,2,\ldots,n$. We assume that $\tau_0 = \sigma_0^{-2}\sigma_u^2$ is known.

It follows from (2.1) and our assumptions on $z_1,\ldots,z_n$ that

$$y_1,\ldots,y_n \text{ are i.i.d. } \mathrm{MVN}_2(\eta,\Omega), \qquad (2.2)$$

where

$$\eta = (\eta_0,\eta_1)', \quad \Omega = \begin{pmatrix} \omega_0^2 & \omega_0\omega_1\rho \\ \omega_0\omega_1\rho & \omega_1^2 \end{pmatrix},$$

and

$$\eta_0 = \mu, \quad \eta_1 = \alpha_1+\beta_1\mu, \quad \omega_0^2 = \sigma_0^2(1+\tau_0), \quad \omega_1^2 = \sigma_1^2+\beta_1^2\sigma_0^2\tau_0,$$

$$\rho = \beta_1 \left(\frac{\sigma_0^2\tau_0}{\sigma_1^2+\beta_1^2\sigma_0^2\tau_0}\right)^{\frac{1}{2}} \left(\frac{\tau_0}{1+\tau_0}\right)^{\frac{1}{2}}. \tag{2.3}$$

It is easily shown that the transformation (2.3) is a one-to-one onto map from the original parameter space

$$\{(\mu,\alpha_1,\beta_1,\sigma_0^2,\sigma_1^2): \ -\infty < \mu,\alpha_1,\beta_1 < \infty; \ \sigma_0^2,\sigma_1^2 > 0\}$$

of the model (2.1) to the space

$$\{(\eta,\Omega): \ -\infty < \eta_0, \ \eta_1 < \infty; \ \omega_0^2,\omega_1^2 > 0; \ \rho^2 \leq \tau_0(1+\tau_0)^{-1}\}. \tag{2.4}$$

Since $\tau_0(1+\tau_0)^{-1} < 1$, the space (2.4) is a proper subset of the usual parameter space for a bivariate normal distribution.

Let

$$\bar{y} = (\bar{y}_0, \bar{y}_1)' = \frac{1}{n} \sum_{j=1}^{n} y_j,$$

$$S = \begin{pmatrix} s_{00} & s_{01} \\ s_{01} & s_{11} \end{pmatrix} = \frac{1}{n} \sum_{j=1}^{n} (y_j - \bar{y})(y_j - \bar{y})',$$

$$r = \frac{s_{01}}{(s_{00}s_{11})^{\frac{1}{2}}}.$$

Theorem 1. The maximum likelihood estimators of η_0, η_1, ω_0^2, ω_1^2 and ρ are the following:

$$\hat{\eta}_i = \bar{y}_i, \quad \hat{\omega}_i^2 = s_{ii}(1-\hat{\rho}^2)/(1-\hat{\rho}r), \ i = 1,2,$$

$$\hat{\rho} = r\left[\min\left\{1, \frac{\tau_0}{r^2(1+\tau_0)}\right\}\right]^{\frac{1}{2}}.$$

The maximum of the likelihood is

$$[n/(2\pi e)]^n \{(1-\hat{\rho}^2)/[s_{00}s_{11}(1-\hat{\rho}r)^2]\}^{\frac{1}{2}n}.$$

Proof. Fix ρ and maximize the likelihood over the remaining parameters, using the results on p. 73 of Anderson (1958). Then maximize over ρ, $\rho^2 \leq \tau_0(1+\tau_0)^{-1}$. □

It follows from Theorem 1, and the invariance property of maximum likelihood estimation under reparameterization, that the maximum likelihood estimators of μ, α_1, β_1, σ_0^2 and σ_1^2 are

$$\hat{\mu} = \bar{y}_0, \quad \hat{\alpha}_1 = \bar{y}_1 - \hat{\beta}_1 \bar{y}_0, \quad \hat{\beta}_1 = \hat{\rho}(1+\tau_0)\tau_0^{-1} \left(s_{11}/s_{00}\right)^{\frac{1}{2}},$$

$$\hat{\sigma}_0^2 = \frac{s_{00}(1-\hat{\rho}^2)}{(1+\tau_0)(1-\hat{\rho}r)}, \quad \hat{\sigma}_1^2 = \frac{s_{11}(1-\hat{\rho}^2)(1-\hat{\rho}^2(1+\tau_0)\tau_0^{-1})}{(1-\hat{\rho}r)},$$

and the maximum likelihood estimators of the precisions π_0, π_1 and of the precision ratio $\psi_1 = \pi_0^{-1}\pi_1$ are:

$$\hat{\pi}_0 = \frac{(1+\tau_0)(1-\hat{\rho}r)}{s_{00}(1-\hat{\rho}^2)}, \quad \hat{\pi}_1 = \frac{(1-\hat{\rho}r)\hat{\rho}^2(1+\tau_0)^2}{\tau_0 s_{00}(1-\hat{\rho}^2)(\tau_0-\hat{\rho}^2(1+\tau_0))},$$

$$\hat{\psi}_1 = \frac{\hat{\rho}^2(1+\tau_0)}{\tau_0(\tau_0-\hat{\rho}^2(1+\tau_0))}.$$

Note that $\hat{\psi}_1$ is a function of the data only through $\hat{\rho}^2$, and thus only through r^2. We could obtain a $100(1-\gamma)\%$ confidence interval for ψ_1 by using the above fact to determine the c.d.f. of $\hat{\psi}_1$. However, the following approach is more convenient because of the availability of tables.

<u>$100(1-\gamma)\%$ Confidence interval for ψ_1</u>

<u>Step I</u>. Use David's (1938) tables of the c.d.f. of r to construct a $100(1-\gamma)\%$ confidence interval $[\rho_L(r), \rho_U(r)]$ for ρ. Let

$$\rho_L^*(r) = \max(\rho_L(r), -(\tfrac{\tau_0}{1+\tau_0})^{\frac{1}{2}}), \quad \rho_U^*(r) = \min(\rho_U(r), (\tfrac{\tau_0}{1+\tau_0})^{\frac{1}{2}}).$$

Step II. Note that

$$\psi_1 = \frac{(1+\tau_0)\rho^2}{\tau_0(\tau_0-(1+\tau_0)\rho^2)}$$

is strictly increasing in ρ^2 for $|\rho| \le \tau_0(1+\tau_0)^{-1}$. Hence,

$$\{c: \ c = \frac{(1+\tau_0)\rho^2}{\tau_0(\tau_0-(1+\tau_0)\rho^2)}, \ \rho_L^*(r) \le \rho \le \rho_U^*(r)\}$$

is a $100(1-\gamma)\%$ confidence interval for ψ_1.

As an example, if $n = 10$, $\gamma = .05$, $r = 0.7952$, $\tau_0 = 10$, we find that $\rho_L(r) = 0.34 = \rho_L^*(r)$, $\rho_U(r) = 0.94 = \rho_U^*(r)$, and the 95% confidence interval for ψ_1 is $[.015, 3.466]$.

Turning now to the test of

$$H_0: \ \psi_1 \le 1 \qquad \text{v.s.} \qquad H_1: \ \psi_1 > 1,$$

note that $\psi_1 \le 1$ if and only if $\rho^2 \le \tau_0^2(1+\tau_0)^{-2}$. Thus, we can equivalently test

$$H_0: \ \rho^2 \le \frac{\tau_0^2}{(1+\tau_0)^2} \qquad \text{vs.} \qquad H_1: \ \rho^2 > \frac{\tau_0^2}{(1+\tau_0)^2}.$$

Using Theorem 1, but with ρ^2 bounded by $\tau_0^2(1+\tau_0)^{-2}$ rather than $\tau_0(1+\tau_0)^{-1}$ in (2.4), we can find the maximum of the likelihood under H_0, and thus determine the likelihood ratio test statistic λ for H_0 v.s. H_1. It is

easily shown that λ is a function of the data only through $|r|$, that λ is a continuous and strictly decreasing function of $|r|$ for $|r| > \tau_0(1+\tau_0)^{-1}$, and that $\lambda = 1$ for $0 \leq |r| \leq \tau_0(1+\tau_0)^{-1}$. Consequently, the likelihood ratio test of H_0 v.s. H_1 has rejection region for H_0 of the form

$$r^2 \geq c^2, \tag{2.5}$$

where $c^2 > \tau_0^2(1+\tau_0)^{-2}$. Since the density function of r^2 is known to have monotone likelihood ratio (in terms of its parameter ρ^2), it follows that for a level γ, $0 < \gamma < 1$, test of H_0 v.s. H_1, the critical constant $c^2 = c^2(\gamma)$ in (2.5) should be chosen to satisfy

$$P\{r^2 \geq c^2 | \rho = \tau_0(1+\tau_0)^{-1}\} = \gamma. \tag{2.6}$$

The resulting test is UMP among all tests depending only upon r^2. Since r^2 is the maximal invariant in the sample space under the group of transformations

$$y_j \to \begin{pmatrix} b_0 & 0 \\ 0 & b_1 \end{pmatrix} y_j + \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \quad -\infty < a_0, a_1, b_0, b_1 < \infty,$$

and since this group of transformations (together with the induced group of transformations on the parameter space) leaves the problem of testing H_0 v.s. H_1 invariant, we conclude that the test defined by (2.5), (2.6) is UMP invariant level γ for testing H_0 v.s. H_1.

Note that c^2 satisfying (2.6) can be found using David's (1938) tables, or the recent extension of these tables by Subrahmanian and Subrahmanian (1980). The value of c^2 satisfying (2.6) is an increasing function of τ_0 (for fixed n, γ), and a decreasing function of n (for fixed τ_0, γ). The following table gives c^2 to four-decimal accuracy for $\gamma = 0.05$, $\tau_0 = 1.0, 2.0, 4.0, 6.0$, n = 10(5)30(10)50.

Table 1

Value of c^2 for $\gamma = .05$

τ_0 \ n	10	15	20	25	30	40	50
1.0	.6927	.6053	.5541	.5198	.4947	.4599	.4364
2.0	.8053	.7429	.7047	.6783	.6587	.6308	.6118
4.0	.8879	.8487	.8241	.8066	.7935	.7745	.7612
6.0	.9214	.8924	.8739	.8606	.8505	.8354	.8245

Note that when τ_0 is large (the standard instrument has large relative precision), a very large value of $|r|$ (exceeding .9) is needed to conclude that instrument 1 is more precise than the standard instrument.

We now investigate the power of the test (2.5), (2.6) at an alternative:

$$H_{1\Delta}: \ \psi_1 = 1 + \Delta, \ \Delta > 0.$$

If $H_{1\Delta}$ is true, then

$$\rho^2 = \rho^2(\Delta) = \frac{(1+\Delta)\tau_0^2}{1+(2+\Delta)\tau_0+(1+\Delta)\tau_0^2} . \tag{2.7}$$

Since the density of r^2 has monotone likelihood ratio in terms of its parameter ρ^2, and since it is easy to see that ρ^2 is increasing in Δ for fixed τ_0, and also increasing in τ_0 for fixed Δ, it follows that the power function

$$\text{Power}(\Delta, \tau_0, n, \gamma) = P\{r^2 \geq c^2 | \rho^2(\Delta)\} \tag{2.8}$$

of the test (2.5), (2.6) is increasing in Δ for fixed τ_0, n, γ, and is increasing in τ_0 for fixed Δ, n, γ. The power function is also easily seen to increase in n for fixed Δ, τ_0, γ.

We have calculated the power of the test (2.5), (2.6) for $\gamma = 0.05$, $\Delta = 3.0, 5.0$, n = 10(5)30,40,50, $\tau_0 = 1.0, 2.0, 4.0, 6.0$. The results are shown in the following table.

Table 2.

The power of the test (2.5), (2.6) for $\gamma = 0.05$, $\Delta = 3.0, 5.0$.

$\Delta = 3.0$

τ_0 \ n	10	15	20	25	30	40	50
1.0	0.13160	0.16951	0.20450	0.23751	0.26917	0.32967	0.38635
2.0	0.14222	0.18629	0.22720	0.26590	0.30301	0.37286	0.43702
4.0	0.14895	0.19738	0.24234	0.28530	0.32638	0.40407	0.47615
6.0	0.15473	0.20772	0.25807	0.30677	0.35426	0.44533	0.52850

$\Delta = 5.0$

τ_0 \ n	10	15	20	25	30	40	50
1.0	0.15512	0.20592	0.25297	0.29721	0.33931	0.41820	0.48983
2.0	0.16630	0.22380	0.27714	0.32725	0.37473	0.46207	0.53940
4.0	0.17316	0.23529	0.29281	0.34718	0.39840	0.49246	0.57559
6.0	0.17947	0.24663	0.30978	0.36971	0.42670	0.53111	0.62051

As can be seen from Table 2, the test (2.5), (2.6) is not very powerful, when n and τ_0 are small, for detecting even six-fold ($1+\Delta=6$) increases in precision of instrument 1 relative to the standard instrument. Since τ_0 is not usually under the control of the experimenter, we must be prepared to use fairly large sample sizes n when τ_0 is small.

It is known that as $n\to\infty$,

$$\sqrt{n}(r^2-\rho^2) \to N(0,\ 4\rho^2(1-\rho^2)^2), \tag{2.9}$$

in distribution. Consequently, it follows from (2.6) that if n is large and

$$c^2 = \frac{2\tau_0(1+2\tau_0)}{n^{\frac{1}{2}}(1+\tau_0)^3}\ z_\gamma + \frac{\tau_0^2}{(1+\tau_0)^2}\ , \tag{2.10}$$

then the test (2.5) has size approximately equal to γ. (Note: Here, z_γ has its usual meaning; that is, z_γ is the $100(1-\gamma)$th percentile of the standard normal distribution.) Using (2.7), (2.8) and (2.9), the

power function of the test (2.5), (2.10) can be approximated. We have used this approximation to find the power of the test (2.5), (2.10) for $\gamma = 0.05$, $\Delta = 3.0, 5.0$, $\tau_0 = 1.0, 2.0, 4.0, 6.0$, and $n = 150, 200(100)500$. The results are shown in Table 3.

Table 3.

The approximate power of the test (2.5), (2.10) for $\gamma = 0.05$, $\Delta = 3.0, 5.0$.

$\Delta = 3.0$

τ_0 \ n	150	200	300	400	500
1.0	0.78665	0.87890	0.96391	0.99002	0.99740
2.0	0.82891	0.91883	0.98413	0.99721	0.99960
4.0	0.84524	0.93514	0.99068	0.99890	1.00000
6.0	0.84892	0.93959	0.99231	0.99920	1.00000

$\Delta = 5.0$

τ_0 \ n	150	200	300	400	500
1.0	0.87893	0.95787	0.99350	0.99910	1.00000
2.0	0.92652	0.97651	0.99807	1.00000	1.00000
4.0	0.93636	0.98266	0.99991	1.00000	1.00000
6.0	0.93853	0.98472	0.99930	1.00000	1.00000

3. The case of three or more instruments

When $p \geq 3$, the model (1.1), (1.2) is identifiable without additional parametric restrictions. In vector-matrix form this model can be written as follows:

$$y_j = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} + \begin{pmatrix} 1 \\ \beta \end{pmatrix} u_j + e_j, \quad j = 1,2,\ldots,n, \tag{3.1}$$

$$\begin{pmatrix} u_j \\ e_j \end{pmatrix}, \ 1 \leq j \leq n \ \text{ are i.i.d. } \ MVN_{p+1}\left(\begin{pmatrix} \mu \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & 0 \\ 0 & D_{\sigma^2} \end{pmatrix}\right),$$

where $y_j = (y_{0j}, y_{1j},\ldots,y_{p-1,j})'$, $e_j = (e_{0j}, e_{ij},\ldots,e_{p-1,j})'$,

$$\alpha = (\alpha_1,\ldots,\alpha_{p-1})', \quad \beta = (\beta_1,\ldots,\beta_{p-1})',$$

$$D_{\sigma^2} = \operatorname{diag}(\sigma_0^2, \sigma_1^2,\ldots,\sigma_{p-1}^2).$$

Let

$$\bar{y} = (\bar{y}_0,\ldots,\bar{y}_{p-1})' = n^{-1} \sum_{j=1}^{n} y_j,$$

$$S = ((s_{uv})), \ s_{uv} = \frac{1}{n} \sum_{j=1}^{n} (y_{uj}-\bar{y}_u)(y_{vj}-\bar{y}_v), \ 0 \leq u, v \leq p-1.$$

When $p = 3$, the maximum likelihood estimators of μ, σ_u^2, α, β, and $\sigma^2 = (\sigma_0^2,\ldots,\sigma_{p-1}^2)'$ have the following form (Barnett, 1969):

$$\hat{\mu} = \bar{y}_0, \quad \hat{\sigma}_u^2 = s_{12}^{-1}s_{01}s_{02}, \quad \hat{\alpha} = (\bar{y}_1,\bar{y}_2)' - \bar{y}\,\hat{\beta}\ ,$$

$$\hat{\beta} = (\hat{\beta}_1,\hat{\beta}_2)' = \hat{\sigma}_u^{-2}(s_{01},s_{02})', \tag{3.2}$$

$$\hat{\sigma}^2 = (\hat{\sigma}_0^2,\hat{\sigma}_1^2,\hat{\sigma}_2^2)' = (s_{00},s_{11},s_{22})' - \hat{\sigma}_u^2(1,\hat{\beta}_1^2,\hat{\beta}_2^2).$$

For $p > 3$, the maximum likelihood estimators of μ, α and σ^2 have the usual form:

$$\hat{\mu} = \bar{y}_0, \quad \hat{\alpha} = (\bar{y}_1,\ldots,\bar{y}_{p-1})' - \bar{y}_0\hat{\beta}\ ,$$

$$\hat{\sigma}^2 = (\hat{\sigma}_0^2,\ldots,\hat{\sigma}_{p-1}^2)' = (s_{00},s_{11},\ldots,s_{p-1,p-1})' - \hat{\sigma}_u^2(1,\hat{\beta}_1^2,\ldots,\hat{\beta}_{p-1}^2)$$

The maximum likelihood estimators of these and the remaining parameters require solution of the equations:

$$\hat{\Sigma}_y^{-1}(S-\hat{\Sigma}_y)\,\hat{\Sigma}_y^{-1}\begin{pmatrix}1\\ \hat{\beta}\end{pmatrix} = 0$$

where

$$\hat{\Sigma}_y = \hat{\sigma}_u^2\begin{pmatrix}1\\ \hat{\beta}\end{pmatrix}\begin{pmatrix}1\\ \hat{\beta}\end{pmatrix}' + \mathrm{Diag}(\hat{\sigma}_0^2,\ldots,\hat{\sigma}_{p-1}^2).$$

Alternatively (Theobald and Mallinson, 1978), we can use standard computer algorithms (Jöreskog and Sorbom, 1982) to find maximum likelihood estimators of the parameters

$$\xi = (\xi_0,\ldots,\xi_{p-1})', \quad \lambda = (\lambda_0,\ldots,\lambda_{p-1})', \quad \theta^2 = (\theta_0^2,\ldots,\theta_{p-1}^2),$$

where $\lambda_0 \geq 0$, $\theta_i^2 \geq 0$, $0 \leq i \leq p-1$, in the factor analysis model:

$$y_j = \xi + \lambda f_j + e_j, \quad j = 1,\ldots,n,$$

$$\begin{pmatrix} f_j \\ e_j \end{pmatrix}, \quad 1 \leq j \leq n, \text{ are i.i.d. } MVN_{p+1}\left(0, \begin{pmatrix} 1 & 0 \\ 0 & D_{\theta^2} \end{pmatrix}\right),$$

where $D_{\theta^2} = \text{diag}(\theta_0^2,\ldots,\theta_{p-1}^2)$. Then, since as noted in Section 1 there is a correspondence between the parameters of this model and the model (1.1),

$$\hat{\sigma}_u^2 = \hat{\lambda}_0^2, \quad \hat{\beta} = \hat{\lambda}_0^{-1}(\hat{\lambda}_1,\ldots,\hat{\lambda}_{p-1})', \quad \hat{\sigma}_i^2 = \hat{\theta}_i^2, \quad 0 \leq i \leq p-1.$$

Because of the computational difficulty in finding the maximum likelihood estimators for $p > 3$, Barnett (1969) suggests method-of-moments estimators for the parameters based on the maximum likelihood solutions for $p = 3$.

Once the maximum likelihood estimators of β and σ^2 have been obtained, the maximum likelihood estimators of the precisions π_i and precision ratios ψ_i are calculated as follows:

$$\begin{aligned} \hat{\pi}_0 &= \hat{\sigma}_0^{-2}, \quad \hat{\pi}_i = \hat{\sigma}_i^{-2}\,\hat{\beta}_i^2, \quad i = 1,\ldots,p-1, \\ \hat{\psi}_i &= \hat{\sigma}_i^{-2}\,\hat{\beta}_i^2\,\hat{\sigma}_0^2, \quad i = 1,\ldots,p-1. \end{aligned} \tag{3.3}$$

3.1 Asymptotic Theory

For notational convenience, we adopt the convention that for any vector $t = (t_1,\ldots,t_r)'$,

$$D_t = \mathrm{diag}(t_1,\ldots,t_r) = \begin{pmatrix} t_1 & & 0 \\ & t_2 & \\ & & \ddots \\ 0 & & t_r \end{pmatrix}.$$

Thus, $D_\beta = \mathrm{diag}(\beta_1,\ldots,\beta_{p-1})$, $D_{\sigma^2} = \mathrm{diag}(\sigma_0^2,\ldots,\sigma_{p-1}^2)$. The symbol 0 will be used to represent the scalar zero, a zero vector, or a matrix of zeroes; the dimensions of such a matrix will always be clear from the context.

Fuller, Pantula, and Amemiya (1982) have considered a generalization of the model (3.1) in which u_j is a k-dimensional vector of normal variables and β is an $r\times k$ matrix of unknown parameters. They have shown that the maximum likelihood estimators are consistent and jointly asymptotically normal ($n\to\infty$), and have given an explicit expression for the covariance matrix of this asymptotic multivariate normal distribution. When $k = 1$, $r = p-1$, their results can be simplified into the following form.

Lemma 1. Under the model (3.1),

$$\sqrt{n}\begin{pmatrix} \hat{\beta} - \beta \\ \hat{\sigma}^2 - \sigma^2 \end{pmatrix} \to \mathrm{MVN}_{2p-1}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} V_{\beta\beta} & V_{\beta\sigma^2} \\ V'_{\beta\sigma^2} & V_{\sigma^2\sigma^2} \end{pmatrix}\right)$$

in distribution as $n \to \infty$, where

$$V_{\beta\beta} = W + PV_{\sigma^2\sigma^2}P', \quad V_{\beta\sigma^2} = -PV_{\sigma^2\sigma^2},$$

$$W = \frac{1}{\sigma_u^2}\left(1 + \frac{1}{\tau_0 H}\right)\left[\mathrm{diag}(\sigma_1^2,\ldots,\sigma_{p-1}^2) + \sigma_0^2\,\beta\beta'\right]$$

$$P = \frac{1}{\tau_0 H}\left[\mathrm{diag}(\sigma_1^2,\ldots,\sigma_{p-1}^2)\right]^{-1}(-\beta, D_\beta)$$

$$H = \sum_{j=0}^{p-1}\psi_j, \quad \psi_0 \equiv 1, \quad V_{\sigma^2\sigma^2} = 2D_{\sigma^2}Q^{-1}D_{\sigma^2},$$

and $Q = ((Q_{ij}))$ is defined by

$$H^2Q_{ij} = \begin{cases} \psi_i\psi_j, & i \neq j \\ (H-\psi_i)^2, & i = j, \quad i,j = 0,1,\ldots,p-1. \end{cases}$$

We are interested in the asymptotic properties of the maximum likelihood estimators $\hat{\pi} = (\hat{\pi}_0,\ldots,\hat{\pi}_{p-1})'$, $\hat{\psi} = (\hat{\psi}_1,\ldots,\hat{\psi}_{p-1})'$ of $\pi = (\pi_0,\ldots,\pi_{p-1})'$ and $\psi = (\psi_1,\ldots,\psi_{p-1})'$, respectively.

Theorem 2. Under the model (3.1)

$$n^{\frac{1}{2}}(\hat{\pi}-\pi) \to \mathrm{MVN}_p(0,L)$$

in distribution as $n\to\infty$, where

$$L = \frac{4}{\sigma_u^2}\left(1+\frac{1}{\tau_0 H}\right)(D_\pi+\pi\pi') + 2\,TQ^{-1}T',$$

$$T = \left(1+\frac{2}{\tau_0 H}\right)D_\pi - \frac{2}{\tau_0 H}(0,\pi).$$

Also,

$$n^{\frac{1}{2}}(\hat{\psi}-\psi) \to MVN_{p-1}(0,M)$$

in distribution as $n\to\infty$, where

$$M = \frac{4}{\tau_0}(1+\frac{1}{\tau_0 H})(D_\psi+\psi\psi') + 2(1+\frac{2}{\tau_0 H})^2(D_\psi, -\psi)Q^{-1}(D_\psi,-\psi)'.$$

Proof. It follows from (3.3) that

$$\hat{\pi} - \pi = \begin{pmatrix} D_{\hat{\beta}} + D_\beta \\ 0 \end{pmatrix} [\mathrm{diag}(\hat{\sigma}_1^2,\ldots,\hat{\sigma}_{p-1}^2)]^{-1}(\hat{\beta}-\beta)$$

$$-\begin{pmatrix} D_\beta & 0 \\ 0 & 1 \end{pmatrix}^2 D_{\sigma^2}^{-1} D_{\hat{\sigma}^2}^{-1}(\hat{\sigma}^2-\sigma^2),$$

and that

$$\hat{\psi}-\psi=\hat{\sigma}_0^2(D_{\hat{\beta}}+D_\beta)[\mathrm{diag}(\hat{\sigma}_1^2,\ldots,\hat{\sigma}_{p-1}^2)^{-1}(\hat{\beta}-\beta)$$

$$-\sigma_0^2[\mathrm{diag}(\hat{\sigma}_1^2,\ldots,\hat{\sigma}_{p-1}^2)]^{-1}D_\beta(-\beta,D_\beta)D_{\sigma^2}^{-1}(\hat{\sigma}^2-\sigma^2).$$

Since $\hat{\beta}$ and $\hat{\sigma}^2$ are consistent etsimators of β and σ^2 respectively, the conclusions of the theorem follow from Lemma 1 and Slutzky's Theorem, after considerable algebraic simplification. □

The covariance matrices L and M of the asymptotic distributions of $n^{\frac{1}{2}}(\hat{\pi}-\pi)$ and $n^{\frac{1}{2}}(\hat{\psi}-\psi)$, respectively, depend on Q^{-1}, which is difficult

to express in a single explicit form which holds for all values of the ψ_i's. One exception is the case p = 3, where

$$Q^{-1} = \frac{H}{2\psi_1\psi_2} \begin{pmatrix} H & 0 & 0 \\ 0 & \psi_1^2 & 0 \\ 0 & 0 & \psi_2^2 \end{pmatrix} + 1_3 1_3' \qquad (3.4)$$

where $1_3 = (1,1,1)'$.

3.2. Asymptotic confidence regions

To construct joint asymptotic 100(1-γ)% confidence regions for the vector parameters π or ψ, we could use the Scheffé method to form confidence ellipsoids:

$$\{\pi: \ n(\hat{\pi}-\pi)' \hat{L}^{-1}(\hat{\pi}-\pi) \leq \chi^2_{p,\gamma}\},$$

$$\{\psi: \ n(\hat{\psi}-\psi)' \hat{M}^{-1}(\hat{\psi}-\psi) \leq \chi^2_{p-1,\gamma}\},$$

where $\chi^2_{r,\gamma}$ is the 100(1-γ) percentile of the χ^2_r-distribution, and $\hat{L}$, $\hat{M}$ are consistent estimates of L, M formed by replacing the parameters π, ψ, σ_u^2, τ_0^2 in the expressions for L and M in Theorem 2 by their maximum likelihood estimators. (Note: $\hat{\tau}_0^2 = \hat{\sigma}_0^{-2}\hat{\sigma}_u^2$.) However, in most practical applications these regions will only be used to provide simultaneous 100(1-γ)% confidence intervals for the individual π_i's or the individual ψ_i's. For this purpose the Scheffé method yields intervals which are much too broad. Instead, using Theorem 2 and

Sidák's (1968) Corollary 3, we can construct the rectangles

$$\hat{\pi}_i - n^{-\frac{1}{2}} z_{\gamma_1} \hat{\ell}_{ii}^{\frac{1}{2}} \le \pi_i \le \hat{\pi}_i + n^{-\frac{1}{2}} z_{\gamma_1} \hat{\ell}_{ii}^{\frac{1}{2}}, \quad 0 \le i \le p-1,$$

$$\hat{\psi}_i - n^{-\frac{1}{2}} z_{\gamma_2} \hat{m}_{ii}^{\frac{1}{2}} \le \psi_i \le \hat{\psi}_i + n^{-\frac{1}{2}} z_{\gamma_2} \hat{m}_{ii}^{\frac{1}{2}}, \quad 1 \le i \le p-1, \tag{3.5}$$

where

$$\gamma_1 = \tfrac{1}{2}(1-(1-\gamma)^{1/p}), \gamma_2 = \tfrac{1}{2}(1 - (1-\gamma)^{1/(p-1)}) . \tag{3.6}$$

z_γ is the $100(1-\gamma)$th percentile of the standard normal distribution, and $\hat{\ell}_{ii}$, $\hat{m}_{ii}$ are the ith diagonal elements of $\hat{L}$, $\hat{M}$, respectively. (That is, $\hat{\ell}_{ii}$ is a consistent estimator of the variance of the asymptotic distribution of $n^{\frac{1}{2}}(\hat{\pi}_i - \pi_i)$, and $\hat{m}_{ii}$ is a consistent estimator of the variance of the asymptotic distribution of $n^{\frac{1}{2}}(\hat{\psi}_i - \psi_i)$.) The rectangles (3.5) provide $100(1-\gamma)\%$ simultaneous confidence regions for the elements of π and ψ, respectively, and yield simultaneous confidence intervals for π_i, $0 \le i \le p-1$, and for ψ_i, $1 \le i \le p-1$, which are shorter than those provided by the Scheffé method.

Note: Alternatively, we could let $\gamma_1 = (2p)^{-1}\gamma$, $\gamma_2 = (2(p-1))^{-1}\gamma$ in (3.5) The resulting Bonferroni rectangles are slightly wider than the rectangles (3.5) with γ_1, γ_2 defined by (3.6); see Seber (1977, Chapter 5), Miller (1981). If the matrices L and M can be shown to have a convenient structure, the rectangles (3.5) can be narrowed through use of more precise inequalities than that of Šidák; see Tong (1980).

3.3. A comment on the variance of the maximum likelihood estimators of $\psi_1,\ldots,\psi_{p-1}$

Recall that m_{ii}, the ith diagonal element of M, is the variance of the asymptotic (normal) distribution of $n^{\frac{1}{2}}(\hat{\psi}_i-\psi_i)$, $i = 1,\ldots,p-1$. When $p = 3$, it follows from (3.4) and Theorem 2 that

$$m_{ii} = \frac{4}{\tau_0}\left(1 + \frac{1}{\tau_0(1+\psi_1+\psi_2)}\right)\psi_i(1+\psi_i)$$

$$+ \left(1 + \frac{2}{\tau_0(1+\psi_1+\psi_2)}\right)^2(1+\psi_1+\psi_2)\,\frac{\psi_i^2(1+\psi_i)^2}{\psi_1\psi_2},$$

$i = 1,2$. Note that m_{11} depends not only upon ψ_1, but also upon ψ_2 and τ_0, and that m_{11} can be arbitrarily large when either ψ_2 is small (instrument 2 has small precision relative to the standard instrument) or τ_0 is small (the relative precision of the standard instrument is small). Similar remarks hold for m_{22}.

When $p > 3$, each m_{ii} depends not only on ψ_i, but also on ψ_j, $j \neq i$, and on τ_0. It can be shown that $m_{ii} \to \infty$ if either $\tau_0 \to 0$ or $\max_{j\neq i} \psi_j \to 0$, and that m_{ii} is increased by decreasing either τ_0 or any ψ_j, $j\neq i$. Consequently, the accuracy of maximum likelihood estimation of any ψ_i is adversely affected by the inclusion of any imprecise instruments in the experiment described in Section 1.

This observation partially justifies Barnett's (1969) decision to forgo maximum likelihood estimation of the parameters of the model (3.1) in favor of estimators which involve data taken from only three instruments at a time. Unfortunately, our criticism can be also levelled (as we have seen) at the maximum likelihood estimators

for the case p = 3. We can try to use the data to judge which instruments are the most precise (so as to always estimate ψ_i from a triad in which the third instrument, other than instruments 0 and i, is a precise one). However, if our estimators are fallible, our choice of a precise isntrument to use may also be fallible. It would be preferable to use information of which we are more certain. In many cases, as noted in Section 1, we may have a good estimate, based on previous experience, of τ_0. This suggests making use of the results of Section 2 (the case p = 2) to estimate and compare the precisions. Indeed, if we know the value of τ_0, there is no need to use all instruments simultaneously on the same units - a series of experiments in which each new instrument (i = 1,2,...,p-1) is tested with the standard instrument may do as well. We hope to study this design question in our future research.

Of course, one advantage of the experiment described in Section 1 is that (particularly when p is large) it can lead to a good estimator of τ_0. This may be helpful when such an estimator is not previously available.

The discussion above also suggests that the likelihood ratio test of H_0: $\psi_i \leq 1$, i = 1,...,p-1 versus H_1: $\psi_i > 1$, some i, may have poor power in cases where some of the instruments included in the experiment for comparison to the standard instrument are imprecise relative to the standard (but where at least one instrument is more precise than the standard). For this reason, and because of its appropriateness for follow-up determination of more precise instruments than the standard when H_0 is rejected, in Section 4 we propose a union-intersection test for H_0 based on the correlation test described in Section 2.

4. A union-intersection Test of H_0 when τ_0 is known

Note that the null hypothesis H_0 that the standard instrument is at least as precise as any of the other instruments can be written as

$$H_0 = \bigcap_{i=1}^{p-1} H_{0i}, \quad H_{0i}: \ \psi_i \leq 1, \ i = 1,\ldots,p-1. \tag{4.1}$$

If we assume that τ_0 is known, then the theory of union-intersection tests (Roy, 1953) and the results of Section 2 suggest the following rejection region for H_0:

$$\max_{1\leq i\leq p-1} r_i^2 \geq c^2, \tag{4.2}$$

where

$$r_i^2 = \frac{s_{0i}^2}{s_{00}s_{ii}} .$$

Theorem 3. Let r have the distribution of the sample correlation statistic based on a random sample of size n from a bivariate normal population with correlation $\rho = \tau_0^{\frac{1}{2}}(1+\tau_0)^{-\frac{1}{2}}$. If c^2 in (4.2) is chosen to satisfy

$$P\{r^2 \geq c^2\} = 1 - (1-\gamma)^{1/(p-1)}, \tag{4.3}$$

then the test of H_0 defined by the rejection region (4.2) has level of significance no greater than γ, $0 < \gamma < 1$.

Proof. It is straightforward to show that

$$x = s_{00}^{-\frac{1}{2}} (s_{01},\ldots,s_{0p-1})' = (x_1,\ldots,x_{p-1})'$$

has, conditional on $y_{01},\ldots,y_{0n}$, a (p-1)-variate normal distribution with mean vector $s_{00}^{\frac{1}{2}}\ \tau_0(1+\tau_0)^{-1}\ \beta$ and covariance matrix

$$\Xi = \mathrm{diag}(\sigma_1^2,\ldots,\sigma_{p-1}^2) + \sigma_0^2\ \tau_0^2(1+\tau_0)^{-1}\ \beta\beta',$$

and is independent of

$$q_i^2 = s_{ii} - \frac{s_{0i}^2}{s_{00}}, \quad i = 1,\ldots,p-1,$$

where $q_1^2,\ldots,q_{p-1}^2$ have the joint distribution of the diagonal entries of a (p-1)-dimensional Wishart matrix with n-2 degrees of freedom and covariance parameter Ξ. Note that

$$r_i^2 \le c^2, \quad i = 1,\ldots,p-1 \quad \Leftrightarrow \quad \frac{r_i^2}{1-r_i^2} \le \frac{c^2}{1-c^2}, \quad i = 1,\ldots,p-1,$$

and that

$$\frac{r_i^2}{1-r_i^2} = \frac{x_i^2}{q_i^2}, \quad i = 1,\ldots,p-1.$$

Note also that the conditional joint distribution of x, $q_1^2,\ldots,q_{p-1}^2$ given $y_{0i}\ldots, y_{0n}$ depends on the y_{0i}'s only through s_{00}.

Thus,

$$P\{r_i^2 \leq c^2,\ i = 1,\ldots,p-1 \mid H_0\} = P\{\frac{|x_i|}{q_i} \leq \left(\frac{c^2}{1-c^2}\right)^{\frac{1}{2}},\ i = 1,\ldots,p-1\}$$

$$= E_{s_{00}}[P(A|s_{00})]$$

where

$$A = \{\frac{|x_i|}{q_i} \leq \left(\frac{c^2}{1-c^2}\right)^{\frac{1}{2}},\ i = 1,\ldots,p-1\}.$$

Further,

$$P(A|s_{00}) = E_{q_1^2,..,q_{p-1}^2}\ [P(A|q_1^2,..,q_{p-1}^2,s_{00})].$$

It is easily seen that the mean vector $s_{00}^{\frac{1}{2}}\ \tau_0(1+\tau_0)^{-1}\ \beta$ and covariance matrix $\equiv$ of x given s_{00} satisfy the conditions of Theorem 3.1 of Das Gupta, Eaton, et al (1972) for every partition of x into components $\dot{x}$, $\ddot{x}$. Consequently, using induction, it can be shown that

$$P(A|q_1^2,\ldots,q_{p-1}^2,s_{00}) \geq \prod_{i=1}^{p-1} P\{\frac{|x_i|}{q_i} \leq \left(\frac{c^2}{1-c^2}\right)^{\frac{1}{2}} |q_i^2,s_{00}\}.$$

Then, noting that $\equiv$ has the form of Example 4.1 in Karlin and Rinott (1981) and that $P\{|x_i| \leq (c^2/(1-c^2))^{\frac{1}{2}} q_i | q_i^2, s_{00}\}$ is increasing in q_i, $i = 1,\ldots,p-1$, we can use the arguments of Section 6 of that paper

to show that

$$E_{q_1^2,\ldots,q_{p-1}^2}\left[\prod_{i=1}^{p-1} P\{\frac{|x_i|}{q_i} \leq \left(\frac{c^2}{1-c^2}\right)^{\frac{1}{2}} | q_i^2, s_{00}\}\right]$$

$$\geq \prod_{i=1}^{p-1} \left[P\{\frac{|x_i|}{q_i} \leq \left(\frac{c^2}{1-c^2}\right)^{\frac{1}{2}} | s_{00}\}\right].$$

Finally, since each

$$P\{\frac{|x_i|}{q_i} \leq \left(\frac{c^2}{1-c^2}\right)^{\frac{1}{2}} | s_{00}\}$$

is decreasing in s_{00}, repeated use of Lemma 3 of Lehmann (1966) shows that

$$E_{s_{00}}\left[\prod_{i=1}^{p-1} P\{\frac{|x_i|}{q_i} \leq \left(\frac{c^2}{1-c^2}\right)^{\frac{1}{2}} | s_{00}\}\right]$$

$$\geq \prod_{i=1}^{p-1} E_{s_{00}}\left[P\{\frac{|x_i|}{q_i} \leq \left(\frac{c^2}{1-c^2}\right)^{\frac{1}{2}} | s_{00}\}\right]$$

$$= \prod_{i=1}^{p-1} P\{r_i^2 \leq c^2 | H_0\}.$$

However, by (4.3) and the results of Section 2,

$$P\{r_i^2 \leq c^2 | H_0\} \geq (1-\gamma)^{1/(p-1)}, \quad i = 1,\ldots,p-1.$$

Consequently,

$$P\{\max_{1\le i\le p-1} r_i^2 \ge c^2 | H_0\} = 1 - P\{r_i^2 \le c^2,\ i = 1,\ldots,p-1 | H_0\}$$

$$\le 1 - \prod_{i=1}^{p-1} (1-\gamma)^{1/(p-1)} = \gamma,$$

proving the assertion of the theorem. □

When n is large, we can let

$$c^2 = \frac{2\tau_0(1+2\tau_0)}{n^{\frac{1}{2}}(1+\tau_0)^3} z_\xi + \frac{\tau_0^2}{(1+\tau_0)^2}, \tag{4.4}$$

where $\xi = 1 - (1-\gamma)^{1/(p-1)}$ and z_ξ is the $100(1-\xi)$th percentile of the standard normal distribution. This fact, and also the large-sample representation of the power of the test defined by (4.2), (4.4), follows from the following theorem and its corollary.

Theorem 4. Let

$$\rho_i^2 = \frac{\psi_i \tau_0^2}{(1+\tau_0)(1+\psi_i\tau_0)}.$$

Then

$$n^{\frac{1}{2}}(r_1^2-\rho_1^2,\ldots,r_{p-1}^2-\rho_{p-1}^2)' \to MVN_{p-1}(0,C)$$

in distribution as $n\to\infty$, where $C = ((c_{ik}))$,

$$c_{ii} = 4\rho_i^2(1-\rho_i^2)^2, \qquad i = 1,\ldots,p-1,$$

$$c_{ik} = \frac{4\rho_i^2\rho_k^2}{\tau_0^2}\left[\tau_0(1+\tfrac{1}{2}\tau_0)(1-\rho_i^2)(1-\rho_k^2) + \tfrac{1}{2}\rho_i^2\rho_k^2\right], \quad i,k = 1,\ldots,p-1,\ i \neq k.$$

Corollary. For large enough n,

$$p\left\{\max_{1\le i\le p-1} r_i^2 \ge c^2 \mid \tau_0, \psi_1,\ldots,\psi_{p-1}\right\}$$

$$\approx 1 - \Phi_{p-1}(n^{\frac{1}{2}}(c^2-\rho_1^2)\,c_{11}^{-\frac{1}{2}},\ldots,n^{\frac{1}{2}}(c^2-\rho_{p-1}^2)\,c_{p-1,p-1}^{-\frac{1}{2}})$$

$$\le 1 - \prod_{i=1}^{p-1}\Phi(n^{\frac{1}{2}}(c^2-\rho_i^2)c_{ii}^{-\frac{1}{2}}),$$

where $\Phi_{p-1}(z)$, $z = (z_1,\ldots,z_{p-1})'$ is the joint c.d.f. of a (p-1)-variate normal distribution with mean vector 0 and covariance matrix $((c_{ik}/c_{ii}^{\frac{1}{2}}c_{kk}^{\frac{1}{2}}))$, and $\Phi(\cdot)$ is the c.d.f. of a standard normal random variable.

Proof of Theorem 4. We can use the results in Section 3 of Olkin and Siotani (1964) to show that $n^{\frac{1}{2}}(r_1-\rho_1,\ldots,r_{p-1}-\rho_{p-1})'$ has asymptotically ($n\to\infty$) a (p-1)-variate normal distribution with mean vector 0 and a certain covariance matrix which depends upon ρ_i, $i = 1,\ldots,p-1$, and also on the correlations ρ_{ik} between y_{ij} and y_{kj}, $i \neq k$, $i,k = 1,\ldots,p-1$, any $j = 1,\ldots,n$. However, the fact that

$$\rho_{ik} = \tau_0^{-1}(1+\tau_0)\rho_i\rho_k$$

simplifies matters considerably, and the conclusion of the theorem follows from a standard Taylor's series argument. □

Proof of Corollary. Since the correlations $c_{ik}c_{ii}^{-\frac{1}{2}}c_{kk}^{-\frac{1}{2}}$ are all nonnegative (see Theorem 4), the stated inequality is a consequence of Slepian's (1962) inequality. □

One advantage of using the union-intersection test (4.2) of H_0 is that when H_0 is rejected, we can search for instruments i which are more precise than the standard instrument. Thus, we can agree that instrument i is more precise than the standard instrument if $r_i^2 \geq c^2$. If H_0 is rejected, at least one such $i \neq 0$ must exist.

Note. Since we are assuming here that the value of τ_0 is known, we can construct confidence rectangles for $\psi_1,\ldots,\psi_{p-1}$, alternative to those given in Section 3, by simultaneously constructing the confidence intervals for $\psi_1,\ldots,\psi_{p-1}$ given in Section 2, each at confidence level $(1-\gamma)^{1/(p-1)}$. That the resulting confidence rectangle formed from such intervals has joint confidence $1-\gamma$ follows from arguments similar to those used to prove Theorem 3.

Appendix.

Since we assume in Sections 2 and 4 that τ_0 is known, we indicate how past experience with the standard instrument can be used to obtain an accurate estimate of τ_0. Assume that the standard instrument has been used K, $K > 1$, times on each of m units selected from the target population of units. Thus we observe

$$y_{0jk} = u_j + e_{0jk}, \qquad k = 1,\ldots,K, \quad j = 1,\ldots,m,$$

where the e_{0jk} are i.i.d. $N(0,\sigma_0^2)$ variables independent of $u_1,\ldots,u_m$, and where $u_1,\ldots,u_m$ are i.i.d. $N(\mu,\sigma_u^2)$ variables. Let $\bar{y}_{0j}$ be the average of the measurements on unit j and $\bar{y}_{0\cdot\cdot}$ be the overall average. Let

$$(m-1)s_1^2 = K \sum_{j=1}^{m} (\bar{y}_{0j\cdot} - \bar{y}_{0\cdot\cdot})^2, \qquad m(K-1)s_2^2 = \sum_{j=1}^{m} \sum_{k=1}^{K} (y_{0jk} - \bar{y}_{0j\cdot})^2.$$

Then when $m(K-1) > 4$,

$$\tilde{\tau}_0 = \frac{1}{K}\left(\frac{m(K-1)-2}{m(K-1)} \frac{s_1^2}{s_2^2} - 1\right)$$

is an unbiased estimator of τ_0 with variance

$$\mathrm{var}(\tilde{\tau}_0) = \frac{(1+K\tau_0)^2}{K^2} \frac{2(mK-3)}{(m-1)(m(K-1)-4)}.$$

In some applications, it is assumed that instead of the value of τ_0, the values of certain other parameters are known. For example, it may be assumed that $\beta_1 = \beta_2 = \ldots = \beta_{p-1} = 1$ (Grubbs, 1973), or that the error-variance ratios $\sigma_i^2\sigma_0^{-2}$, $1 \le i \le p-1$, are known. In each of these two cases, a union intersection test of H_0: $\psi_i \le 1$, $i = 1, 2,\ldots,p-1$, using a test statistic based on the sample correlations taken between each instrument and the standard instrument can be constructed (see Shyr (1984)).

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