LOCALLY OPTIMAL SUBSET SELECTION RULES BASED ON RANKS UNDER JOINT TYPE II CENSORING

by

Shanti S. Gupta and Ta Chen Liang
Purdue University

Technical Report #84-33

Department of Statistics
Purdue University

September 1984
(Revised January 1986)

*This research was supported by the Office of Naval Research Contract N00014-84-0167 at Purdue University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

LOCALLY OPTIMAL SUBSET SELECTION RULES BASED ON RANKS UNDER JOINT TYPE II CENSORING

Shanti S. Gupta and TaChen Liang

Received:

Abstract. This paper deals with the derivation of subset selection rules which locally maximize the probability of a correct selection among all invariant subset selection rules satisfying the basic condition that the probability of a correct selection is at least equal to a specified value P*. Based on the ranks under the joint type II censoring, a locally optimal subset selection rule is derived. The property of local monotonicity related to this selection rule is also studied.

AMS 1980 subject classifications. 62 F 07, 62 G 99

Key words and phrases. Subset selection, locally optimal, locally strongly monotone, joint type II censoring.

1. Introduction

Let $\pi_1,\ldots,\pi_k$ be k ($\geq$ 2) independent populations where π_i has the associated distribution function $F(x, \theta_i)$ and density $f(x, \theta_i)$ with the unknown parameter θ_i belonging to an interval (a, b) of the real line. Our goal is to select a subset (preferably small in size) of the k populations $\pi_1,\ldots, \pi_k$ that will contain the best (suitably defined) among them.

In practice, it sometimes happens that the actual values of the random variables can only be observed under great cost, or not at all, while their ordering is readily observable. This occurs for instance in life-testing when one only observes the order in which the parts under investigation fail without being able to record the actual time of failure. In problems of this type, one may desire to investigate decision rules based on ranks.

In dealing with the goal specified above, Gupta and McDonald [3] studied three classes of subset selection rules based on ranks for selecting a subset containing the best among k populations when the underlying distributions are unknown. When the form of the underlying distribution is known but the values of the parameter θ_i, $i = 1,\ldots,k$, are unknown, Gupta, Huang and Nagel [2] studied some locally optimal subset selection rules based on ranks. The latter study leads to the conclusion that the class of subset selection rules R_3 of Gupta and McDonald [3] is locally optimal in some sense. Huang and Panchapakesan [5] also studied the problem of deriving some subset selection rules, based on ranks, which are locally optimal in the sense that the rules have the property of strong monotonicity. All the studies mentioned above only considered the situation where the ranks are completely observed.

We now consider a problem as follows: Suppose that there are k different devices and we want to select the best among them. From each kind of device, say π_i, n prototypes are taken for experiment and the $N = kn$ prototypes are simultaneously put on a life test. Due to design reasoning or cost consideration, the experiment terminates as soon as the first r failures among the N devices are observed for some predetermined value r, where $1 \leq r \leq N$. Based on these r observations, we want to ascertain which device is associated with the largest (expected) lifetime. Since we are only concerned with the first r failures, no matter what devices, we call this censoring scheme as a joint type II censoring.

In the case of hypothesis testing, locally most powerfully rank tests (LMPR) based on censored data have been considered in a very general setup by Basu, Ghosh and Sen [1]. In particular, these authors have provided a useful lemma. This lemma enables one to construct LMPR tests based on data from various censoring schemes.

In this paper, we are interested in deriving subset selection rules which satsify the basic P*-condition and locally maximize the probability of a correct selection among all invariant subset selection rules based on ranks under the joint type II censoring. We assume that the functional form of the density function $f(x,\theta)$ is known but the value of the parameter θ is unknown. In Section 2, the problem is formulated. Following the earlier setup of Gupta, Huang and Nagel [2], a locally optimal subset selection rule R_1 is derived in Section 3. The property of local monotonicity related to the rule R_1 is also discussed in Section 4.

2. Formulation of the problem

Let $\pi_1,\ldots, \pi_k$ be k ($\geq$ 2) populations and let $f(x, \theta_i)$ be the density function associated with the population π_i for i = 1,...,k. Let $\theta_{[1]} \leq\cdots\leq \theta_{[k]}$ be the ordered parameters $\theta_1,\ldots, \theta_k$. Of course, the correct pairing of the ordered and unordered θ_i is unknown to us. The population associated with $\theta_{[k]}$ is called the best population. In case of a tie, one of the contenders is tagged and is called the best. Let X_{ij}, j = 1,...,n, be independent observations from π_i and let R_{ij} denote the rank of X_{ij} in the pooled sample of N (= kn) observations so that the smallest observation has rank 1 and the largest one has rank N.

A rank configuration is defined as an N-tuple $\underset{\sim}{\Delta} = (\Delta_1,\ldots,\Delta_N)$, $\Delta_i \in \{1,\ldots,k\}$ where $\Delta_i = j$ means that the ith smallest observation in the pooled sample comes from π_j. Let $\mathscr{L} = \{\underset{\sim}{\Delta}\}$ denote the set of all rank configurations.

Let r be a predetermined integer such that $1 \le r \le N$. Under the joint type II censoring scheme, only the first r smallest observations in the pooled sample of size N (X_{ij}, $j = 1,\ldots,n$; $i = 1,\ldots,k$) are observed. For the preassigned value r, let C_r be a function defined on $\mathcal{L}$ such that for each $\underset{\sim}{\Delta} = (\Delta_1,\ldots,\Delta_N) \in \mathcal{L}$, $C_r(\underset{\sim}{\Delta}) = (\Delta_1,\ldots,\Delta_r) = \underset{\sim}{\Delta}(r)$. Let $\mathcal{L}_r = C_r(\mathcal{L})$.

Hence, for each $\underset{\sim}{\Delta}(r) \in \mathcal{L}_r$, $\max(0, r-(k-1)n) \le r_i \equiv \sum_{j=1}^{r} I_{\{i\}}(\Delta_j) \le \min(r,n)$ for each $i = 1,\ldots,k$, and $\sum_{i=1}^{k} r_i = r$. We call $\underset{\sim}{\Delta}(r)$ a joint type II censored rank configuration. For each $\underset{\sim}{\Delta}(r) \in \mathcal{L}_r$, define the set $\mathcal{L}(\underset{\sim}{\Delta}(r)) = \{\underset{\sim}{\Delta} \in \mathcal{L} \mid C_r(\underset{\sim}{\Delta}) = \underset{\sim}{\Delta}(r)\}$. Let $|A|$ denote the number of elements in the set A. Then, $|\mathcal{L}(\underset{\sim}{\Delta}(r))| = \prod_{m=1}^{k} \binom{N-r-c_m}{n-r_m}$, where $c_m = \sum_{i=1}^{m-1} (n-r_i)$, $m = 1,\ldots,k$ and $\sum_{i=1}^{0} \equiv 0$. Also, $\sum_{\underset{\sim}{\Delta}(r)\in\mathcal{L}_r} |\mathcal{L}(\underset{\sim}{\Delta}(r))| = N!/(n!)^k$.

For the observed censored rank configuration $\underset{\sim}{\Delta}(r)$, let $\alpha_i(\underset{\sim}{\Delta}(r))$ denote the probability of including population π_i in the selected subset.

<u>DEFINITION 2.1</u>. A subset selection rule R based on the censored ranks is a measurable mapping from $\mathcal{L}_r$ into $[0, 1]^k$ such that

$$R(\underset{\sim}{\Delta}(r)) = (\alpha_1(\underset{\sim}{\Delta}(r)),\ldots,\alpha_k(\underset{\sim}{\Delta}(r))).$$

Let $P_i(\underset{\sim}{\theta})$ denote the probability of including the population π_i in the selected subset when $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k)$ are the true parameters. That is, $P_i(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}[\alpha_i(\underset{\sim}{\Delta}(r))]$ where the expectation is over the set $\mathcal{L}_r$. Any decision that includes the selection of the best population is called a correct selection (CS). The probability of a correct selection is denoted by $P_{\underset{\sim}{\theta}}(CS|R)$ when the subset selection rule R is applied.

Let G denote the group of permutations g of the integers $1,\ldots,k$. We write $g(1,\ldots,k) = (g1,\ldots,gk)$. Let h denote the inverse of g and define $g(\theta_1,\ldots,\theta_k) = (\theta_{h1},\ldots,\theta_{hk})$. For each $\underset{\sim}{\Delta} \in \mathscr{L}$, $\underset{\sim}{\Delta}(r) \in \mathscr{L}_r$, let $\bar{g}$ and $\bar{\bar{g}}$ be defined by $\bar{g}\underset{\sim}{\Delta} = (g\Delta_1,\ldots,g\Delta_N)$ and $\bar{\bar{g}}\underset{\sim}{\Delta}(r) = (g\Delta_1,\ldots,g\Delta_r)$, respectively. Thus, both $\bar{g}$ and $\bar{\bar{g}}$ are induced from g. Let $\bar{G} = \{\bar{g}\}$ and $\bar{\bar{G}} = \{\bar{\bar{g}}\}$. It is easy to see that $C_r(\bar{g}\underset{\sim}{\Delta}) = \bar{\bar{g}}(C_r(\underset{\sim}{\Delta}))$. Also, $\underset{\sim}{\Delta} \in \mathscr{L}(\underset{\sim}{\Delta}(r))$ iff $\bar{g}\underset{\sim}{\Delta} \in \mathscr{L}(\bar{\bar{g}}\underset{\sim}{\Delta}(r))$. Hence, $|\mathscr{L}(\underset{\sim}{\Delta}(r))| = |\mathscr{L}(\bar{\bar{g}}\underset{\sim}{\Delta}(r))|$ for each $\underset{\sim}{\Delta}(r) \in \mathscr{L}_r$ and for each $\bar{\bar{g}} \in \bar{\bar{G}}$.

DEFINITION 2.2. A subset selection rule R on $\mathscr{L}_r$ is invariant under permutation if and only if $(\alpha_1(\bar{\bar{g}}\underset{\sim}{\Delta}(r)),\ldots,\alpha_k(\bar{\bar{g}}\underset{\sim}{\Delta}(r))) = g(\alpha_1(\underset{\sim}{\Delta}(r)),\ldots,\alpha_k(\underset{\sim}{\Delta}(r)))$ for all $\underset{\sim}{\Delta}(r) \in \mathscr{L}_r$, $g \in G$ and $\bar{\bar{g}}$ induced from g.

Let $f(x, \theta_i)$ be the density function associated with population π_i, with the parameter θ_i belonging to some interval (a, b) of the real line, where $-\infty \le a < b \le \infty$. Let $\Omega = \{\underset{\sim}{\theta} | \underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k)\}$, $\Omega_0 = \{\underset{\sim}{\theta} \in \Omega | \theta_1 = \ldots = \theta_k\}$. Furthermore, let the density $f(x, \theta)$ have the following properties:

Condition A:

(i) $f(x, \theta)$ is absolutely continuous in θ for every x;

(ii) $\dot{f}(x, \theta) = \frac{\partial}{\partial\theta} f(x, \theta)$ exists and is continuous in θ for every x;

(iii) $\lim_{\theta\to\theta_0} \int_{-\infty}^{\infty} |\dot{f}(x, \theta)|dx = \int_{-\infty}^{\infty} |\dot{f}(x, \theta_0)|dx < \infty$ holds for every $\theta_0 \in (a, b)$.

Now, under the assumptions of Condition A, our goal is to derive an invariant subset selection rule R, based on the joint type II censored ranks, such that

(i) $\inf_{\underset{\sim}{\theta}_0 \in \Omega_0} P_{\underset{\sim}{\theta}_0}(CS|R) = P^*$ where $P^* \in (\frac{1}{k}, 1)$ is prespecified;

(ii) $P_{\underset{\sim}{\theta}}(CS|R)$ is as large as possible for all $\underset{\sim}{\theta}$ in a neighborhood of $\underset{\sim}{\theta}_0 \in \Omega_0$.

Note that for each $\underset{\sim}{\theta}_0 \in \Omega_0$, $P_{\underset{\sim}{\theta}_0}(CS|R)$ will be interpreted as the probability of selecting a specified population.

3. <u>A locally optimal subset selection rule</u>

For each $\underset{\sim}{\theta} \varepsilon \Omega$, $\underset{\sim}{\Delta}(r) \varepsilon \mathcal{L}_r$, let $P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}(r))$ denote the probability that the joint type II censored rank configuration $\underset{\sim}{\Delta}(r)$ is observed under $\underset{\sim}{\theta}$. Also, let $P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta})$, $\underset{\sim}{\Delta} \varepsilon \mathcal{L}$, denote the probability that the complete rank configuration $\underset{\sim}{\Delta}$ is observed under $\underset{\sim}{\theta}$. Then,

$$(3.1) \qquad P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}) = (n!)^k \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \prod_{j=1}^{N} f(x_j, \theta_{\Delta_j}) dx_1 \ldots dx_N,$$

and

$$(3.2) \qquad P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}(r)) = \sum_{\underset{\sim}{\Delta} \varepsilon \mathcal{L}(\underset{\sim}{\Delta}(r))} P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}).$$

Let $\underset{\sim}{\theta}_0 = (\theta_0, \ldots, \theta_0) \varepsilon \Omega_0$, where $\theta_0 \varepsilon (a, b)$. By applying a simple algebraic computation, $P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta})$ can be written as follows:

$$(3.3) \qquad P_{\underset{\sim}{\theta}}(\underset{\sim}{\Delta}) = (n!)^k [A_0(\theta_0) + \sum_{i=1}^{k} (\theta_i - \theta_0) A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, \underset{\sim}{\theta})]$$

where $A_0(\theta_0) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} \prod_{j=1}^{N} f(x_j, \theta_0) dx_1 \ldots dx_N = \frac{1}{N!}$ which is independent of θ_0,

$$(3.4) \qquad A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, \underset{\sim}{\theta}) = \sum_{\substack{j=1 \\ \Delta_j = i}}^{N} \int_{-\infty}^{\infty} \int_{-\infty}^{x_N} \cdots \int_{-\infty}^{x_2} q(i,j,\underset{\sim}{\theta}_0,\underset{\sim}{\theta},\underset{\sim}{x}) dx_1 \ldots dx_N$$

for each $i = 1,\ldots,k$, where $\underset{\sim}{x} = (x_1,\ldots,x_N)$ and

$$q(i,j,\underset{\sim}{\theta}_0,\underset{\sim}{\theta},\underset{\sim}{x}) = \frac{f(x_j,\theta_i) - f(x_j,\theta_0)}{\theta_i - \theta_0} \prod_{m=1}^{j-1} f(x_m,\theta_0) \prod_{m=j+1}^{N} f(x_m,\theta_{\Delta_m}).$$

Here, we define $\prod_{j=1}^{0} \equiv 1$, $\prod_{j=N+1}^{N} \equiv 1$ and $[f(x_j,\theta_i) - f(x_j,\theta_0)]/(\theta_i - \theta_0) = 0$ if $\theta_i = \theta_0$.

Let $\underset{\sim}{\theta}_0 \varepsilon \Omega_0$ and let $||\underset{\sim}{\theta} - \underset{\sim}{\theta}_0|| = \max_{1 \leq i \leq k} |\theta_i - \theta_0|$. Thus, if $\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k)$ is in the neighborhood of $\underset{\sim}{\theta}_0$ with $\theta_i \neq \theta_0$ for all $i = 1,\ldots,k$, then, under

Condition A, following an argument analogous to a theorem (page 71) of Hajek and Sidak [4], we have

$$\text{(3.5)} \qquad \lim_{||\underset{\sim}{\theta}-\underset{\sim}{\theta}_0||\to 0} A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, \underset{\sim}{\theta}) = A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = \sum_{\substack{j=1\\ \Delta_j=i}}^{N} B_j(\underset{\sim}{\theta}_0),$$

and

$$|A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)| \le \sum_{\substack{j=1\\ \Delta_j=i}}^{N} |B_j(\underset{\sim}{\theta}_0)| < \infty$$

for each $i = 1,\ldots,k$, where

$$\text{(3.6)} \quad B_j(\underset{\sim}{\theta}_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_N}\cdots\int_{-\infty}^{x_2} \dot{f}(x_j,\theta_0)\prod_{\substack{m=1\\ m\neq j}}^{N} f(x_m,\theta_0)dX_1 \ldots dX_N.$$

That is, there exists an $\varepsilon > 0$ such that as $0 < ||\underset{\sim}{\theta} - \underset{\sim}{\theta}_0|| < \varepsilon$, $A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, \underset{\sim}{\theta})$ is approximately equal to $A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$ for each $i = 1,\ldots,k$.

We see that $\sum_{j=1}^{k} A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = \sum_{m=1}^{N} B_m(\underset{\sim}{\theta}_0)$. Also, under the Condition A, it is easy to see that $\sum_{m=1}^{N} B_m(\underset{\sim}{\theta}_0) = 0$. Hence, $\sum_{j=1}^{k} A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = 0$.

<u>LEMMA 3.1</u>. Let $\underset{\sim}{\theta} \in \Omega$ and let $P_i(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}[\alpha_i(\underset{\sim}{\Delta}(r))]$ be the probability of including population π_i in the selected subset under $\underset{\sim}{\theta}$ by applying an invariant subset selection rule R. Let $G(i) = \{g \in G|gi=i\}$. Then,

$$P_i(\underset{\sim}{\theta}) = \sum_{\underset{\sim}{\Delta}(r)\in\mathcal{L}_r}\left[\frac{(n!)^k}{N!}|\mathcal{L}(\underset{\sim}{\Delta}(r))| + \frac{(n!)^k}{(k-1)!}W(\underset{\sim}{\Delta}(r),\underset{\sim}{\theta}, \underset{\sim}{\theta}_0, G(i))\right]\alpha_i(\underset{\sim}{\Delta}(r))$$

where

$$W(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}, \underset{\sim}{\theta}_0, G(i)) = \sum_{\underset{\sim}{\Delta}\in\mathcal{L}(\underset{\sim}{\Delta}(r))}\ \sum_{g\in G(i)}\ \sum_{j=1}^{k}(\theta_{hj}-\theta_0)A_j(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, g\underset{\sim}{\theta}),$$

h is the inverse of $g \in G(i)$ and $\underset{\sim}{\theta}_0 = (\theta_0,\ldots,\theta_0) \in \Omega_0$.

Proof: This lemma can be verified by following an argument analogous to that of Gupta, Huang and Nagel [2]. We omit the details here.

LEMMA 3.2. Suppose that the density function $f(x, \theta)$ satisfies the requirements of Condition A. Let $G(i) = \{g \varepsilon G | gi = i\}$. Then,

$$\sum_{g \varepsilon G(i)} \sum_{j=1}^{k} (\theta_{hj} - \theta_0) A_j^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0) = (k-2)!(k\theta_i - U) A_i^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)$$

for each $i = 1, \ldots, k$, for each $\underset{\sim}{\theta} \varepsilon \Omega$, $\underset{\sim}{\theta}_0 \varepsilon \Omega_0$ where $U = \sum_{j=1}^{k} \theta_j$ and h is the inverse of $g \varepsilon G(i)$ and $\underset{\sim}{A} \varepsilon \mathscr{L}$.

Proof: Note that

$$\sum_{g \varepsilon G(i)} \sum_{j=1}^{k} (\theta_{hj} - \theta_0) A_j^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)$$

$$= \sum_{\substack{j=1 \\ j \neq i}}^{k} A_j^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0) \sum_{g \varepsilon G(i)} \theta_{hj} + A_i^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0) \sum_{g \varepsilon G(i)} \theta_{hi}$$

$$= (k-2)!(U - \theta_i) \sum_{\substack{j=1 \\ j \neq i}}^{k} A_j^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0) + (k-1)! \theta_i A_i^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)$$

$$= (k-2)!(k\theta_i - U) A_i^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)$$

where the first and the last inequalities are due to the fact that $\sum_{j=1}^{k} A^*{}_j(\underset{\sim}{A}, \underset{\sim}{\theta}_0) = 0$. This completes the proof of Lemma 3.2.

THEOREM 3.1. Let $\underset{\sim}{\theta} \varepsilon \Omega$ be any point in the neighborhood of $\underset{\sim}{\theta}_0 \varepsilon \Omega_0$. Let $P_i(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}[\alpha_i(\underset{\sim}{A}(r))]$ be the probability of including population π_i in the selected subset under $\underset{\sim}{\theta}$ by applying an invariant subset selection rule R. Then, under Condition A, for each $i = 1, 2, \ldots, k$,

$$(3.7) \qquad P_i(\underset{\sim}{\theta}) \approx E_{\underset{\sim}{\theta}_0}\{[1 + \frac{(k\theta_i - U)N!}{k-1} T_i^*(\underset{\sim}{A}(r), \underset{\sim}{\theta}_0)] \alpha_i(\underset{\sim}{A}(r))\},$$

where the expectation is taken over the set $\mathscr{L}_r$, and

$$(3.8) \qquad T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) = \frac{1}{|\mathscr{L}(\underset{\sim}{\Delta}(r))|} \sum_{\underset{\sim}{\Delta}\varepsilon\mathscr{L}(\underset{\sim}{\Delta}(r))} A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0).$$

Proof: Note that $||\underset{\sim}{\theta}-\underset{\sim}{\theta}_0|| = ||g\underset{\sim}{\theta}-\underset{\sim}{\theta}_0||$ for all $g \varepsilon G$. Then, by Condition A and (3.5), we can choose $\varepsilon > 0$ so small that as $||\underset{\sim}{\theta}-\underset{\sim}{\theta}_0|| < \varepsilon$, $A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, g\underset{\sim}{\theta}) \approx A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$ for all $g \varepsilon G$ and so $(\theta_{hj}-\theta_0)A_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, g\underset{\sim}{\theta}) \approx (\theta_{hj}-\theta_0)A_i^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$ for all $g \varepsilon G$ where h is the inverse of g. Thus, if $||\underset{\sim}{\theta}-\underset{\sim}{\theta}|| < \varepsilon$, we have

$$\sum_{g\varepsilon G(i)} \sum_{j=1}^{k} (\theta_{hj}-\theta_0)A_j(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0, g\underset{\sim}{\theta})$$

$$(3.9) \qquad \approx \sum_{g\varepsilon G(i)} \sum_{j=1}^{k} (\theta_{hj}-\theta_0)A_j^*(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0)$$

$$= (k-2)!(k\theta_i-U)A_i^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)$$

where the last equality is due to Lemma 3.2. Then, from Lemma 3.1, (3.9), and a straightforward computation, the proof follows.

Now, define subset selection rule R_1 as follows:

$$(3.10) \qquad \alpha_i(\underset{\sim}{\Delta}(r)) = \begin{cases} 1 & \text{if } T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) > c(\underset{\sim}{\theta}_0); \\ \rho(\underset{\sim}{\theta}_0) & \text{if } T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) = c(\underset{\sim}{\theta}_0); \\ 0 & \text{if } T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) < c(\underset{\sim}{\theta}_0); \end{cases}$$

where the constants $c(\underset{\sim}{\theta}_0)$ and $\rho(\underset{\sim}{\theta}_0)$, $(0 \leq \rho(\underset{\sim}{\theta}_0) < 1)$, depend on the parameter θ_0, and can be determined by

$$(3.11) \quad P_{\underset{\sim}{\theta}_0}\{T_1^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) > c(\underset{\sim}{\theta}_0)\} + \rho(\underset{\sim}{\theta}_0)P_{\underset{\sim}{\theta}_0}\{T_1^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0) = c(\underset{\sim}{\theta}_0)\} = P^*.$$

We then have the following theorem.

<u>THEOREM 3.2</u>. Suppose that the density function $f(x,\theta)$ satisfies Condition A. Then, the subset selection rule R_1 maximizes $P_{\underset{\sim}{\theta}}(CS|R)$ in a neighborhood of $\underset{\sim}{\theta}_0 \varepsilon \Omega_0$, among all invariant subset selection rules, based on the joint type II censored ranks, satisfying $\inf_{\underset{\sim}{\theta}\varepsilon\Omega_0} P_{\underset{\sim}{\theta}}(CS|R) = P^*$.

Proof: Without loss of generality, we assume that π_k is the best population. Then by Theorem 3.1, for any $\underset{\sim}{\theta} \in \Omega$ in a neighborhood of $\underset{\sim}{\theta}_0 \in \Omega_0$,

$$P_{\underset{\sim}{\theta}}(CS|R) = P_k(\underset{\sim}{\theta})$$

$$\approx E_{\underset{\sim}{\theta}_0}\{[1 + \frac{(k\theta_k - U)N!}{k-1} T_k^*(\underset{\sim}{A}(r), \underset{\sim}{\theta}_0)]\alpha_k(\underset{\sim}{A}(r))\}.$$

Since $k\theta_k - U = \sum_{j=1}^{k-1} (\theta_k - \theta_j) \geq 0$, then by Neyman-Pearson lemma, we conclude this theorem.

4. Locally strong monotonicity of the subset selection rule R_1

Let R be a subset selection rule and $P_i(\underset{\sim}{\theta})$ be the associated probability of including population π_i in the selected subset for each i = 1,...,k, when $\underset{\sim}{\theta}$ is the true parameter. By the definition of $P_i(\underset{\sim}{\theta})$,

$$(4.1) \qquad P_i(\underset{\sim}{\theta}) = \sum_{\underset{\sim}{A}(r)\in\mathcal{L}_r} \left[\sum_{\underset{\sim}{A}\in\mathcal{L}(\underset{\sim}{A}(r))} P_{\underset{\sim}{\theta}}(\underset{\sim}{A})\right] \alpha_i(\underset{\sim}{A}(r))$$

where $P_{\underset{\sim}{\theta}}(\underset{\sim}{A})$ is defined in (3.1).

For each $\underset{\sim}{\theta}_0 \in \Omega_0$, under Condition A, $\left.\frac{\partial P_{\underset{\sim}{\theta}}(\underset{\sim}{A})}{\partial \theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0}$ exists and is equal to $(n!)^k A_j^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)$ for each j = 1,...,k. Therefore, for each j = 1,...,k,

$$(4.2) \qquad \left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} = (n!)^k \sum_{\underset{\sim}{A}(r)\in\mathcal{L}_r} \left[\sum_{\underset{\sim}{A}\in\mathcal{L}(\underset{\sim}{A}(r))} A_j^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)\right] \alpha_i(\underset{\sim}{A}(r)).$$

<u>DEFINITION 4.1</u>. A subset selection rule R is locally strongly monotone at point $\underset{\sim}{\theta}_0 \in \Omega_0$ if for each i = 1,...,k, $\left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_i}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \geq 0$ and $\left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial \theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \leq 0$ for all $j \neq i$.

The following lemmas are needed for deriving the locally strong monotonicity of the subset selection rule R_1.

LEMMA 4.1. Let $g \varepsilon G$ and $\bar{g} \varepsilon \bar{G}$, where $\bar{g}$ is induced from g. Then, for any $\underset{\sim}{\Delta} \varepsilon \mathscr{L}$, $\underset{\sim}{\Delta}(r) \varepsilon \mathscr{L}_r$, $i = 1,\ldots,k$, we have

$$A^*_{gi}(\bar{g}\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = A^*_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0).$$

Proof: From (3.5), we have

$$A^*_i(\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0) = \sum_{\substack{j=1 \\ \Delta_j = i}}^{N} B_j(\underset{\sim}{\theta}_0) = \sum_{\substack{j=1 \\ g\Delta_j = gi}}^{N} B_j(\underset{\sim}{\theta}_0)$$

$$= \sum_{\substack{j=1 \\ (\bar{g}\underset{\sim}{\Delta})_j = gi}}^{N} B_j(\underset{\sim}{\theta}_0) = A^*_{gi}(\bar{g}\underset{\sim}{\Delta}, \underset{\sim}{\theta}_0).$$

From (3.5), (3.8) and (3.10), we see that for the subset selection rule R_1, $\alpha_i(\underset{\sim}{\Delta}(r))$ depends on $\underset{\sim}{\Delta}(r)$ only through whether $\Delta_j = i$ or not for each $j = 1,\ldots,r$. Now, according to the value of $\alpha_i(\underset{\sim}{\Delta}(r))$, the set $\mathscr{L}_r$ can be partitioned into three classes, say, $\mathscr{L}_r = \mathscr{L}^i_r(0) \cup \mathscr{L}^i_r(1) \cup \mathscr{L}^i_r(\rho(\underset{\sim}{\theta}_0))$ where $\mathscr{L}^i_r(\beta) = \{\underset{\sim}{\Delta}(r) \varepsilon \mathscr{L}_r | \alpha_i(\underset{\sim}{\Delta}(r)) = \beta\}$ for $\beta = 0$, 1 or $\rho(\underset{\sim}{\theta}_0)$.

LEMMA 4.2. Let $g \varepsilon G(i)$ and $\bar{\bar{g}} \varepsilon \bar{\bar{G}}$, where $\bar{\bar{g}}$ is induced from g. Then

$$\bar{\bar{g}}(\mathscr{L}^i_r(\beta)) = \mathscr{L}^i_r(\beta) \quad \text{for} \qquad \beta = 0, 1 \text{ or } \rho(\underset{\sim}{\theta}_0).$$

Proof: For $g \varepsilon G(i)$, g does not change the position of index i; thus, for each $\underset{\sim}{\Delta}(r) \varepsilon \mathscr{L}_r$, $\alpha_i(\bar{\bar{g}}\underset{\sim}{\Delta}(r)) = \alpha_i(\underset{\sim}{\Delta}(r))$. Therefore, for each $\beta = 0$, 1 or $\rho(\underset{\sim}{\theta}_0)$, $\bar{\bar{g}}(\mathscr{L}^i_r(\beta)) \subset \mathscr{L}^i_r(\beta)$. Also, $\bar{\bar{g}}(\mathscr{L}_r) = \mathscr{L}_r$. Thus, if $\bar{\bar{g}}(\mathscr{L}^i_r(\beta)) \subsetneq \mathscr{L}^i_r(\beta)$ for some β, we then have $\bar{\bar{g}}(\mathscr{L}_r) \subsetneq \mathscr{L}_r$ which is a contradiction. Therefore, $\bar{\bar{g}}(\mathscr{L}^i_r(\beta)) = \mathscr{L}^i_r(\beta)$ for each $\beta = 0$, 1 or $\rho(\underset{\sim}{\theta}_0)$.

LEMMA 4.3. For each fixed i and $m \neq i$, $j \neq i$, we have

$$\sum_{\underset{\sim}{A}(r)\varepsilon\mathcal{L}_r^i(\beta)} \left[\sum_{\underset{\sim}{A}\varepsilon\mathcal{L}(\underset{\sim}{A}(r))} A_j^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)\right] \alpha_i(\underset{\sim}{A}(r))$$

$$= \sum_{\underset{\sim}{A}(r)\varepsilon\mathcal{L}_r^i(\beta)} \left[\sum_{\underset{\sim}{A}\varepsilon\mathcal{L}(\underset{\sim}{A}(r))} A_m^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)\right] \alpha_i(\underset{\sim}{A}(r))$$

for $\beta = 0$, 1 or $\rho(\underset{\sim}{\theta}_0)$.

Proof: Let $g \varepsilon G(i)$ with $gj = m$. Then,

$$\sum_{\underset{\sim}{A}(r)\varepsilon\mathcal{L}_r^i(\beta)} \left[\sum_{\underset{\sim}{A}\varepsilon\mathcal{L}(\underset{\sim}{A}(r))} A_j^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)\right] \alpha_i(\underset{\sim}{A}(r))$$

$$= \beta \sum_{\underset{\sim}{A}(r)\varepsilon\mathcal{L}_r^i(\beta)} \left[\sum_{\underset{\sim}{A}\varepsilon\mathcal{L}(\underset{\sim}{A}(r))} A_{gj}^*(\bar{g}\underset{\sim}{A}, \underset{\sim}{\theta}_0)\right] \quad \text{(by Lemma 4.1)}$$

$$= \beta \sum_{\underset{\sim}{A}(r)\varepsilon\bar{\bar{g}}\mathcal{L}_r^i(\beta)} \left[\sum_{\underset{\sim}{A}\varepsilon\mathcal{L}(\underset{\sim}{A}(r))} A_m^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)\right]$$

$$= \beta \sum_{\underset{\sim}{A}(r)\varepsilon\mathcal{L}_r^i(\beta)} \left[\sum_{\underset{\sim}{A}\varepsilon\mathcal{L}(\underset{\sim}{A}(r))} A_m^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0)\right] \quad \text{(by Lemma 4.2).}$$

This completes the proof of Lemma 4.3.

From (4.2) and Lemma 4.3, we can see that for each fixed i,

$$\left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial\theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} = \left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial\theta_m}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0}, \quad \text{for } m \neq i,\ j \neq i. \tag{4.3}$$

THEOREM 4.1. Suppose that the density function $f(x,\theta)$ satisfies Condition A. Then, the subset selection rule R_1 is locally strongly monotone at each $\underset{\sim}{\theta}_0 \varepsilon \Omega_0$.

Proof: By (4.3) and the fact that $\sum_{j=1}^{k} A_j^*(\underset{\sim}{A}, \underset{\sim}{\theta}_0) = 0$, for each $m \neq i$,

$$\left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial\theta_m}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} = \frac{1}{k-1}\sum_{\substack{j=1\\ j\neq i}}^{k} \left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial\theta_j}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0}$$

$$= \frac{(n!)^k}{k-1}\sum_{\underset{\sim}{\Delta}(r)\in\mathcal{L}_r}\left[\sum_{\underset{\sim}{\Delta}\in\mathcal{L}(\underset{\sim}{\Delta}(r))}\sum_{\substack{j=1\\ j\neq i}}^{k} A_j^*(\underset{\sim}{\Delta},\underset{\sim}{\theta}_0)\right]\alpha_i(\underset{\sim}{\Delta}(r))$$

$$= -\frac{1}{k-1}\frac{\partial}{\partial\theta_i}P_i(\underset{\sim}{\theta})\Big|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0}.$$

Therefore, it suffices to prove that $\frac{\partial}{\partial\theta_i} P_i(\underset{\sim}{\theta})\Big|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \geq 0$ for each $\underset{\sim}{\theta}_0 \in \Omega_0$. Now, by (3.5), (3.8) and a straightforward computation,

$$\sum_{\underset{\sim}{\Delta}(r)\in\mathcal{L}_r} |\mathcal{L}(\underset{\sim}{\Delta}(r))|\, T_i^*(\underset{\sim}{\Delta}(r), \underset{\sim}{\theta}_0)$$

$$(4.4) \qquad = \sum_{\underset{\sim}{\Delta}\in\mathcal{L}}\sum_{j=1}^{N} B_j(\underset{\sim}{\theta}_0) I_{\{i\}}(\Delta_j)$$

$$= \frac{(N-1)!}{(n!)^{k-1}(n-1)!}\sum_{j=1}^{N} B_j(\underset{\sim}{\theta}_0)$$

$$= 0,$$

since $\sum_{j=1}^{N} B_j(\underset{\sim}{\theta}_0) = 0$ under Condition A.

Then, by (3.10), (4.2) and (4.4), we see that $\left.\frac{\partial P_i(\underset{\sim}{\theta})}{\partial\theta_i}\right|_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_0} \geq 0$. Hence, the subset selection rule R_1 is locally strongly monotone at each $\underset{\sim}{\theta}_0 \in \Omega_0$.

Acknowledgement

This research was supported by the Office of Naval Research Contract N00014-84-C-0167 at Purdue University, West Lafayette, IN, U.S.A.

References

[1] Basu, A. P., Ghosh, J. K. and Sen, P. K.: A unified way of deriving LMP rank tests from censored data. J. Roy. Statist. Soc. Ser. B 45, 384-390 (1983).

[2] Gupta, S. S., Huang, D. Y. and Nagel, K.: Locally optimal subset selection procedures based on ranks. In J. S. Rustagi (ed.): Optimizing Methods in Statistics, Academic Press, New York, 251-260 (1979).

[3] Gupta, S. S. and McDonald, G. C.: On some classes of selection procedures based on ranks. In M. L. Puri (ed.): Nonparametric Techniques in Statistical Inference, Cambridge University Press, London, 491-514 (1970).

[4] Hajek, J. and Sidak, Z.: Theory of Rank Tests. Academic Press, New York (1967).

[5] Huang, D. Y. and Panchapakesan, S.: Some locally optimal subset selection rules based on ranks. In S. S. Gupta and J. O. Berger (eds.): Statistical Decision Theory and Related Topics III, Vol. 2, Academic Press, New York, 1-14 (1982).

Shanti S. Gupta
Department of Statistics
Purdue University
West Lafayette, Indiana 47907
U.S.A.

TaChen Liang
Department of Mathematics
Southern Illinois University
Carbondale, Illinois 62901
U.S.A.

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: Technical Report #84-33	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Locally Optimal Subset Selection Rules Based on Ranks Under Joint Type II Censoring	5. TYPE OF REPORT & PERIOD COVERED: Technical
	6. PERFORMING ORG. REPORT NUMBER: Technical Report #84-33
7. AUTHOR(s): Shanti S. Gupta and Ta Chen Liang	8. CONTRACT OR GRANT NUMBER(s): N00014-84-C-0167
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Purdue University, Department of Statistics, West Lafayette, IN 47907	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Office of Naval Research, Washington, D.C.	12. REPORT DATE: September 1984
	13. NUMBER OF PAGES: 14
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Subset Selection; Locally Optimal; Locally Strongly Monotone, Joint Type II Censoring

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper deals with the derivation of subset selection rules which satisfy the basic P*-condition and which locally maximize the probability of a correct selection among all invariant subset selection rules based on the ranks under the joint type II censoring. Following the earlier setup of Gupta, Huang and Nagel (1979), a locally optimal subset selection rule R_1 is derived. The property of local monotonicity related to the rule R_1 is discussed.

DD FORM 1 JAN 73 1473

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)