COMPARATIVE PRECISION IN LINEAR STRUCTURAL RELATIONSHIPS*

by

Jing Y. Shyr
Purdue University

Technical Report #84-30

Department of Statistics
Purdue University

August 1984

*This work was partially supported by National Science Foundation Grant No. DMS-8121948.

CHAPTER 1

INTRODUCTION

1.0 The Experimental Model and Definition of Precision

The problem of comparing the precisions of several measuring instruments, or methods of measurement, arises in many practical and scientific contexts. Miller (1980), for example, considers the problem of comparing two methods for measuring Kanamycin levels in premature babies. An example where four instrument - operator combinations designed to measure human lung function are compared is discussed by Barnett (1969). Grubbs (1973) gives a non-medical example in which three velocity chronographs are compared. Numerous other examples occur in educational and psychological measurement (see Lord and Novick, 1968), environmental monitoring, and in the physical and agricultural sciences.

The concept of "precision" must be distinguished from that of "accuracy". Precision refers to the repeatability of measurements (how close they tend to be to each other), while accuracy refers to how close the measurements are to the true value measured. Thus, accuracy is related to lack of bias, while precision is related to the size of random errors of measurement. (Synonyms used in the psychological measurements literature for precision and accuracy are "reliability" and "validity", respectively.) An analogy given by Murphy (1969) may help clarify the distinction between precision and accuracy. We can regard the measurement situation as being similar to that of a marksman aiming at a target. If the marksman can place all his (or her) shots in a rather small circle, then we would call him a precise marksman. However, the center of the circle may be far from the bullseye of the target. In this case, the marksman would be precise, but inaccurate. It is also possible that the marksman would place all his shots in a very wide circle, but with the center of the circle exactly on the bullseye. In this case, the marksman would be imprecise, but might be regarded as being accurate. In this dissertation, we will be interested in estimating and comparing the precisions of several measuring instruments, or methods. Finding precise instruments is typically more difficult than making such instruments accurate (provided, of course, that the instruments actually measure what is desired). If an instrument has a fixed bias (inaccuracy), there are standard methods for aligning (rescaling) the instrument so that it accurately measures the desired quantity. Reducing measurement variance (imprecision) cannot be accomplished by rescaling.

Before we formally define the precision of an instrument, we need to give the statistical model underlying this definition. Suppose that we wish to measure a quantity u which is a property of some experimental unit (a person, a physical object, etc.). Each instrument which can be used to measure this quantity provides a reading (measurement) y, which is a random variable with mean $E(y)$ linearly related to u, and with a variance σ^2. Thus, we can write

$$y = \alpha + \beta u + e \equiv t(u) + e, \tag{1.0.1}$$

where e is a random error of measurement with mean 0 and variance σ^2. The bias α, scaling factor (slope) β, and error variance σ^2 differ from instrument to instrument, but all instruments have in common the property that their "true scores" $t(u)$ are linearly related to the quantity u being measured.

Once an instrument is chosen, we assume that it can be calibrated (rescaled) to eliminate the bias and scaling factors. If this calibration is done without error, such rescaling would produce a rescaled reading

$$y^* = \frac{1}{\beta}(y-\alpha) = u + \beta^{-1}e = u + e^* \tag{1.0.2}$$

whose true score would be u, and whose measurement error e^* would be

$$\text{var}(e^*) = \sigma^2/\beta^2. \tag{1.0.3}$$

The smaller var(e^*) is, the more precise is the (rescaled) instrument. Since it is customary to think of the precision of an instrument as increasing when the error variance decreases, a natural definition for the index of precision (or simply precision) of the instrument is

$$\pi = 1/\text{var}(e^*) = \sigma^{-2}\beta^2.$$

Now, consider an experiment for comparing $p(p \geq 2)$ such instruments. We assume that n units are available, with the ith unit having the true value u_i of the quantity measured, $1 \leq i \leq n$. Each such unit is measured by all p instruments. (Alternatively, if measurement is destructive or changes the unit measured, we may assume that each unit can be divided into p homogeneous specimens - one for each instrument.) Let y_{ij} be the reading on the ith instrument when measuring the jth unit. Then our model is

$$y_{ij} = \alpha_i + \beta_i u_j + e_{ij} \tag{1.0.4}$$

$i = 0,1,\ldots,p-1$, $j = 1,2,\ldots,n$. We assume that the random errors of measurement e_{ij} are mutually statistically independent, and that for each i ($0 \le i \le p-1$) the random variables $e_{i1},\ldots,e_{in}$ are identically distributed with

$$E(e_{ij}) = 0,\ \mathrm{var}(e_{ij}) = \sigma_i^2,\ i = 0,1,\ldots,p-1. \tag{1.0.5}$$

Note that α_i, β_i and σ_i^2 are the bias, scale factor and error variance, respectively, of instrument i, $i = 0,1,\ldots,p-1$. The precision of instrument i is

$$\pi_i = \sigma_i^{-2}\beta_i^2, \quad i = 0,1,\ldots,p-1. \tag{1.0.6}$$

Note that one instrument is indexed by $i = 0$. This instrument is assumed to be the standard or accurate instrument, and will be called the control. We assume that this instrument has already been rescaled so that its true score is u; that is, we assume that

$$\alpha_0 = 0, \quad \beta_0 = 1. \tag{1.0.7}$$

If this is not the case, or if no standard instrument can be identified, we will regard the true score $\alpha_0 + \beta_0 u$ of the instrument labelled 0 as being the unknown quantity to be measured. This does not affect the analysis, although, it does, of course, somewhat affect interpretation of the results. Note that it follows from (1.0.3) and (1.0.7) that the precision π_0 of the control equals σ_0^{-2}.

In order to state our model in vector-matrix form, let

$$\underset{\sim}{y}_j = (y_{0j}, y_{1j}, \ldots, y_{p-1,j})', \quad \underset{\sim}{e}_j = (e_{0j}, e_{1j}, \ldots, e_{p-1,j})'$$

$$\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_{p-1})', \quad \underset{\sim}{\beta} = (\beta_1, \ldots, \beta_{p-1})'.$$

Then

$$\underset{\sim}{y}_j = \begin{pmatrix} 0 \\ \underset{\sim}{\alpha} \end{pmatrix} + \begin{pmatrix} 1 \\ \underset{\sim}{\beta} \end{pmatrix} u_j + \underset{\sim}{e}_j, \quad j = 1,2,\ldots,n, \tag{1.0.8}$$

where the vectors $\underset{\sim}{e}_j$ are i.i.d., with mean vector $\underset{\sim}{0}$ and covariance matrix $\Sigma_e = \mathrm{diag}(\sigma_0^2, \sigma_1^2, \ldots, \sigma_{p-1}^2)$. This model is recognizable as a special case of a linear errors-in-variables regression model (Kendall and Stuart, 1979, Chapter 29; Gleser, 1981), and also (Theobald and Mallinson, 1978) as a one-factor factor analysis model. In the literature on errors-in-variables models, two ways of modeling the unknown true quantities u_j are considered:

(I) The u_j's are unknown constants (parameters).

(II) The u_j's are a random sample from a population having mean μ and variance σ_u^2, and $(u_1, \ldots, u_n)$ is statistically independent of $(\underset{\sim}{e}_1, \ldots, \underset{\sim}{e}_n)$.

The model described by (I) and (1.0.8) is a "linear functional errors-in-variables model", while the model described by (II) is a "linear structural errors-in-variables" model. Since assumption (II) is commonly adopted in the literature on comparison of instrumental precisions, we will make this assumption here. In addition, we will make the usual assumptions that the vectors $\underset{\sim}{e}_j$ have a common p-variate normal distribution, and that the scalars u_j have a common $N(\mu, \sigma_u^2)$ distribution. Consequently, the $\underset{\sim}{y}_j$ vectors are independent and identically distributed with

$$\underset{\sim}{y}_j \sim MVN(\begin{pmatrix} 0 \\ \underset{\sim}{\alpha} \end{pmatrix} + \begin{pmatrix} 1 \\ \underset{\sim}{\beta} \end{pmatrix} \mu, \Sigma_y) \tag{1.0.9}$$

and

$$\Sigma_y = \Sigma_e + \sigma_u^2 \begin{pmatrix} 1 \\ \underset{\sim}{\beta} \end{pmatrix} \begin{pmatrix} 1 \\ \underset{\sim}{\beta} \end{pmatrix}' \tag{1.0.10}$$

It should be noted that the model (1.0.8) we have adopted here is parameterized somewhat differently than that of Theobald and Mallinson (1978). The model of Theobald and Mallinson does not distinguish a control instrument (and thus treats all scale factors β_i, $0 \leq i \leq p-1$, as unknown), but assumes that $\sigma_u^2 = 1$. One constraint on the parameters is necessary to identify the parameters of the linear structural errors-in-variables model when $p > 2$ (two constraints are needed when $p = 2$). Theobald and Mallinson's formulation has the merit (when no standard instrument exists) of treating all instruments symmetrically, but at the expense of creating a standardized true value (i.e. $\sigma_u^2 = 1$) which is not necessarily the quantity we wish to measure. Our formulation is both more appropriate for the many situations where a standard instrument exists (and in which a change of instrument is desired only if some other instrument is clearly more precise), and also expresses all parameters in natural units of measurement. Additionally, by allowing σ_u^2 to be unknown (and estimated), information is obtained concerning the efficiency of the experiment used to compare the instruments; such efficiency is known to increase with σ_u^2. Indeed, it is well known that a well designed comparative calibration experiment should utilize units for which the corresponding true scores u_i vary as widely as possible over the range of values where the instruments will be used. Thus, if σ_u^2 is small, clear comparisons among the instruments will be difficult, while if σ_u^2 is large, differences in precision among instruments will be more apparent. This is not to say that similar information cannot be obtained from the Theobald-Mallinson model, since their factor loading λ_i corresponds to the quantities $\sigma_u\beta_i$ in our model, so that $\lambda_0 = \sigma_u$. However, we feel that our parameterization expresses this information in more natural terms.

Since we will compare precisions $\pi_i = \sigma_i^{-2}\beta_i^2$ by taking ratios π_i/π_j, it does not matter whether we compare π_i and π_j, or $\pi_i\sigma_u^2$ and $\pi_j\sigma_u^2$. The quantities $\tau_i = \pi_i\sigma_u^2$ are the squares of the precisions $\lambda_i\sigma_i^{-1}$ defined in Theobald and Mallinson (1978). As they remark, the term "precision" can be applied equally well to $\lambda_i\sigma_i^{-1}$ or to $\tau_i = \lambda_i^2\sigma_i^{-2}$. However, since an instrument's precision conceptually should be independent of the value of σ_u^2 in the experiment used to measure that precision, it seems more appropriate to call π_i the precision of instrument i. We will call the quantities τ_i the <u>relative precisions</u> of the instruments. This terminology seems to accord with standard usage (see Cochran, 1968; Thompson, 1963).

1.1 The Case p = 2: History and Summary of Results

The comparison of two instruments is a problem which has received a good deal of attention over the years. Consequently, a complete list of references would be excessively cumbersome to reproduce. We therefore summarize only results most closely related to our own contributions.

When p = 2, unique maximum likelihood estimators for the parameters of the model (1.0.8) do not exist, since the parameters of this model are not identifiable. The usual resolution of this difficulty is to impose a functional constraint on the six parameters μ, α_1, β_1, σ_u^2, σ_0^2, σ_1^2, although various alternative approaches (grouping the data, use of instrumental variables, replicating measurements for each instrument on each unit) have been proposed (see Moran, 1971).

The most common constraints imposed on the parameters are:

(a) to specify the value of one or both of the residual variances, σ_0^2 or σ_1^2,

(b) to specify the ratio $\sigma_1^{-2}\sigma_0^2$ of the residual variances,

(c) to specify $\beta_1 = 1$.

Constraint (a), with σ_0^2 specified (or, equivalently, π_0 specified) is meaningful in practice in cases where we have considerable experience with the control (standard) instrument. Alternatively, we might be able to specify the value of the relative precision $\tau_0 = \sigma_u^2\sigma_0^{-2}$ of the control. This last situation would be the case, for example, when we had used the control instrument many times previously on the same population of units as used in the experiment modeled by (1.0.8), with repeated measurements taken on each unit used. In this case, a standard model II ANOVA method exists (see Cochran, 1968) for forming an exact $100(1-\nu)\%$ confidence interval for τ_0. If such an interval is narrow enough, we would be willing to assume that τ_0 is known. Although confidence intervals for σ_0^2 are also obtainable from such data, these intervals are not exact, and consequently one would probably require much more data before feeling confident that σ_0^2 was sufficiently well estimated to be assumed known.

Constraint (b) is commonly adopted in textbook discussions of errors-in-variables models. The special case $\sigma_1^2 = \sigma_0^2$ has some practical appeal in situations where the source of measurement errors for both instruments is assumed to be the same (but the instruments are thought to measure u on different scales, i.e., β_1 is not necessarily equal to one). Examples of such an assumption occur in psychological testing, geophysical measurement (see Gleser and Watson, 1973), and in engineering (where measurement error is often assumed to result mainly from visual errors in reading the scale).

Constraint (c) has been frequently used in industrial and agricultural examples, where it can be assumed that all instruments measure the unknown u on the same scale ($\beta_i = 1$). This special case of the model (1.0.8) has been studied by Thompson (1962, 1963), Cochran (1968), Grubbs (1948, 1973), Maloney and Rastogi (1970), and others.

Finally, we can assume that instead of knowing σ_0^2 or σ_1^2 (as in constraint (a)), we are able to obtain independent and consistent estimators $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$ of σ_0^2 and σ_1^2, either from prior experience with the instruments, or by repeated measurements by each instrument on each unit in the context of the experiment described in Section 1.0 (see Thompson, 1963; Cochran, 1968). Using $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$, we can estimate (and pretend that we know) σ_0^2 or σ_1^2 as in constraint (a), or we can estimate (and pretend we know) $\sigma_1^{-2}\sigma_0^2$ as in constraint (b). However, it is possible that the amount of information in the data used to form the estimators $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$ may be of the same order of magnitude as the amount of information in the data of the experiment described in Section 1.0. Consequently, errors in these estimators can result in additional errors in our estimates of the precisions (apart from those inherent in the data $\underset{\sim}{y}_1,\ldots,\underset{\sim}{y}_n$), and our inferences concerning the precisions must take account of such additional variation in our estimates of π_0 and π_1.

Thus, in Chapter 2, we consider the following special cases of the model (1.0.8) when p = 2:

(1) the ratio $R = \sigma_1^{-2}\sigma_0^2$ of the error variances is known,

(2) the slope β_1 is equal to 1,

(3) the relative precision τ_0 of the control instrument is known,

(4) there exist independent consistent estimators of σ_0^2 and σ_1^2.

In each such case, we discuss estimation of the precisions π_0 and π_1 (and also of the relative precisions τ_0 and τ_1), and derive both a test statistic for comparing π_0 and π_1 and a confidence interval for the ratio $\pi_0^{-1}\pi_1$ of the precisions. We also evaluate the power function of our tests at alternatives $\pi_1 = (1+\Delta)\pi_0$, $\Delta > 0$. In the case $\beta_1 = 1$, our test statistic is identical to that proposed by Maloney and Rastogi (1970), and by Grubbs (1973). When $\beta_1 = 1$, Thompson (1963) has obtained a joint confidence region for τ_0 and τ_1. Our results for the other three cases appear to be new (when $R = \sigma_1^{-2}\sigma_0^2$ is equal to 1, our test comparing π_0 to π_1 is equivalent to testing the null hypothesis that $\beta_1^2 \leq 1$.) Even in the case $\beta_1 = 1$, we are able to state some properties of the test comparing π_0 to π_1 that have not previously been mentioned, and our tables of the exact and approximate power functions of the test may be helpful to investigators planning comparative calibration experiments.

1.2 The Case $p \geq 3$: History and Summary of Results

When $p \geq 3$, the model (1.0.8) is identifiable, and no constraints on the parameters are needed. This model was used by Mandel (1959), for the analysis of inter-laboratory round robins, by Mosteller (see Cochran, 1968) for ratings of individuals by different judges, and later by Barnett (1969), who used it in comparing four instrument-operator combinations designed to measure human lung function. An equivalent formulation of the model, in factor-analytic terms, is given by Theobald and Mallinson (1978).

Since the u_i are not known in these examples, both Mandel and Mosteller suggest using a least squares method to estimate the parameters by treating $\bar{y}_{\cdot j} = p^{-1} \sum_{i=0}^{p-1} y_{ij}$, the average over all instruments, as u_j. Because the maximum likelihood estimators for the parameters of (1.0.8) have no closed form for $p > 3$, Barnett suggests using consistent method-of-moment estimators for the parameters. Alternatively, Theobald and Mallinson (1978) reparameterize the model (1.0.8) as a factor analysis model with one factor, so that maximum likelihood estimators can be found by using a computer algorithm for factor analysis.

In Chapter 3, we discuss estimation of the parameters for the model (1.0.8), and derive the asymptotic joint distributions of the maximum likelihood estimators of the precisions $\pi_0,\ldots,\pi_{p-1}$, and of the ratios of the precisions $\psi_1,\ldots,\psi_{p-1}$, respectively. Using these results, we find joint confidence regions for the π_i's and for the ψ_i's, respectively. We also attempt to apply a type of rule originally suggested by Paulson (1952) for choosing the largest mean among the means of p independent normal populations to here select the most precise instrument among p instruments in large samples. However, our rule is not applicable because of the dependence of the large-sample variances and covariances of the statistics used upon the unknown parameters τ_0, $\psi_1,\ldots,\psi_{p-1}$. To overcome these difficulties, it seems necessary to impose some constraints on the parameter space.

Thus, in Chapter 4, we consider some special cases generalized from the special cases we discussed for $p = 2$. They are as follows:

(1) the error variance ratios $R_1,\ldots,R_{p-1}$ are known, where $R_i = \sigma_i^{-2}\sigma_0^2$,

(2) the slopes $\beta_1,\ldots,\beta_{p-1}$ are all equal to 1,

(3) the relative precision τ_0 of the control instrument is known.

In each of cases (1) to (3), we discuss the estimation of the parameters. We also use the test statistics derived in Chapter 2 for comparing each instrument with the control as the basis of a decision rule for selecting the most precise instrument. Each such procedure is of the type considered in Chapter 3, and satisfies the P_0^* requirement. That is, the probability of selecting the control as the best is at least P_0^*, where P_0^* is a predetermined number, whenever the control is actually at least as precise as any other instrument. We also attempt to bound the probability of correct selection for these procedures from below. That is, we seek a lower bound for the probability of choosing one of the (p-1) instruments as the best when that instrument is actually more precise than the others (including the control). However, this problem appears to be very complicated and remains unsolved at present.

CHAPTER 2

ESTIMATION AND COMPARISON OF THE PRECISION OF TWO INSTRUMENTS

2.0 Introduction

In the two instrument case (p = 2), the model (1.0.8) considered in Chapter 1 becomes the following:

$$\underset{\sim}{y}_j = \begin{pmatrix} y_{0j} \\ y_{1j} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} + \begin{pmatrix} 1 \\ \beta \end{pmatrix} u_j + \begin{pmatrix} e_{0j} \\ e_{1j} \end{pmatrix}, \quad j = 1,\ldots,n, \tag{2.0.1}$$

where $u_1,\ldots,u_n$ is a random sample from a normal distribution with mean μ and variance σ_u^2, and $\underset{\sim}{e}_1,\ldots,\underset{\sim}{e}_n$, $\underset{\sim}{e}_j = (e_{0j},e_{1j})'$, is an independent random sample from the BVN$(0,\text{diag}(\sigma_0^2,\sigma_1^2))$ distribution. Here, α, β, μ, σ_u^2, σ_0^2, σ_1^2 are unknown parameters. We refer to instrument 0 as "the control".

In this chapter, we are concerned with the problem of comparing the precisions $\pi_0 = \sigma_0^{-2}$ and $\pi_1 = \beta^2\sigma_1^{-2}$ of the two instruments. The relevant hypotheses can be formulated as

$$H_0: \ \pi_1 \leq \pi_0, \qquad H_1: \ \pi_1 > \pi_0. \tag{2.0.2}$$

Thus, H_1 is the hypothesis that instrument 1 is better (more precise) than the control.

As noted in Chapter 1, the model (2.0.1) is not identifiable unless a constraint is placed on the parameters. In Sections 1 through 4, respectively, we will consider the following cases:

(1) $R = \sigma_1^{-2}\sigma_0^2$ is known,

(2) $\beta = 1$,

(3) $\tau_0 = \pi_0\sigma_u^2$, the relative precision of the control, is known,

(4) there exist consistent independent estimators $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$ of σ_0^2 and σ_1^2, respectively.

In each of the above cases, we discuss estimation of the parameters of the model, with particular attention to forming point and confidence interval estimators of the ratio $(\pi_0)^{-1}\pi_1$ of the precisions π_0 and π_1. We also derive test statistics in each case for testing (2.0.2), and obtain power functions for our tests. For case (2), some of our results were anticipated by Grubbs (1948), Cochran (1968), Maloney and Rastogi (1970), and Thompson (1962, 1963).

2.1 The Case Where R Is Known

This case has been widely discussed in the econometric and biometric literature, particularly the situation where R is known to be 1 ($\sigma_0^2 = \sigma_1^2$). Note that

$$\frac{\pi_1}{\pi_0} = \beta^2 R.$$

Thus, (2.0.2) can be equivalently stated as

$$H_0: \ R\beta^2 \leq 1, \quad H_1: \ R\beta^2 > 1. \tag{2.1.1}$$

It is well known (see Moran, 1971) that the maximum likelihood estimators of the free parameters α, β, μ, σ_0^2 and σ_u^2 can be expressed in the following form. Let $\bar{y}_i = n^{-1} \sum_{j=1}^{n} y_{ij}$, $i = 0,1$, and

$$S = \begin{pmatrix} S_{00} & S_{01} \\ S_{01} & S_{11} \end{pmatrix} = \frac{1}{n} \sum_{j=1}^{n} \begin{pmatrix} y_{0j} - \bar{y}_0 \\ y_{1j} - \bar{y}_1 \end{pmatrix} \begin{pmatrix} y_{0j} - \bar{y}_0 \\ y_{1j} - \bar{y}_1 \end{pmatrix}' .$$

Further, let d_1 and d_2, $d_1 \geq d_2$, be the eigenvalues of

$$S \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} .$$

Then $\hat{\mu} = \bar{y}_0$, and

$$\hat{\alpha} = \bar{y}_1 - \hat{\beta}\bar{y}_0, \quad \hat{\beta} = \frac{(RS_{11}-S_{00}) + [(RS_{11}-S_{00})^2 + 4RS_{01}^2]^{\frac{1}{2}}}{2RS_{01}},$$

$$\hat{\sigma}_u^2 = \frac{d_1 - d_2}{1+R\hat{\beta}^2}, \quad \hat{\sigma}_0^2 = d_2. \tag{2.1.2}$$

By the invariance property of maximum likelihood, the maximum likelihood estimators of π_0, π_1 and $\pi_0^{-1}\pi_1$ are

$$\hat{\pi}_0 = \frac{1}{d_2}, \quad \hat{\pi}_1 = \frac{R\hat{\beta}^2}{d_2}, \quad (\hat{\pi}_1/\hat{\pi}_0) = R\hat{\beta}^2. \tag{2.1.3}$$

The exact distributions of these estimators can be obtained from the known joint distribution of $\hat{\beta}$ and d_2, but are too complicated to be of much help. It is known that $E|\hat{\beta}|^t = \infty$ for $t \geq 1$, so that the mean and variance of $\hat{\pi}_0^{-1}\hat{\pi}_1$ are infinite. Further, although $E(\hat{\pi}_0)$ exists when $n > 2$, it can be shown that $E(\hat{\pi}_0) > \pi_0$, so that $\hat{\pi}_0$ has a positive bias. Although no results have been published for the mean and variance of $\hat{\pi}_1$, we conjecture that both such moments are infinite. Despite these results, there is evidence in the literature to show that the maximum likelihood estimators (2.1.2), (2.1.3), are good point estimators of their respective parameters, particularly when $\tau_0 = \sigma_0^{-2}\sigma_u^2$ is large. We remark that the maximum likelihood estimators of the relative precisions τ_0 and τ_1 are

$$\hat{\tau}_0 = \hat{\sigma}_u^2\hat{\pi}_0 = \frac{d_2^{-1}d_1-1}{1+R\hat{\beta}^2}\ ,\ \hat{\tau}_1 = \hat{\sigma}_u^2\hat{\pi}_1 = \frac{R\hat{\beta}^2(d_2^{-1}d_1-1)}{1+R\hat{\beta}^2}.$$

Since $\tau_0^{-1}\tau_1 = \pi_0^{-1}\pi_1$, the maximum likelihood estimator of $\tau_0^{-1}\tau_1$ is the same as that of $\pi_0^{-1}\pi_1$.

Since the large sample $(n \to \infty)$ joint distribution of the estimators (2.1.2) is known, standard techniques of analysis can be used to obtain asymptotic joint confidence regions for any collection of these parameters, and also for (π_0,π_1), or for (τ_0,τ_1). All such regions for vectors of parameters (e.g. for (π_0,π_1)) have ellipsoidal form with centers equal to the maximum likelihood estimators and shape determined by a consistent estimator of the asymptotic covariance matrix of the estimators. For individual parameters, large sample confidence intervals can be obtained centered at the maximum likelihood estimators. In the rest of this section, we concern ourselves with the methods of forming confidence intervals for $\pi_0^{-1}\pi_1$, and for testing H_0 and H_1, which are appropriate in samples of moderate (as well as large) size.

Our method is analogous to that used by Pitman (1939) and Morgan (1939), and applied by Maloney and Rastogi (1970) in Case (2) of the model (2.0.1). Thus, let

$$\begin{pmatrix} v_j \\ w_j \end{pmatrix} = \begin{pmatrix} 1 & R^{\frac{1}{2}} \\ -1 & R^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} y_{0j} \\ y_{1j} \end{pmatrix} = \begin{pmatrix} y_{0j} + R^{\frac{1}{2}} y_{1j} \\ R^{\frac{1}{2}} y_{1j} - y_{0j} \end{pmatrix}, \quad j = 1,2,\ldots,n. \tag{2.1.4}$$

Note that (2.1.4) is an observable nonsingular (1-1 onto) transformation of the data, so that no information is lost by this transformation. The vectors $(v_j, w_j)'$, $1 \leq j \leq n$, are a random sample of size n from a bivariate normal distribution with mean vector $(\mu_v, \mu_w)'$ and covariance matrix C given by

$$(\mu_v, \mu_w) = (\alpha + (R^{\frac{1}{2}}\beta + 1)\mu,\ \alpha + (R^{\frac{1}{2}}\beta - 1)\mu),$$

$$C = \begin{pmatrix} \sigma_{vv} & \sigma_{vw} \\ \sigma_{vw} & \sigma_{ww} \end{pmatrix} = 2\sigma_0^2 I_2 + \sigma_u^2 \begin{pmatrix} 1 + R^{\frac{1}{2}}\beta \\ R^{\frac{1}{2}}\beta - 1 \end{pmatrix} \begin{pmatrix} 1 + R^{\frac{1}{2}}\beta \\ R^{\frac{1}{2}}\beta - 1 \end{pmatrix}', \tag{2.1.5}$$

respectively. Hence, the correlation coefficient ρ_{vw} of v and w is

$$\rho_{vw} = \frac{(R\beta^2 - 1)\tau_0}{\sqrt{(1 - R\beta^2)^2 \tau_0^2 + 4\tau_0(1 + R\beta^2) + 4}}, \tag{2.1.6}$$

where $\tau_0 = \sigma_0^{-2}\sigma_u^2$. The hypotheses (2.1.1) are equivalent to

$$H_0:\ \rho_{vw} \leq 0, \qquad H_1:\ \rho_{vw} > 0. \tag{2.1.7}$$

It is well known that a good test statistic for testing the hypotheses (2.1.7) is

$$T = \frac{(n-2)^{\frac{1}{2}} r_{vw}}{(1-r_{vw}^2)^{\frac{1}{2}}} ,$$

where r_{vw} is the sample correlation between v and w. In terms of the original data (y_{0j}, y_{1j}), $1 \leq j \leq n$,

$$T = \frac{(n-2)^{\frac{1}{2}}(RS_{11}-S_{00})}{2R^{\frac{1}{2}}|S|^{\frac{1}{2}}}. \qquad (2.1.8)$$

A size ν test of the hypotheses (2.1.7) has rejection region:

$$\text{Reject } H_0 \text{ if } T > t_\nu, \qquad (2.1.9)$$

where t_ν is the $100(1-\nu)$th percentile of the t distribution with n-2 degrees of freedom.

Although (2.1.9) is known to be the likelihood ratio test (LRT) of the hypotheses (2.1.7), and also a uniformly most powerful unbiased (UMPU) test of size ν for these hypotheses, what properties does this test have as a test of the hypotheses (2.1.1) in the context of the model (2.0.1)?

Theorem 2.1.1. Under the assumptions of the model (2.0.1), with $R = \sigma_0^2\sigma_1^{-2}$ known, the test (2.1.9) is a LRT of the hypotheses (2.1.1), and is also the UMPU size ν test of (2.1.1).

Proof: We have already noted that (2.1.4) is an observable nonsingular transformation of the data, and that (2.1.1) and (2.1.7) are equivalent hypotheses. We now show that the transformation

$$(\mu, \alpha, \beta, \sigma_0^2, \sigma_u^2) \rightarrow (\mu_v, \mu_w, \sigma_{vv}, \sigma_{ww}, \sigma_{vw})$$

defined by (2.1.5) defines a nonsingular mapping from the parameter space

$$\Omega = \{(\mu, \alpha, \beta, \sigma_0^2, \sigma_u^2): -\infty < \mu, \alpha, \beta < \infty, \sigma_0^2 \geq 0, \sigma_u^2 \geq 0\}$$

to the parameter space

$$\Omega^* = \{(\mu_v, \mu_w, \sigma_{vv}, \sigma_{ww}, \sigma_{vw}): -\infty < \mu_v, \mu_w < \infty, \sigma_{vv}, \sigma_{ww} \geq 0, |\sigma_{vw}|^2 \leq \sigma_{vv}\sigma_{ww}\}.$$

The assertions of our theorem will then follow from the invariance of LRT and UMPU tests under nonsingular transformations of data and parameters.

To see that (2.1.5) is 1-1 onto, note that for given $(\mu_v, \mu_w, \sigma_{vv}, \sigma_{ww}, \sigma_{vw})$ in Ω^*, the inverse image $(\mu, \alpha, \beta, \sigma_0^2, \sigma_u^2)$ of this point under (2.1.5) is defined by

$$\mu = \frac{1}{2}(\mu_v - \mu_w), \quad \alpha = \frac{1}{2}[(\mu_v + \mu_w) - R^{\frac{1}{2}}\beta(\mu_v - \mu_w)],$$

$$\sigma_0^2 = \frac{1}{2}\lambda_{min}(C) = \frac{1}{4}\{(\sigma_{vv}+\sigma_{ww}) - [(\sigma_{vv}-\sigma_{ww})^2 + 4\sigma_{vw}^2]^{\frac{1}{2}}\},$$

$$\sigma_u^2 = \frac{[(\sigma_{vv}-\sigma_{ww})^2 + 4\sigma_{vw}^2]^{\frac{1}{2}}}{2(1+R\beta^2)},$$

where

$$\beta = \frac{[(\sigma_{vv}-\sigma_{ww})^2 + 4\sigma_{vw}^2]^{\frac{1}{2}} + 2\sigma_{vw}}{R^{\frac{1}{2}}(\sigma_{vv}-\sigma_{ww})}.$$

Note that $\sigma_0^2 \geq 0$, $\sigma_u^2 \geq 0$ as required. This completes the proof. □

We now consider the power function of the test (2.1.9) against alternatives $H_{1\Delta}$ defined by

$$R\beta^2 = 1+\Delta, \quad \Delta > 0. \tag{2.1.10}$$

That is, $H_{1\Delta}$ states that $\pi_0^{-1}\pi_1 = 1+\Delta$, $\Delta > 0$. Note that it follows from (2.1.6) that

$$\rho_{vw} = \frac{\Delta\tau_0}{[(\Delta\tau_0+2)^2+8\tau_0]^{\frac{1}{2}}} . \tag{2.1.11}$$

The power function of the test (2.1.9) is known to depend on the parameters μ_v, μ_w, σ_{vv}, σ_{ww}, σ_{vw} only through ρ_{vw}. It is also known that the power of the test increases with ρ_{vw}. However, we see from (2.1.11) that ρ_{vw} is a function not only of Δ, but also of the relative precision τ_0 of the control instrument.

Lemma 1. If ρ_{vw} is given by (2.1.11), then

(1) ρ_{vw} is strictly increasing in Δ for fixed τ_0,

(2) ρ_{vw} is strictly increasing in τ_0 for fixed Δ.

Proof: Observe that

$$\frac{\partial\rho_{vw}}{\partial\Delta} = \frac{2\tau_0[(\Delta+4)\tau_0+2]}{[(\Delta\tau_0+2)^2+8\tau_0]^{3/2}} > 0$$

and

$$\frac{\partial\rho_{vw}}{\partial\tau_0} = \frac{2\Delta[(\Delta+2)\tau_0+2]}{[(\Delta\tau_0+2)^2+8\tau_0]^{3/2}} > 0. \quad \square$$

Theorem 2.1.2. For fixed ν, n, Δ and τ_0, the power of the test (2.1.9) against the alternative $H_{1\Delta}$ is given by

$$G(\nu,\Delta,\tau_0,n) = \int_L^\infty f(r|\rho(\Delta,\tau_0))\,dr, \tag{2.1.12}$$

where

$$L = [n-2 + t_\nu^2]^{-\frac{1}{2}} t_\nu,$$

$$f(r|\rho) = \pi^{-1}(n-2)(1-\rho^2)^{\frac{1}{2}(n-1)}(1-r^2)^{\frac{1}{2}(n-4)}\int_0^\infty (\cosh w - \rho r)^{-(n-1)} dw, \tag{2.1.13}$$

and $\rho(\Delta,\tau_0) = \rho_{VW}$ is defined by (2.1.11). For fixed ν, $G(\nu,\Delta,\tau_0,n)$ is strictly increasing in each of the arguments Δ, τ_0, n, when the remaining arguments are held fixed.

Proof: Note that

$$G(\nu, \Delta, \tau_0, n) = P\{T > t_\nu\} = P\{r_{VW} > L\}.$$

Let $r = r_{VW}$. The probability density function (2.1.12) for r is given by Graybill (1976, p. 392). The fact that for fixed ν, Δ, τ_0, the function $G(\nu, \Delta, \tau_0, n)$ is strictly increasing in n is well known. The remaining monotonicity assertions follow from the fact that $P\{r_{VW} > L\}$, for fixed L, n, is increasing in ρ_{VW}, and from Lemma 1.□

It follows directly from (2.1.11) that for fixed $\Delta > 0$,

$$\lim_{\tau_0 \downarrow 0} \rho_{VW} = 0.$$

Consequently, for fixed ν, Δ, n, it can be shown that

$$\inf_{\tau_0 \geq 0} G(\nu, \Delta, \tau_0, n) = \nu.$$

We see that in order to insure that the test (2.1.9) has a specified power against $H_{1\Delta}$, τ_0 must be bounded below ($\tau_0 \geq \tau_0^*$) by a positive number τ_0^*. That is, a lower bound to the relative precision of the control instrument must be known.

The cumulative distribution of r,

$$P(r \leq r^*) = F(r^*|n,\rho),$$

has been tabulated by F. N. David (1938) for $\rho = 0(0.1)0.9$, $n = 3(1)25$, 50, 100, 200, 400, and $r^* = -1(0.05)1$. However, the tables are not easy to find. In order to evaluate the performance of the test (2.1.9), we have recalculated the power of the test (2.1.9) for different values of τ_0, n and Δ. Tables A1 and A2 show the power of the test (2.1.9) for $\nu = 0.05$, $\tau_0 = 1.0, 2.0, 4.0, 6.0$, $n = 10(5)50$, $\Delta = 1.0$ and 2.0 respectively.

From Table A1 and A2, we can see that when both Δ and τ_0 are small, the power of the test is fairly low. For a better power, it is necessary to increase the sample size n.

Table A1. The power of the test (2.1.9) for $\nu = 0.05$, $\Delta = 1.0$.

τ_0 \ n	10	15	20	25	30	35	40	45	50
1.0	0.1675	0.2227	0.2739	0.3220	0.3679	0.4171	0.4522	0.4933	0.5274
2.0	0.2684	0.3749	0.4688	0.5512	0.6230	0.6846	0.7377	0.7826	0.8205
4.0	0.4376	0.6054	0.7285	0.8163	0.8776	0.9193	0.9474	0.9661	0.9783
6.0	0.5679	0.7521	0.8623	0.9254	0.9605	0.9794	0.9895	0.9947	0.9973

Table A2. The power of the test (2.1.9) for $\nu = 0.05$, $\Delta = 2.0$.

τ_0 \ n	10	15	20	25	30	35	40	45	50
1.0	0.3321	0.4663	0.5779	0.6694	0.7435	0.8011	0.8488	0.8825	0.9130
2.0	0.5475	0.7311	0.8449	0.9128	0.9519	0.9739	0.9861	0.9927	0.9962
4.0	0.7867	0.9278	0.9769	0.9929	0.9979	0.9994	0.9998	1.0000	1.0000
6.0	0.8914	0.9773	0.9956	0.9992	0.9999	1.0000	1.0000	1.0000	1.0000

When n is sufficiently large, and $\rho_{VW} = 0$, the distribution of T is approximated by that of a standard normal random variable. Therefore, the cutting point in (2.1.9) becomes z_ν instead of t_ν, where z_ν is the $100(1-\nu)\%$ percentile of the standard normal distribution. For $\rho_{VW} \neq 0$ and large n, the distribution of T also approximates to a normal distribution. To evaluate the power of the test (2.1.9), we need to find the mean and variance of this asymptotic normal distribution.

<u>Theorem 2.1.3</u>. For fixed ν, Δ, τ_0 and a sufficiently large n, the power of the test (2.1.9) against the alternative $H_{1\Delta}$ is given by

$$\int_{z_\nu}^{\infty} g_1(t|\Delta,\pi_0)\,dt, \tag{2.1.14}$$

where $g_1(t|\Delta,\pi_0)$ is the probability density function of a normal distribution with mean $(n-2)^{\frac{1}{2}}\rho(1-\rho^2)^{-\frac{1}{2}}$ and variance $(1-\rho^2)^{-1}$, where $\rho = \rho_{VW}$ is defined in (2.1.11).

Proof: By Theorem 4.2.6 in Anderson (1958), $\sqrt{n}\,(r-\rho)/(1-\rho^2)$ is asymptotically distributed according to N(0,1). Thus, using the result 6a. 2 (iv) in Rao (1973), we find that the asymptotic distribution of T is normal with mean $(n-2)^{\frac{1}{2}}\rho[1-\rho^2]^{-\frac{1}{2}}$ and variance $(1-\rho^2)^{-1}$. For fixed Δ and τ_0, $\rho = \rho_{VW}$ is defined in (2.1.11). □

After we compare the probabilities calculated from the exact distribution of r and the probabilities computed by assuming T is normally distributed with mean and variance shown in Theorem 2.1.3, we find that the approach of T to normality is reasonably fast. For $n \geq 100$ the approximation is accurate to five decimal places. We have calculated the (approximate) power of the test (2.1.9) for $\nu = 0.05$, n = 100(100)500, τ_0 = 1.0, 2.0, 4.0, 6.0, Δ = 0.5, 1.0. The results are shown in Table A3 and A4.

Table A3. The power of the test (2.1.9) for $\nu = 0.05$, $\Delta = 0.5$, and a large n.

τ_0 \ n	100	200	300	400	500
1.0	0.37477	0.59225	0.74412	0.84424	0.90745
2.0	0.64361	0.885398	0.96715	0.99126	0.99781
4.0	0.90020	0.99354	0.9997	1.0000	1.0000
6.0	0.973541	0.9997	1.0000	1.0000	1.0000

Table A4. The power of the test (2.1.9) for $\nu = 0.05$, $\Delta = 1.0$, and a large n.

τ_0 \ n	100	200	300	400	500
1.0	0.78958	0.96538	0.99521	0.9994	1.0000
2.0	0.97507	0.9997	1.0000	1.0000	1.0000
4.0	0.9996	1.0000	1.00000	1.0000	1.0000
6.0	1.0000	1.00000	1.00000	1.0000	1.0000

The last question we consider now is to obtain a confidence interval for the ratio $\pi_1\pi_0^{-1}$ of the precisions π_0, π_1 of the two instruments. Note that $\pi_1\pi_0^{-1}$ is equal to $R\beta^2$. We first consider a confidence region for $R^{\frac{1}{2}}\beta$, then from this region derive a confidence region for $R\beta^2$. The confidence region for $R^{\frac{1}{2}}\beta$ proposed independently by Creasy (1957) and by Williams (1969) has the advantage that the region for $R^{\frac{1}{2}}\beta$ is free from the unknown parameter τ_0. Let $r(R^{\frac{1}{2}}\beta)$ be the sample correlation coefficient between $R\beta y_{1i} + y_{0i}$ and $y_{1i} - \beta y_{0i}$, $i = 1,\dots,n$, and let $F_{1,n-2}(\nu)$ be the $100(1-\nu)$ percentile of the F distribution with 1 and n-2 degrees of freedom. The Creasy-William (CW) confidence region for $R^{\frac{1}{2}}\beta$ is then

$$CW = \{R^{\frac{1}{2}}\beta:\ (n-2)r(R^{\frac{1}{2}}\beta)[1-r^2(R^{\frac{1}{2}}\beta)]^{-1} \le F_{1,n-2}(\nu)\}. \tag{2.1.15}$$

In terms of the original data (y_{0i}, y_{1i}), $1 \le i \le n$, we have

$$r^2(R^{\frac{1}{2}}\beta)[1-r^2(R^{\frac{1}{2}}\beta)]^{-1} = \frac{[-S_{01}(R^{\frac{1}{2}}\beta)^2 + (R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})R^{\frac{1}{2}}\beta + S_{01}]^2}{(S_{11}S_{00}-S_{01}^2)(1+(R^{\frac{1}{2}}\beta)^2)}. \tag{2.1.16}$$

Substituting (2.1.16) into (2.1.15), we obtain the following inequality:

$$(S_{01}^2-c)(R^{\frac{1}{2}}\beta)^4 - 2S_{01}(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})(R^{\frac{1}{2}}\beta)^3 + [(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})^2$$

$$- 2S_{01}^2-2c](R^{\frac{1}{2}}\beta)^2 + 2S_{01}(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})(R^{\frac{1}{2}}\beta) + (S_{01}^2-c) \leq 0, \quad (2.1.17)$$

where

$$c = (n-2)^{-1}(S_{11}S_{00}-S_{01}^2)F_{1,n-2}(\nu). \quad (2.1.18)$$

The four roots of the equation obtained by setting the left side of (2.1.17) equal to 0 are

$$A_1 = \frac{(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00}) + [(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})^2 + 4(S_{01}^2-c)]^{\frac{1}{2}}}{2(S_{01} + \sqrt{c})},$$

$$A_2 = \frac{(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00}) + [(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})^2 + 4(S_{01}^2-c)]^{\frac{1}{2}}}{2(S_{01} - \sqrt{c})},$$

$$A_3 = \frac{(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00}) - [(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})^2 + 4(S_{01}^2-c)]^{\frac{1}{2}}}{2(S_{01} - \sqrt{c})},$$

$$A_4 = \frac{(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00}) - [(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})^2 + 4(S_{01}^2-c)]^{\frac{1}{2}}}{2(S_{01} + \sqrt{c})}. \quad (2.1.19)$$

It is easy to check that the order of the four roots A_1, A_2, A_3, A_4 which depends on the values of $S_{01} + \sqrt{c}$, $S_{01} - \sqrt{c}$ and $R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00}$, are as follows:

$$A_3 < A_4 < 0 < A_1 < A_2, \text{ if } S_{01} > \sqrt{c},$$

$$A_1 < A_2 < 0 < A_3 < A_4, \text{ if } S_{01} < - \sqrt{c},$$

$$A_2 < A_4 < 0 < A_3 < A_1, \text{ if } S_{01}^2 < c,\ R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00} > 2(c-S_{01}^2)^{\frac{1}{2}} > 0,$$

$$A_1 < A_4 < 0 < A_3 < A_2, \text{ if } S_{01}^2 < c,\ R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00} < -2(c-S_{01}^2)^{\frac{1}{2}} < 0.$$

When $S_{01}^2 < c$ and $(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})^2 < 4(c-S_{01}^2)$, all the four roots are imaginary. Solving (2.1.17) with respect to $R^{\frac{1}{2}}\beta$, we obtain a $1-\nu$ confidence region for $R^{\frac{1}{2}}\beta$:

$$R^{\frac{1}{2}}\beta \in (A_3,A_4) \cup (A_1,A_2), \text{ if } S_{01} > \sqrt{c},$$

$$R^{\frac{1}{2}}\beta \in (A_1,A_2) \cup (A_3,A_4), \text{ if } S_{01} < -\sqrt{c},$$

$$R^{\frac{1}{2}}\beta \in (A_4,A_3) \cup (-\infty,A_2) \cup (A_1,\infty), \text{ if } S_{01}^2 < c,\quad R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00} > 2(c-S_{01}^2)^{\frac{1}{2}}, \tag{2.1.20}$$

$$R^{\frac{1}{2}}\beta \in (A_4,A_3) \cup (-\infty,A_1) \cup (A_2,\infty), \text{ if } S_{01}^2 < c,\ R^{\frac{1}{2}}S_{11} -R^{-\frac{1}{2}}S_{00} < -2(c-S_{01}^2)^{\frac{1}{2}}.$$

Note that the maximum likelihood estimator of $R^{\frac{1}{2}}\beta$ can be expressed as

$$\frac{(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00}) + [(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00})^2 + 4S_{01}^2]^{\frac{1}{2}}}{2S_{01}},$$

which is known to have the same sign as S_{01}. When $S_{01}^2 > c$, as can be seen from the definition of A_1 and A_2, the values of A_1 and A_2 also have the same sign as S_{01}. In fact, we can show that if $S_{01}^2 > c$, the interval (A_1,A_2) covers the maximum likelihood estimator $R^{\frac{1}{2}}\hat{\beta}$ of $R^{\frac{1}{2}}\beta$. We can also show that when $S_{01}^2 < c$ and $(R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00}) > 2(c-S_{01}^2)$, $R^{\frac{1}{2}}\hat{\beta}$

belongs to $(-\infty, A_2)$ and (A_1, ∞) for $S_{01} < 0$ and $S_{01} > 0$, respectively; while for $S_{01}^2 < c$ and $(R^{\frac{1}{2}}S_{11} - R^{-\frac{1}{2}}S_{00}) < -2(c-S_{01}^2)^{\frac{1}{2}}$, $R^{\frac{1}{2}}\hat{\beta}$ belongs to $(-\infty, A_1)$ and (A_2, ∞) for $S_{12} < 0$ and $S_{12} > 0$, respectively. If we only choose that interval which covers the maximum likelihood estimator of $R^{\frac{1}{2}}\beta$ as our confidence interval for $R^{\frac{1}{2}}\beta$, the coverage probability of this modified C-W region will be less than $1-\nu$. However, as pointed out by Jolicoeur and Mosimann (1968), when $|\rho|$ is large, where ρ is the correlation coefficient of y_0 and y_1, the coverage probability of the interval containing $R^{\frac{1}{2}}\hat{\beta}$ is very near to $1-\nu$. The square of the population correlation coefficient ρ of y_0 and y_1 is increasing in τ_0. Hence, we conjecture that when τ_0 is large, choosing the interval shown in (2.1.20) which covers the maximum likelihood estimator $R^{\frac{1}{2}}\hat{\beta}$ as the confidence interval for $R^{\frac{1}{2}}\beta$ will have coverage probability close to ν.

The modified CW region for $R^{\frac{1}{2}}\beta$ (that is the interval which covers $R^{\frac{1}{2}}\beta$) is as follows:

$$0 < A_1 < R^{\frac{1}{2}}\beta < A_2, \quad \text{if } S_{01} > \sqrt{c},$$

$$A_1 < R^{\frac{1}{2}}\beta < A_2 < 0, \quad \text{if } S_{01} < -\sqrt{c}$$

$$0 < A_1 < R^{\frac{1}{2}}\beta < \infty, \quad \text{if } 0 < S_{01} < \sqrt{c},\ R^{\frac{1}{2}}S_{11} - R^{-\frac{1}{2}}S_{00} > 2\,(c-S_{01}^2)^{\frac{1}{2}},$$

$$-\infty < R^{\frac{1}{2}}\beta < A_2 < 0, \text{ if } -\sqrt{c} < S_{01} < 0,\ R^{\frac{1}{2}}S_{11} - R^{-\frac{1}{2}}S_{00} > 2(c-S_{01}^2)^{\frac{1}{2}},$$

$$0 < A_2 < R^{\frac{1}{2}}\beta < \infty, \quad \text{if } 0 < S_{01} < \sqrt{c},\ R^{\frac{1}{2}}S_{11} - R^{-\frac{1}{2}}S_{00} < -2(c-S_{01}^2)^{\frac{1}{2}},$$

$$-\infty < R^{\frac{1}{2}}\beta < A_1 < 0, \text{ if } -\sqrt{c} < S_{01} < 0,\ R^{\frac{1}{2}}S_{11} - R^{-\frac{1}{2}}S_{00} < -2(c-S_{01}^2)^{\frac{1}{2}}.$$

From the above modified CW region for $R^{\frac{1}{2}}\beta$, we obtain the following confidence interval for $R\beta^2$:

$$A_1^2 < R\beta^2 < A_2^2, \quad \text{if } S_{01} > \sqrt{c}$$

$$A_2^2 < R\beta^2 < A_1^2, \quad \text{if } S_{01} < -\sqrt{c}$$

$$A_1^2 < R\beta^2 < \infty, \quad \text{if } 0 < S_{01} < \sqrt{c},\ R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00} > 2(c-S_{01}^2)^{\frac{1}{2}},$$

$$A_2^2 < R\beta^2 < \infty, \quad \text{if } -\sqrt{c} < S_{01} < 0,\ R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00} > 2(c-S_{01}^2)^{\frac{1}{2}},$$

$$A_2^2 < R\beta^2 < \infty, \quad \text{if } 0 < S_{01} < \sqrt{c},\ R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00} < -2(c-S_{01}^2)^{\frac{1}{2}},$$

$$A_1^2 < R\beta^2 < \infty, \quad \text{if } -\sqrt{c} < S_{01} < 0,\ R^{\frac{1}{2}}S_{11}-R^{-\frac{1}{2}}S_{00} < -2(c-S_{01}^2)^{\frac{1}{2}}.$$

This confidence interval (which can be infinite in length) is defined in the usual way from the confidence region for $R^{\frac{1}{2}}\beta$; that is,

$$\{R\beta^2:\ R\beta^2 = x^2,\ x \text{ in confidence region for } R^{\frac{1}{2}}\beta\}.$$

2.2 The Case Where $\beta = 1$

For $\beta = 1$, the model (1.0.8) is known to be a variance component model. As can be seen from the definition of the precisions π_i, the hypotheses (2.0.2) are equivalent to

$$H_0:\ \sigma_0^2 \le \sigma_1^2, \quad H_1:\ \sigma_0^2 > \sigma_1^2. \tag{2.2.1}$$

As usual for a variance component model, the estimators of the variances sometimes take negative values. The estimation problem has been considered by Thompson (1962), Grubbs (1948) and Cochran (1968). It is known that if $\min(S_{11},S_{00}) \ge S_{01} \ge 0$, the maximum likelihood estimators of σ_0^2, σ_u^2 and σ_1^2 can be expressed in the following form:

$$\hat{\sigma}_0^2 = S_{00} - S_{01},\ \hat{\sigma}_1^2 = S_{11} - S_{01},\ \hat{\sigma}_u^2 = S_{01}. \tag{2.2.2}$$

Hence, the maximum likelihood estimators of π_0, π_1 and $\pi_1\pi_0^{-1}$ are

$$\hat{\pi}_0 = \frac{1}{S_{00}-S_{01}}, \quad \hat{\pi}_1 = \frac{1}{S_{11}-S_{01}}, \quad \hat{\pi}_1\hat{\pi}_0^{-1} = \frac{S_{00}-S_{01}}{S_{11}-S_{01}}. \qquad (2.2.3)$$

The exact distribution of $\hat{\pi}_1\hat{\pi}_0^{-1}$ can be obtained from the joint distribution of $S_{11.0} = S_{11} - S_{01}^2S_{00}^{-1}$, $S_{01}S_{00}^{-\frac{1}{2}}$ and S_{00}, but is too complicated to be useful. However, in large sample cases the joint distribution of the estimators $\hat{\sigma}_0^2$, $\hat{\sigma}_1^2$ and $\hat{\sigma}_u^2$ is multivariate normal (see Anderson, 1968). Thus, the asymptotic joint confidence region for any collection of these parameters (σ_0^2, σ_u^2 and σ_1^2), or for (π_0,π_1), can be obtained by standard methods. All such regions for collections of parameters, and all confidence intervals for individual parameters, have their centers equal to maximum likelihood estimators.

For testing the hypotheses (2.1.1), the method used by Pitman (1939) and Morgan (1939), and applied by Maloney and Rasotgi (1970) is appropriate in both small and large sample cases. The test statistic is derived by transforming the original data (y_{0i}, y_{1i}), $i = 1,\ldots,n$, into new data (v_i^*, w_i^*), $i = 1,\ldots,n$. The transformation is almost identical to that of (2.1.4) except R is equal to 1 here. Thus,

$$\begin{pmatrix} v_j^* \\ w_j^* \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_{0j} \\ y_{1j} \end{pmatrix} = \begin{pmatrix} y_{0j} + y_{1j} \\ y_{1j} - y_{0j} \end{pmatrix}, \quad j = 1,\ldots,n. \qquad (2.2.4)$$

Then the vectors $(v_j^*, w_j^*)'$, $1 \leq j \leq n$, are a random sample of size n from a bivariate normal distribution with mean vector $(\mu_v^*, \mu_w^*)'$ and covariance matrix $\underset{\sim}{C}^*$ given by

$$(\mu_v^*, \mu_w^*) = (\alpha + 2\mu, \alpha)',$$

$$\underset{\sim}{C}^* = \begin{pmatrix} \sigma_0^2 + \sigma_1^2 + 4\sigma_u^2 & \sigma_1^2 - \sigma_0^2 \\ \sigma_1^2 - \sigma_0^2 & \sigma_0^2 + \sigma_1^2 \end{pmatrix}, \tag{2.2.5}$$

respectively. Hence, the correlation coefficient ρ_{vw}^* of v* and w* is

$$\rho_{vw}^* = \frac{1 - \dfrac{\sigma_0^2}{\sigma_1^2}}{\sqrt{\left(1 + \dfrac{\sigma_0^2}{\sigma_1^2}\right)^2 + 4\dfrac{\sigma_0^2}{\sigma_1^2}\left(1 + \dfrac{\sigma_0^2}{\sigma_1^2}\right)\tau_0}}, \tag{2.2.6}$$

where $\tau_0 = \sigma_0^{-2}\sigma_u^2$. Note that the hypotheses (2.2.1) are equivalent to

$$H_0:\ \rho_{vw}^* \geq 0, \qquad H_1:\ \rho_{vw}^* < 0. \tag{2.2.7}$$

The test statistic suggested by Maloney and Rastogi (1970) for testing the hypotheses (2.2.1) is

$$T^* = \frac{(n-2)^{\frac{1}{2}} r_{vw}^*}{[1-(r_{vw}^*)^2]^{\frac{1}{2}}} = \frac{(n-2)^{\frac{1}{2}}(S_{11}-S_{00})}{2\ (S_{11}S_{00}-S_{01}^2)^{\frac{1}{2}}}, \tag{2.2.8}$$

where r_{vw}^* is the sample correlation coefficient, which is known to be an appropriate test statistic for the hypotheses (2.2.7). A size ν test for the hypotheses (2.2.7) is as follows:

$$\text{Reject } H_0 \text{ if } T^* < -t_\nu, \tag{2.2.9}$$

where t_ν is the 100 $(1-\nu)$ percentile of the t distribution with n-2 degrees of freedom.

It is known that the test (2.2.9) is the LRT, and also the UMPU size test, for the hypotheses (2.2.7) in cases where the covariance matrix C* of (v*, w*)' is unrestricted. However, observe from (2.2.5) that the variances σ_{vv}, σ_{ww} and covariance σ_{vw} of v, w are required to satisfy the inequalities

$$\sigma_{v*v*} \geq \sigma_{w*w*} \geq |\sigma_{v*w*}|.$$

It can be shown that the likelihood ratio test statistic for the hypotheses (2.2.7) under these inequality restrictions agrees with the unrestricted LRT statistic whenever the sample variances S_{vv}, S_{ww} and covariance S_{vw} of v and w satisfy the inequality restrictions; that is, when

$$S_{vv} \geq S_{ww} \geq |S_{vw}|.$$

Since

$$\lim_{n\to\infty} S_{ij} = \sigma_{ij}, \qquad i,j = v,w$$

with probability one as $n \to \infty$, it follows that the restricted LRT is asymptotically equivalent to the test defined by (2.2.9). That is, the test (2.2.9) is asymptotically equivalent to the LRT for the hypotheses (2.2.1).

The test (2.2.9) can easily be shown to be an unbiased level ν test of the hypotheses (2.2.1) - see Theorem 2.2.1 below. However, it is possible that there may exist a test of (2.2.7) of level ν which is unbiased for the restricted parameter space defined by the inequalities on σ_{vv}, σ_{ww}, σ_{vw} mentioned above (but biased when σ_{vv}, σ_{ww}, σ_{vw} is unrestricted), and which has greater power than (2.2.9) for some alternative to H_0. That is, the test (2.2.9) need not be the UMP unbiased level ν test for the hypotheses (2.2.1). Indeed, no such UMP unbiased test may exist. However, the fact that this test is asymptotically equivalent to the likelihood ratio test of (2.2.1) can be used to show that it is asymptotically (as $n \to \infty$) UMP unbiased level ν.

In discussing the above properties, we have assumed that the relative precision τ_0 of the control instrument is unknown. However, in practice, some information (perhaps in terms of bounds on τ_0) is usually known about the control instrument. Indeed, if τ_0 were not sufficiently large, the control instrument would likely not have been of previous interest, and thus hardly could serve as a standard for comparison to instrument 1.

Keeping this fact in mind, we now investigate the power function of the test (2.2.9). We will demonstrate that this test, despite its good properties mentioned above, has the somewhat disturbing property of having a power function which, for a fixed alternative to H_0, is <u>decreasing</u> in the relative precision τ_0 of the control instrument. It follows from this property, which does not seem to have previously been noted in the literature, that the more precise is the control instrument, the larger must be the sample size n of an experiment designed to have a specified probability of detecting that another instrument (instrument 1) has superior precision.

Let the alternative $H_{1\Delta}$ to H_0 be defined by

$$\frac{\sigma_0^2}{\sigma_1^2} = 1 + \Delta, \ \Delta > 0. \tag{2.2.10}$$

That is, $H_{1\Delta}$ states that $\pi_1\pi_0^{-1} = 1 + \Delta$, $\Delta > 0$. Following from (2.2.6), we have

$$\rho^*_{vw} = \frac{-\Delta}{[(2+\Delta)^2 + 4(1+\Delta)(2+\Delta)\tau_0]^{\frac{1}{2}}}. \tag{2.2.11}$$

It is known that the power function of the test (2.2.9) depends only on ρ^*_{vw}, while ρ^*_{vw} in turn is a function of Δ and τ_0.

<u>Lemma 2</u>. For fixed ν, n, Δ and τ_0, the power of the test (2.2.9) against the alternative $H_{1\Delta}$ is decreasing in τ_0 for fixed Δ and increasing in Δ for fixed τ_0.

<u>Proof</u>: It is easy to see that ρ^*_{vw} in (2.2.11) is increasing in τ_0 for fixed Δ ($\Delta > 0$) and decreasing in Δ for fixed τ_0. It is also known that the power function of the test (2.2.9) given by

$$G^*(\nu, \Delta, \tau_0, n) = P\{T^* < -t_\nu | \rho^*_{vw}\}, \tag{2.2.12}$$

is decreasing in ρ^*_{vw}. Combining these two results, the Lemma now follows. □

<u>Theorem 2.2.1</u>. For fixed ν, n, Δ and τ_0, the power function of the test (2.2.9) against the alternative $H_{1\Delta}$, $G^*(\nu, \Delta, \tau_0, n)$, is equal to

$$\int_L^\infty f(r|\rho(\Delta,\tau_0))dr, \tag{2.2.13}$$

where L and $f(r|\rho)$ are defined in Theorem 2.1.2, and $\rho(\Delta,\tau_0) = -\rho^*_{vw}$ is defined by (2.2.11). For fixed ν, the minimum of $G^*(\nu, \Delta, \tau_0, n)$ is ν when $\rho^*_{vw} = 0$, while the maximum of $G^*(\nu, \Delta, \tau_0, n)$ is achieved when $\rho(\Delta, \tau_0) = \rho_u$, where

$$\rho_u = \frac{\Delta}{2+\Delta}. \tag{2.2.14}$$

Proof: Note that

$$G^*(\nu, \Delta, \tau_0, n) = P\{r_{vw} < -L|\rho^*_{vw}\} = P\{r_{vw} > L|-\rho^*_{vw}\}.$$

Following Theorem 2.2.1, $G^*(\nu, \Delta, \tau_0, n)$ is equal to (2.2.13). From Lemma 2, we know that the minimum and maximum of $G^*(\nu, \Delta, \tau_0, n)$ are achieved at $\tau_0 = \infty$ and $\tau_0 = 0$, respectively, for fixed Δ. However, $\tau_0 = \infty$ and $\tau_0 = 0$ give $\rho^*_{vw} = 0$ and $\rho^*_{vw} = -\frac{\Delta}{2+\Delta}$, respectively. Thus, the theorem follows. □

For fixed $\nu = 0.05$, we have calculated the upper bound of the power of the test (2.2.9) for $\Delta = 1.0, 2.0$, $n = 10(5)50$, and the power of the test (2.2.9) for $n = 10(5)50$, $\tau_0 = 0.2, 0.4, 0.6$, $\Delta = 1.0, 2.0$. The results are shown in the following tables.

Table A5. The upper bound of the power of the test (2.2.9) for $\nu = 0.05$.

Δ \ n	10	15	20	25	30	35	40	45	50
1.0	0.2473	0.3437	0.4299	0.5070	0.5758	0.6362	0.6894	0.7358	0.7758
2.0	0.4602	0.6328	0.7557	0.8404	0.8974	0.9348	0.9591	0.9746	0.9846

Table A6. The power of the test (2.2.9) for $\nu = 0.05$, $\Delta = 1.0$.

τ_0 \ n	10	15	20	25	30	35	40	45	50
0.2	0.1885	0.2547	0.3159	0.3729	0.4265	0.4761	0.5226	0.5657	0.6054
0.4	0.1597	0.2107	0.2580	0.3027	0.3454	0.3858	0.4245	0.4613	0.4960
0.6	0.1423	0.1842	0.2230	0.2580	0.2952	0.3289	0.3617	0.3931	0.4232

Table A7. The power of the test (2.2.9) for $\nu = 0.05$, $\Delta = 2.0$.

τ_0 \ n	10	15	20	25	30	35	40	45	50
0.2	0.3162	0.4439	0.5518	0.6419	0.7163	0.7766	0.8254	0.8642	0.8949
0.4	0.2511	0.3494	0.4371	0.5152	0.5846	0.6453	0.6987	0.7448	0.7845
0.6	0.2141	0.2938	0.3665	0.4332	0.4946	0.5503	0.6012	0.6472	0.6885

From Table A5, we can see that the maximum of the power is relatively small even when $\tau_0 \leq 0.6$. A rule of thumb mentioned by Thompson (1963) suggests that if the instrumentation of an experiment is to be effective, τ_0 should be ≥ 100. Note that the power of the test (2.2.9) is decreasing in τ_0, thus, for a $\tau_0 \geq 100$, the maximum of the power of the test (2.2.9) would be very small. Hence, we conclude that although the test (2.2.9) is appropriate for testing the hypotheses (2.2.1), the test is very insensitive for detecting the difference of the precisions when $n \leq 50$. It is well known that the power increases with n. In order to improve the power of the test (2.2.9) for a large τ_0, it is necessary to increase the sample size n.

For a large n, the asymptotic distribution of the test statistic T* defined by (2.2.8) is known to be a normal distribution. As a direct consequence of Theorems 2.1.3 and 2.2.1, we can obtain the power of the test (2.2.9) for a large n. These results are summarized in the following theorem.

Theorem 2.2.2. For fixed ν, Δ, τ_0 and a large n, the power of the test (2.2.9) is given approximately by

$$\int_{z_\nu}^{\infty} g_2(t|\Delta,\tau_0)dt, \qquad (2.2.15)$$

where $g_2(t|\Delta,\tau_0)$ is the probability density function of a normal random variable with mean $(n-2)^{\frac{1}{2}}\rho(1-\rho^2)^{-\frac{1}{2}}$ and variance $(1-\rho^2)^{-1}$, and $\rho = -\rho^*_{vw}$ is defined by (2.2.11).

For $\nu = 0.05$, we have used Theorem 2.2.2 to calculate the power of the test (2.2.9) for n = 100(100)500, Δ = 1.0, 2.0, τ = 0.2, 0.4, 0.6, 1.0, 5.0, 10.0. The results are shown in Table A8 and A9.

Table A8. The power of the test (2.2.9) for $\nu = 0.05$, $\Delta = 1.0$, and a large n.

τ_0 \ n	100	200	300	400	500
0.2	0.86003	0.98717	0.9989	1.0000	1.0000
0.4	0.75652	0.95175	0.99185	0.9987	1.0000
0.6	0.66892	0.90310	0.97495	0.99405	0.9987
1.0	0.54116	0.79659	0.91697	0.96731	0.98824
5.0	0.22150	0.34471	0.45271	0.54695	0.62785
10.0	0.155202	0.22661	0.29198	0.35296	0.40998

Table A9. The power of the test (2.2.9) for $\nu = 0.05$, $\Delta = 2.0$, and a large n.

τ_0 \ n	100	200	300	400	500
0.2	0.99184	1.0000	1.0000	1.0000	1.0000
0.4	0.96317	0.9993	1.0000	1.0000	1.0000
0.6	0.91743	0.99579	1.0000	1.0000	1.0000
1.0	0.81117	0.9729	0.99673	0.9997	1.0000
5.0	0.34658	0.55046	0.70045	0.80554	0.87645
10.0	0.22679	0.35395	0.46485	0.56088	0.64281

From the above two tables, we can see that for a large τ_0, the test (2.2.9) is still not very powerful for $100 \leq n \leq 500$. This suggests that for a large τ_0 (i.e. when the control instrument is effective), to obtain a specified power (say .95) for using the test (2.2.9), a very large sample is needed.

Instead of testing the hypotheses (2.2.1), we might consider obtaining confidence regions for the precisions. A joint confidence region for τ_0 and τ_1 has been found by Thompson (1963). Here, we consider the confidence interval for $\pi_1\pi_0^{-1}$. Note that $\pi_1\pi_0^{-1} = \tau_1\tau_0^{-1} = \sigma_0^2\sigma_1^{-2} = R$.

From the model (2.0.1) with $\beta = 1$, rescaling the data y_{1i}, $i = 1,\ldots,n$, to $R^{\frac{1}{2}}y_{1i}$, $i = 1,\ldots,n$, gives

$$\begin{pmatrix} y_{0i} \\ R^{\frac{1}{2}}y_{1i} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha R^{\frac{1}{2}} \end{pmatrix} + \begin{pmatrix} 1 \\ R^{\frac{1}{2}} \end{pmatrix} u_i + \begin{pmatrix} e_{0i} \\ R^{\frac{1}{2}}e_{1i} \end{pmatrix}, \quad i = 1,\ldots,n.$$

Note that e_{0i} and $R^{\frac{1}{2}}e_{1i}$ are independent with equal variance σ_0^2. It is thus easily shown that $y_{0i} - y_{1i}$ and $y_{0i} + Ry_{1i}$ are uncorrelated. Hence, denoting by $r(R)$ the sample correlation coefficient between these two variates, we know that $(n-2)r^2(R)\{1-[r(R)]^2\}^{-1}$ has the F distribution with degrees of freedom (1, n-2). It follows from this result that we can construct the following $1-\nu$ confidence region for R:

$$\{R: \ (n-2)r^2(R)[1-(r(R))^2]^{-1} \leq F_{1,n-2}(\nu)\}, \qquad (2.2.16)$$

where $F_{1,n-2}(\nu)$ is the $100(1-\nu)$ percentile of the F distribution with (1, n-2) degrees of freedom. Actually, since we know that $R \geq 0$, our region is the intersection of (2.2.16) with the half line $[0,\infty)$.

In terms of the original data (y_0, y_1), $r(R)$ can be expressed as

$$r(R) = \frac{(S_{00}-S_{01}) - R(S_{11}-S_{01})}{[(S_{11} + S_{00} - 2S_{01})(R^2S_{11} + 2RS_{01} + S_{00})]^{\frac{1}{2}}},$$

and hence

$$r^2(R)[1-(r(R))^2]^{-1} = \frac{[(S_{00}-S_{01}) - R(S_{11}-S_{01})]^2}{(1+R)^2(S_{11}S_{00}-S_{01}^2)}. \qquad (2.2.17)$$

Substituting (2.2.17) into (2.2.16), we get the following inequality:

$$R^2[(S_{11}-S_{01})^2-c] - 2R[(S_{00}-S_{01})(S_{11}-S_{01})+c] + [(S_{00}-S_{01})^2-c] \leq 0, \qquad (2.2.18)$$

where $c = (n-2)^{-1}(S_{11}S_{00}-S_{12}^2)F_{1,n-2}(\nu)$. The two roots of the equation obtained by setting the left side of (2.2.18) equal to 0 are

$$B_1 = \frac{(S_{00}-S_{01})(S_{11}-S_{01}) + c + c^{\frac{1}{2}}|S_{11}+S_{00}-2S_{01}|}{(S_{11}-S_{01})^2 - c}$$

and

$$B_2 = \frac{(S_{00}-S_{01})(S_{11}-S_{01}) + c - c^{\frac{1}{2}}|S_{11}+S_{00}-2S_{01}|}{(S_{11}-S_{01})^2 - c}.$$

Consequently, the $1-\nu$ confidence region for R defined by the intersection of (2.2.16) with $[0,\infty)$ is as follows:

(1) $B_2 < R < B_1$, if $(S_{11}-S_{01})^2 > c$ and $k_1 \geq k_2$,

(2) $0 < R < B_1$, if $(S_{11}-S_{01})^2 > c$, $|k_1| < k_2$,

(3) $0 < R < B_1$, $R > B_2$, if $(S_{11}-S_{01})^2 < c$, $k_1 \le -k_2$

(4) $R > B_2 > 0$, if $(S_{11}-S_{01})^2 < c$, $|k_1| < k_2$

(5) no interval for R, otherwise, where

$$k_1 = (S_{00}-S_{01})(S_{11}-S_{01}) + c,$$

$$k_2 = c^{\frac{1}{2}}|S_{00} + S_{11} - 2S_{01}|.$$

2.3 The Case Where the Relative Precision τ_0 of the Control Is Known

As discussed in Chapter 1, it is sometime reasonable to assume that τ_0 is known. For example, the control instrument may have previously been used many times on the same population of units as used in the experiment modeled by (2.0.1), with repeated measurements taken on each unit used. In this case, an exact $1-\nu$ confidence interval can be formed by a standard method. If such an interval is narrow enough, we can assume that τ_0 is known.

Thus, assume that τ_0 is known. Note that $\pi_1\pi_0^{-1} = \tau_1\tau_0^{-1}$. Thus, the hypotheses (2.0.2) are equivalent to

$$H_0:\ \tau_1 \le \tau_0, \qquad H_1:\ \tau_1 > \tau_0. \tag{2.3.1}$$

When τ_0 is known, and $S_{01}^2 S_{00}^{-1} S_{11}^{-1} \le \tau_0(1+\tau_0)^{-1}$, then the maximum likelihood estimators for the model (2.0.1) are the following:

$$\hat{\mu} = \bar{y}_0,\ \hat{\alpha} = \bar{y}_1 - \hat{\beta}\bar{y}_0,\ \ \hat{\beta} = S_{01}S_{00}^{-1}(1+\tau_0)\tau_0^{-1},$$

$$\hat{\sigma}_u^2 = S_{00}\tau_0(1+\tau_0)^{-1},\ \hat{\sigma}_0^2 = S_{00}(1+\tau_0)^{-1},\ \hat{\sigma}_1^2 = S_{11}-S_{01}^2S_{00}^{-1}(1+\tau_0)\tau_0^{-1}. \tag{2.3.2}$$

Hence, the maximum likelihood estimator of the relative precision τ_1 is

$$\hat{\tau}_1 = \hat{\beta}^2\hat{\sigma}_u^2\,\hat{\sigma}_1^{-2} = \frac{(1+\tau_0)\tau_0^{-1}}{r^{-2} - (1+\tau_0)\tau_0^{-1}}, \tag{2.3.3}$$

where $r = S_{01}(S_{11}S_{22})^{-\frac{1}{2}}$ is the sample correlation coefficient between y_0 and y_1. Note that $\hat{\tau}_1$ is a function of r and τ_0. Since τ_0 is a known constant, the distribution of $\hat{\tau}_1$ can be obtained from the known distribution of r.

We might think of using $\hat{\tau}_1$ as the test statistic for testing the hypotheses (2.3.1). However, because $\hat{\tau}_1$ is increasing in r^2, an equivalent test is as follows:

$$\text{Reject } H_0 \text{ if } r^2 > c_3^2, \tag{2.3.4}$$

where $c_3 > 0$ is chosen so that the test (2.3.4) is of size ν.

In order to determine the value of c_3, we need to know the relation between ρ and (τ_1,τ_0), where ρ is the population correlation coefficient of y_0 and y_1. It is easily shown that

$$\rho = \frac{\beta\sigma_u^2}{\sqrt{(\sigma_0^2 + \sigma_u^2)(\sigma_1^2 + \beta^2\sigma_u^2)}}\,. \tag{2.3.5}$$

Note that in terms of τ_1 and τ_0, ρ^2 can be expressed as

$$\rho^2 = \frac{\tau_0\tau_1}{(1+\tau_0)(1+\tau_1)}. \tag{2.3.6}$$

It is apparent that ρ^2 is increasing in τ_1 for fixed τ_0. Since τ_0 is a known constant, the hypotheses (2.3.1) can be equivalently stated as

$$H_0:\ \rho^2 \le \delta_0^2, \quad H_1:\ \rho^2 > \delta_0^2, \tag{2.3.7}$$

where

$$\delta_0 = \frac{\tau_0}{1+\tau_0}. \tag{2.3.8}$$

It is known that if c_3 satisfies the following condition:

$$P\{r > c_3 \text{ or } r < -c_3 | \rho = \delta_0\} = \nu, \tag{2.3.9}$$

where δ_0 is defined by (2.3.8), then the test (2.3.4) is the LRT, and also the UMPU size ν test, for the hypotheses (2.3.7) in cases where the covariance matrix Σ_y of $(y_0, y_1)'$ is unrestricted. However, the covariance matrix Σ_y is restricted by the model (2.0.1). To see this fact, first let

$$\Sigma_y = \begin{pmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{01} & \sigma_{11} \end{pmatrix}.$$

For any point $(\sigma_{00}, \sigma_{11}, \sigma_{01})$, the inverse image $(\beta, \sigma_0^2, \sigma_1^2)$ of this point is defined by

$$\sigma_0^2 = \sigma_{00}(1+\tau_0)^{-1}, \quad \sigma_1^2 = \sigma_{11} - \sigma_{01}^2\sigma_{00}^{-1}(1+\tau_0)\tau_0^{-1},$$

$$\beta = \sigma_{01}\sigma_{00}^{-1}\tau_0^{-1}(1+\tau_0).$$

The requirement that $\sigma_0^2 \geq 0$, $\sigma_1^2 \geq 0$ restricts $\rho^2 = \sigma_{00}^{-1}\sigma_{11}^{-1}\sigma_{01}^2$ to be less than or equal to $\tau_0(1+\tau_0)^{-1}$. However, the known constant τ_0 is positive, so that $\tau_0(1+\tau_0)^{-1} < 1$. Hence, the test (2.3.4) need not be a LRT for the hypotheses (2.3.1).

It can be shown that the likelihood ratio test statistic for the hypotheses (2.3.7) under the inequality restriction $\rho^2 \leq (1+\tau_0)^{-1}\tau_0$ agrees with the LRT statistic whenever the sample variances S_{00}, S_{11}

and covariance S_{01} of y_0 and y_1 satisfy

$$r^2 = S_{01}^2 \; S_{11}^{-1} \; S_{22}^{-1} \le \tau_0(1+\tau_0)^{-1}.$$

Since S_{ij} is a consistent estimator of σ_{ij}, i,j = 0,1, it follows that the test (2.3.4) is asymptotically equivalent to the LRT for the hypotheses (2.3.1). Later, we will see that the test (2.3.4) is an unbiased test of size ν with c_3 satisfying the condition (2.3.9). However, for similar reasons to those mentioned in Section 2.2., the test (2.3.4) is not necessarily a uniformly most powerful unbiased test.

For fixed ν, n and τ_0, the value of c_3 which satisfies (2.3.9) can be obtained from the probability density function of r. Tables A10 and A11 provide the values of c_3 for ν = 0.01, 0.05, τ_0 = 1.0, 2.0, 4.0, 6.0, n = 10(5)30, 40, 50.

Table A10. The value of c_3 for ν = 0.05.

τ_0 \ n	10	15	20	25	30	40	50
1.0	0.83228	0.77802	0.74438	0.72095	0.70337	0.67816	0.66064
2.0	0.8974	0.86194	0.83948	0.82361	0.81158	0.79425	0.78216
4.0	0.94229	0.92126	0.90778	0.89813	0.89078	0.88007	0.87244
6.0	0.9599	0.94467	0.93481	0.9277	0.9222	0.914	0.90802

Table A11. The value of c_3 for $\nu = 0.01$.

τ_0 \ n	10	15	20	25	30	40	50
1.0	0.90039	0.84766	0.81177	0.7854	0.76514	0.73511	0.71387
2.0	0.93982	0.90649	0.8833	0.86621	0.85278	0.83276	0.81836
4.0	0.96558	0.94629	0.93292	0.92297	0.91504	0.90344	0.89514
6.0	0.97498	0.96027	0.95013	0.94244	0.93622	0.92639	0.91864

From Tables A10 and A11, we can see that the value of c_3 is increasing in τ_0. Thus, the greater the relative precision of the control, the harder it is to reject the null hypothesis. For fixed n and τ_0, the value of c_3 decreases with the Type I error ν. The value of c_3 is decreasing in n for fixed τ_0 and ν.

We now investigate the power of the test (2.3.4). Let the alternative $H_{1\Delta}$ to H_0 be defined by

$$\pi_1\pi_0^{-1} = \tau_1\tau_0^{-1} = 1 + \Delta, \quad \Delta > 0. \tag{2.3.10}$$

From (2.3.6), for fixed Δ and τ,

$$\rho^2 = \frac{(1+\Delta)\tau_0^2}{1 + (2+\Delta)\tau_0 + (1+\Delta)\tau_0^2}. \tag{2.3.11}$$

It is known that the power function of the test (2.3.4), given by

$$\text{Power}(\rho) = P\{r > c_3 \text{ or } r < -c_3 | \rho\}, \tag{2.3.12}$$

is symmetric about $\rho = 0$ and increasing in $|\rho|$, and hence in ρ^2, for fixed c_3. It is easy to see that ρ^2 is increasing in Δ for fixed τ_0 and also increasing in τ_0 for fixed Δ. Thus, we now know that for fixed ν, the power of the test (2.3.4) is increasing in Δ for fixed n and τ_0 and increasing in τ_0 for fixed Δ and n. The power function is also known to be increasing in n for fixed Δ, τ_0.

We have calculated the power of the test (2.3.4) for $\nu = 0.05$, $\Delta = 3.0, 5.0$, $n = 10(5)30,40,50$, $\tau_0 = 1.0, 2.0, 4.0, 6.0$. The results are shown in the following tables.

Table A12. The power of the test (2.3.4) for $\nu = 0.05$, $\Delta = 3.0$.

π_0 \ n	10	15	20	25	30	40	50
1.0	0.13160	0.16951	0.20450	0.23751	0.26917	0.32967	0.38635
2.0	0.14222	0.18629	0.22720	0.26590	0.30301	0.37286	0.43702
4.0	0.14895	0.19738	0.24234	0.28530	0.32638	0.40407	0.47615
6.0	0.15473	0.20772	0.25807	0.30677	0.35426	0.44533	0.52850

Table A13. The power of the test (2.3.4) for $\nu = 0.05$, $\Delta = 5.0$.

π_0 \ n	10	15	20	25	30	40	50
1.0	0.15512	0.20592	0.25297	0.29721	0.33931	0.41820	0.48983
2.0	0.16630	0.22380	0.27714	0.32725	0.37473	0.46207	0.53940
4.0	0.17316	0.23529	0.29281	0.34718	0.39840	0.49246	0.57559
6.0	0.17947	0.24663	0.30978	0.36971	0.42670	0.53111	0.62051

From the cases we have computed, the power of the test (2.3.4) is increasing in τ_0 for fixed ν, n and Δ. However, the test (2.3.4) is rather insensitive in detecting the differences of π_0 and π_1 for $n \leq 50$ as can be seen from the tables. For a better power, increasing n is needed.

It is known that the asymptotic distribution of r^2 is normal with mean ρ^2 and variance $4n^{-1}\rho^2(1-\rho^2)^2$, where ρ^2 is defined in (2.3.6). As a direct consequence of this result and (2.3.9), we can show that if c_3 satisfies the following condition:

$$c_3^2 = \frac{2z_\nu \tau_0(1+2\tau_0)}{n^{\frac{1}{2}}\,(1+\tau_0)^3} + \frac{\tau_0^2}{(1+\tau_0)^2}, \qquad (2.3.13)$$

then the test (2.3.4) is of asymptotic size ν.

Similar to the proof of Theorem 2.1.3, it can be shown that for a large n, the power of the test (2.3.4) against the alternative $H_{1\Delta}$ given by (2.3.12) can be approximated by

$$\int_{c_3^2}^{\infty} g_3(x|\Delta,\ \tau_0)dx,$$

where c_3^2 satisfies (2.3.13) and $g_3(x|\Delta,\ \tau_0)$ is the probability density function of a normal random variable with mean ρ^2 and variance $4n^{-1}\rho^2(1-\rho^2)^2$, and ρ^2 is defined by (2.3.11). Using this result, we have calculated the power of the test (2.3.4) for $\nu = 0.05$, $\Delta = 3.0$, 5.0, $\tau_0 = 1.0$, 2.0, 4.0, 6.0, $n = 150$, 200(100)500. The results are shown in Tables A14 and A15.

Table A14. The power of the test (2.3.4) for $\nu = 0.05$, $\Delta = 3.0$, and a large n.

τ_0 \ n	150	200	300	400	500
1.0	0.78665	0.87890	0.96391	0.99002	0.9974
2.0	0.82891	0.91883	0.98413	0.99721	0.9996
4.0	0.84524	0.93514	0.99068	0.9989	1.0000
6.0	0.84892	0.93959	0.99231	0.9992	1.0000

Table A15. The power of the test (2.3.4) for $\nu = 0.05$, $\Delta = 5.0$, and a large n.

τ_0 \ n	150	200	300	400	500
1.0	0.87893	0.95787	0.99350	0.9991	1.0000
2.0	0.92652	0.97651	0.99807	1.0000	1.0000
4.0	0.93636	0.98266	0.99991	1.0000	1.0000
6.0	0.93853	0.98472	0.9993	1.0000	1.0000

Turning now to the problem of finding a confidence interval for $\pi_0^{-1}\pi_1$, note that since τ_0 is assumed to be a known constant, and $\pi_1\pi_0^{-1} = \tau_1\tau_0^{-1}$, finding a confidence interval for $\pi_1\pi_0^{-1}$ is equivalent to finding a confidence interval for τ_1. First, let us consider a two-sided $1-\nu$ confidence for ρ. For fixed ν and n, and the observed sample correlation coefficient r, a two-sided $1-\nu$ confidence interval given by Graybill (1976, p. 400) is

$$b_0 \leq \rho \leq b_1, \tag{2.3.14}$$

where b_0 and b_1 are the values of ρ that satisfy (2.315) and (2.3.16) respectively,

$$\int_{-1}^{r} f(r|\rho)dr = 1 - \frac{\nu}{2}, \tag{2.3.15}$$

$$\int_{-1}^{r} f(r|\rho)dr = \frac{\nu}{2}. \tag{2.3.16}$$

From the interval for ρ in (2.3.14), we therefore obtain a $1-\nu$ confidence interval for ρ^2, that is

$$\begin{aligned} b_0^2 \leq \rho^2 \leq b_1^2, &\quad \text{if } b_0 \geq 0 \\ 0 \leq \rho^2 \leq \max(b_0^2, b_1^2), &\quad \text{if } b_0 < 0 < b_1 \\ b_1^2 \leq \rho^2 \leq b_0^2, &\quad \text{if } b_1 < 0. \end{aligned} \tag{2.3.17}$$

Substituting (2.3.6) into (2.3.17), we obtain a $1-\nu$ confidence interval for τ_1:

$$\frac{b_0^2}{\tau_0(1+\tau_0)^{-1} - b_0^2} \leq \tau_1 \leq \frac{b_1^2}{\tau_0(1+\tau_0)^{-1} - b_1^2}, \text{ if } b_0 \geq 0 \text{ and } b_1^2 \leq \tau_0(1+\tau_0)^{-1},$$

$$\frac{b_1^2}{\tau_0(1+\tau_0)^{-1} - b_1^2} \leq \tau_1 \leq \frac{b_0^2}{\tau_0(1+\tau_0)^{-1} - b_0^2}, \text{ if } b_1 < 0 \text{ and } b_0^2 \leq \tau_0(1+\tau_0)^{-1},$$

$$\frac{b_0^2}{\tau_0(1+\tau_0)^{-1} - b_0^2} \leq \tau_1 < \infty, \quad \text{if } b_0 \geq 0,\ b_0^2 \leq \tau_0(1+\tau_0)^{-1} \leq b_1^2,$$

$$\frac{b_1^2}{\tau_0(1+\tau_0)^{-1} - b_1^2} \leq \tau_1 < \infty, \quad \text{if } b_1 < 0,\ b_1^2 \leq \tau_0(1+\tau_0)^{-1} \leq b_0^2,$$

$$0 < \tau_1 < \frac{\max(b_0^2,b_1^2)}{\tau_0(1+\tau_0)^{-1} - \max(b_0^2,b_1^2)}, \text{ if } b_0 < 0 < b_1 \text{ and}$$

$$\tau_0(1+\tau_0)^{-1} > \max(b_0^2,b_1^2).$$

However, if b_0 and b_1 have the same sign and $\tau_0(1+\tau_0)^{-1} \leq \min(b_0^2,b_1^2)$, or if $b_0 < 0 < b_1$ and $\tau_0(1+\tau_0)^{-1} < \max(b_0^2,b_1^2)$, then there does not exist an interval for τ_1.

2.4 The Case Where Consistent Independent Estimators of σ_0^2 and σ_1^2 Exist

In this section, we will assume that in addition to the data (y_{0i}, y_{1i}), $1 \leq i \leq n$, modeled by (2.0.1), we also have estimators $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$ which are independent both of the vectors (y_{0i}, y_{1i}), $1 \leq i \leq n$, and of each other. We also will assume that $\sigma_i^2 \sim n_i^{-1}\sigma_i^2\chi_{n_i}^2$, $i = 0, 1$. Note that as $n_0, n_1 \to \infty$,

$$\lim \hat{\sigma}_i^2 = \sigma_i^2, \quad i = 0, 1,$$

with probability one.

Such independent consistent estimators of σ_0^2, σ_1^2 are usually obtained in practice in one of two ways. First, we could have data

$$y_{ijk} = \alpha_i + \beta_i u_j^{(i)} + e_{ijk},\ k = 1,\ldots,r_{ij};\ j = 1,\ldots,m_i,$$

from past independent experiments on instruments $i = 0$, $i = 1$. Here, y_{ijk} is the kth measurement on unit $u_j^{(i)}$ obtained by the ith instrument, where the units $u_j^{(0)}$ and $u_j^{(1)}$ are not necessarily the same, but are obtained from the same population. For fixed i ($i = 0,1$), the true values $u_j^{(i)}$ are assumed to be i.i.d. $N(\mu, \sigma_u^2)$ and independent of the measurement errors e_{ijk}, where the e_{ijk} are i.i.d. $N(0, \sigma_i^2)$. In this case,

$$\hat{\sigma}_i^2 = n_i^{-1} \sum_{j=1}^{m_i} \sum_{k=1}^{r_{ij}} (y_{ijk} - \bar{y}_{ij})^2, \quad \bar{y}_{ij} = r_{ij}^{-1} \sum_{k=1}^{r_{ij}} y_{ijk}, \quad i = 0,1,$$

are known to be independent, with

$$n_i \hat{\sigma}_i^2 \sim \sigma_i^2 \chi_{n_i}^2, \quad n_i = \sum_{j=1}^{m_i} (r_{ij} - 1).$$

Alternatively, in place of the experiment modeled by (2.0.1), we could actually have data y_{ijk} obtained by taking, for both instruments, an equal number r, $r > 1$, of repeated observations on each unit. If we assume that errors of measurement are independent over both units and replications, our model for the data is

$$\begin{pmatrix} y_{0jk} \\ y_{ijk} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} + \begin{pmatrix} 1 \\ \beta \end{pmatrix} u_j + \begin{pmatrix} e_{0jk} \\ e_{ijk} \end{pmatrix}, \; k = 1,\ldots,r, \; j = 1,\ldots,n,$$

where the u_j are i.i.d. $N(\mu, \sigma_u^2)$ independent of the vectors $(e_{0jk}, e_{1jk})'$, and

$$\begin{pmatrix} e_{0jk} \\ e_{1jk} \end{pmatrix} \text{ are i.i.d. } BVN(\underset{\sim}{0}, \operatorname{diag}(\tilde{\sigma}_0^2, \tilde{\sigma}_1^2)).$$

In this case, a sufficient statistic for the parameters is

$$\bar{y}_{ij} = r^{-1} \sum_{k=1}^{r} y_{ijk}, \quad w_i = n^{-1}(r-1)^{-1} \sum_{j=1}^{n} \sum_{k=1}^{r} (y_{ijk}-\bar{y}_{ij})^2, \quad i = 0,1.$$

Note that the means $\bar{y}_{ij}$ follow the model (2.0.1) with $\sigma_i^2 = r^{-1}\hat{\sigma}_i^2$, while

$$\hat{\sigma}_i^2 = \frac{w_i}{r} \sim \frac{\sigma_i^2}{n(r-1)} \chi^2_{n(r-1)}$$

are consistent (as $n \to \infty$) estimators of σ_i^2, $i = 0,1$, independent of $\bar{y}_{ij}$, $1 \leq j \leq n$, $i = 0,1$.

In both of the above cases, it is possible to write down the likelihood function, and maximize this likelihood function with respect to the unknown parameters. However, the maximum likelihood estimators are complicated functions of the data, and a computer algorithm is required to obtain such estimators.

Instead, in this section we take a simple, but possibly less efficient approach. Using $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$, we estimate $R = \sigma_1^{-2}\sigma_0^2$ by $\hat{R} = \hat{\sigma}_1^{-2}\hat{\sigma}_0^2$, and treat $\hat{R}$ as if it were the true value of R, thus reducing the problem to that treated in Section 2.1. We then consider the (asymptotic) properties of the resulting procedures as $n \to \infty$, assuming that the quantities n_0 and n_1 also tend to infinity as $n \to \infty$. Assume that the limits

$$t_i = \lim_{n\to\infty} \frac{n_i}{n}, \quad i = 0,1, \tag{2.4.1}$$

exist and are both positive. We can identify four cases of practical interest:

<u>Case 1</u>: $t_0 = t_1 = \infty$,

Case 2: $t_0 < \infty$, $t_1 = \infty$,

Case 3: $t_0 = \infty$, $t_1 < \infty$,

Case 4: $t_0, t_1 < \infty$.

It is intuitively obvious that in Case 1, the error involved in assuming that $R = \hat{R}$ is (asymptotically) negligible when compared to other sources of errors in the estimation procedure, so that the results in Section 2.1 are directly applicable. It is hard to imagine Case 2 occuring in practice, since this would imply greater knowledge of (or experience with) instrument 1 than with the control instrument. However, in such a situation, we could assume that $\sigma_1^2 = \hat{\sigma}_1^2$ with only negligible error. Case 3 is certainly possible in practice, and in this case we can assume that $\sigma_0^2 = \hat{\sigma}_0^2$ with negligible error asymptotically. After switching the roles of the instruments in Case 2, so that instrument 1 becomes the "control", both Case 2 and 3 reduce to the consideration of the model (2.0.1) where the error variance σ_0^2 of the "control" instrument is known. There is a modest literature on estimation of the parameters of such a model (see, for example, Kendall and Stuart, 1979, and also Moran, 1971). From this literature, tests and estimators for $\pi_1\pi_0^{-1}$ can be obtained. However, this model will not be considered further here.

The last case, Case 4, seems to be of greatest practical merit. It is likely to be the case that some prior experience with both instruments has been obtained, but that the estimators $\hat{\sigma}_0^2$, $\hat{\sigma}_1^2$ are based on data for which the degrees of freedom n_0, n_1 are of the same order of magnitude as the sample size n of the experiment modeled in (2.0.1). (In this circumstance, it is also likely that $0 < t_1 < 1$, where t_0 can be any number in the interval $(0, \infty)$. Of course, a fifth case, where t_0 or t_1 is equal to 0, is possible, but for this case, asymptotic analysis as $n \to \infty$ is not appropriate.) Alternatively, $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$ can be obtained from the replicated extension of the model (2.0.1) discussed above, in which case $t_0 = t_1 = r-1$.

Letting $R = \hat{R}$ and using the result of Section 2.1, we estimate β and σ_u^2 by

$$\hat{\beta} = \frac{(\hat{R}S_{11}-S_{00}) + [(\hat{R}S_{11}-S_{00})^2 + 4\hat{R}S_{01}^2]^{\frac{1}{2}}}{2\hat{R}S_{01}}, \quad \hat{\sigma}_u^2 = \frac{d_1^*-d_2^*}{1+\hat{R}\hat{\beta}^2}, \qquad (2.4.2)$$

respectively, where d_1^* and d_2^*, $d_1^* \geq d_2^*$, are the eigenvalues of

$$S \begin{pmatrix} 1 & 0 \\ 0 & \hat{R} \end{pmatrix}.$$

Then we can estimate the ratio $\psi = \pi_1\pi_0^{-1}$ of the precisions π_0, π_1 of the instruments by

$$\hat{\psi} = \hat{\beta}^2\hat{R}. \qquad (2.4.3)$$

Since we are only interested in forming a confidence interval for ψ and testing the hypotheses (2.0.2), that is, H_0: $\psi \leq 1$, H_1: $\psi > 1$, the estimation of the other parameters, for example (π_0,π_1), will not be discussed here.

Because $\hat{R}$ is a consistent estimator of R, and S is a consistent estimator of

$$\Sigma_y = \begin{pmatrix} \sigma_0^2 + \sigma_u^2 & \beta\sigma_u^2 \\ \beta\sigma_u^2 & \sigma_1^2 + \beta^2\sigma_u^2 \end{pmatrix},$$

as $n \to \infty$, it is straightforward to show that $\hat{\psi}$ is a consistent estimator of ψ. If we use $\hat{\psi}$ as the test statistic for testing the hypotheses (2.0.2), then the test is

$$\text{Reject } H_0 \text{ if } \hat{\psi} > c_4; \tag{2.4.4}$$

where $c_4(> 1)$ is chosen so that the test (2.4.4) is of asymptotic size ν. Since the statistic $\hat{\psi}$ has no well known finite sample distribution, we discuss the asymptotic distribution of $\hat{\psi}$.

Theorem 2.4.1. Assume that $\hat{\sigma}_i^2 \sim n_i^{-1}\sigma_i^2\chi_{n_i}^2$, $i = 0,1$. Then, the asymptotic distribution of $n^{\frac{1}{2}}(\hat{\psi}-\psi)$ is normal with mean 0 and variance equal to E_1 and E_2 for $t_0 = t_1 = \infty$ and t_0, $t_1 < \infty$, respectively, where t_i is defined by (2.4.1), and

$$E_1 = \frac{4\psi(1 + \tau_0^{-1} + \psi)}{\tau_0},$$

$$E_2 = E_1 + \frac{2\psi^2(1 + 2\tau_0^{-1} + \psi)^2}{(1 + \psi)^2}\left(\frac{1}{t_0} + \frac{1}{t_1}\right). \tag{2.4.5}$$

Proof: Call $\hat{\beta}$ in (2.4.2) $\hat{B}(\hat{R})$. Note that

$$n^{\frac{1}{2}}(\hat{R}\hat{\beta}^2(\hat{R}) - R\beta^2) = n^{\frac{1}{2}}[\hat{R}(\hat{\beta}^2(\hat{R})-\beta^2) + \hat{R}\beta^2 - R\beta^2]$$

$$= R\, n^{\frac{1}{2}}(\hat{\beta}^2(\hat{R})-\beta^2) + n^{\frac{1}{2}}(\hat{R}-R)(\hat{\beta}^2(\hat{R})-\beta^2) + \beta^2 n^{\frac{1}{2}}(\hat{R}-R). \tag{2.4.6}$$

$$= R(\hat{\beta}(\hat{R})+\beta)n^{\frac{1}{2}}(\hat{\beta}(\hat{R})-\beta) + n^{\frac{1}{2}}(\hat{R}-R)(\hat{\beta}^2(\hat{R})-\beta^2) + \beta^2 n^{\frac{1}{2}}(\hat{R}-R),$$

and

$$n^{\frac{1}{2}}(\hat{\beta}(\hat{R})-\beta) = n^{\frac{1}{2}}(\hat{\beta}(\hat{R}) - \hat{\beta}(R)) + n^{\frac{1}{2}}(\hat{\beta}(R)-\beta)$$

$$\approx \left.\frac{\partial\hat{\beta}(\hat{R})}{\partial\hat{R}}\right|_{\substack{\hat{R}=R \\ S=\Sigma_y}} n^{\frac{1}{2}}(\hat{R}-R) + n^{\frac{1}{2}}(\hat{\beta}(R)-\beta) ,$$

where

$$\left.\frac{\partial\hat{\beta}(\hat{R})}{\partial\hat{R}}\right|_{\substack{\hat{R}=R \\ S=\Sigma_y}} = \frac{1}{\beta\tau_0 R^2}\,\frac{\psi}{1+\psi}.$$

Hence

$$n^{\frac{1}{2}}(\hat{\psi}-\psi) \approx (\beta^2 + \frac{1}{\beta\tau_0 R^2})n^{\frac{1}{2}}(\hat{R}-R) + n^{\frac{1}{2}}(\hat{\beta}(R)-\beta).$$

It is known that $n^{\frac{1}{2}}(\hat{R}-R)$ and $n^{\frac{1}{2}}(\hat{\beta}(R)-\beta)$ are independent, and the asymptotic distribution of $n^{\frac{1}{2}}(\hat{\beta}(R)-\beta)$ is normal with mean 0 and variance $R^{-1}\tau_0^{-2}(1 + (1+\psi)\tau_0)$. Using the fact that the asymptotic distribution of $n_i^{\frac{1}{2}}(\hat{\sigma}_i^2 - \sigma_i^2)$ is normal with mean 0 and variance $2\sigma_i^4$, $i = 0,1$, we obtain the asymptotic distribution of $n^{\frac{1}{2}}(\hat{R}-R)$ is normal with mean 0 and variance $2R^2(t_0^{-1} + t_1^{-1})$. Combining the results above, we obtain the asymptotic variance of $n^{\frac{1}{2}}(\hat{\psi}-\psi)$ as shown in (2.4.5). □

Before we determine the value of c_4 such that the test (2.4.4) has an asymptotic size ν, we need one more lemma.

Lemma 3. For fixed τ_0, and a large n, if $c_4 > 1$ and $\psi \leq 1$, then $P(\hat{\psi} > c_4)$ is increasing in ψ.

Proof: Note that for a large n,

$$P(\hat{\psi} > c_4) = \begin{cases} P\{x > n^{\frac{1}{2}}(\frac{c_4-\psi}{E_1^{\frac{1}{2}}})\}, & \text{if } t_0 = t_1 = \infty, \\ P\{x > n^{\frac{1}{2}}(\frac{c_4-\psi}{E_2^{\frac{1}{2}}})\}, & \text{if } t_0, t_1 < \infty, \end{cases}$$

where x is a standard normal random variable. For fixed τ_0, it is easy to see that E_1 and E_2 defined in (2.4.5) are increasing in ψ. Thus, for $c_4 > 1$ and $\psi \le 1$, $P(\hat{\psi} > c_4)$ is increasing in ψ. □

From the above lemma, we therefore know that under H_0: $\psi \le 1$, for fixed τ_0 and a large n,

$$\max_{\psi \le 1} P(\hat{\psi} > c_4) = P(\hat{\psi} > c_4 | \psi = 1).$$

Note that when $\psi = 1$, $E_1 = 4\tau_0^{-2}(1+2\tau_0)$ and $E_2 = 2\tau_0^{-2}[2(1+2\tau_0) + (1+\tau_0)^2(t_0^{-1} + t_1^{-1})]$. We summarize the result so far in the following theorem.

Theorem 2.4.2. For fixed τ_0, and a large n, if c_4 satisfies the conditions

$$c_4 = 1 + \frac{2z_\nu}{n^{\frac{1}{2}}} \tau_0^{-1}(1 + 2\tau_0)^{\frac{1}{2}}, \tag{2.4.7}$$

$$c_4 = 1 + \frac{2^{\frac{1}{2}} z_\nu \tau_0^{-1}}{n^{\frac{1}{2}}} [2(1+2\tau_0) + (1+\tau_0)^2(t_0^{-1} + t_1^{-1})]^{\frac{1}{2}}, \tag{2.4.8}$$

for $t_0 = t_1 = \infty$ and t_0, $t_1 < \infty$, respectively, then the test (2.4.4) has an asymptotic size ν.

However, as can be seen from (2.4.7) and (2.4.8), the value of c_4 depends on the unknown parameter τ_0. However, the unknown parameter τ_0 can be consistently estimated. Note that when $\psi = 1$ ($\tau_1 = \tau_0$), σ_u^2 is consistently estimated by $(\hat{R}\, S_{01}^2)^{\frac{1}{2}}$. Thus,

$$\hat{\tau}_0 = \hat{\sigma}_0^{-2}(\hat{R}\, S_{01}^2)^{\frac{1}{2}} = (\hat{\sigma}_0\hat{\sigma}_1)^{-1}(S_{01}^2)^{\frac{1}{2}} \tag{2.4.9}$$

is a consistent estimator of τ_0 when $\tau_1 = \tau_0$. If we substitute $\hat{\tau}_0$ for τ_0 in (2.4.7) and (2.4.8), standard large sample theory shows that the results of Theorem 2.4.2 still hold.

Another test statistic which can be used to test the hypotheses (2.0.2) is the T statistic defined by (2.1.8) with $\hat{R}$ substituted for R. Call this statistic $T(\hat{R})$, thus,

$$T(\hat{R}) = \frac{(n-2)^{\frac{1}{2}}(\hat{R}S_{11}-S_{00})}{2\hat{R}^{\frac{1}{2}}|S|^{\frac{1}{2}}}. \tag{2.4.10}$$

The results in Section 2.1, suggest that if we use $T(\hat{R})$ as the test statistic for the hypotheses (2.0.2), the rejection region for H_0 should have the form

$$\text{Reject } H_0 \text{ if } T(\hat{R}) > c_5, \tag{2.4.11}$$

where the value of c_5 (> 0) is chosen so that the test (2.4.11) has an asymptotic size ν. Since the proof of the asymptotic distribution of $T(\hat{R})$ given in Theorem 2.4.2 is very similar to the proof of Theorem 2.4.1, we omit the proof.

<u>Theorem 2.4.3</u>. Assume that $\hat{\sigma}_i^2 \sim n_i^{-1}\sigma_i^2\chi_{n_i}^2$, $i = 0,1$. Then, the asymptotic distribution of $(T(\hat{R}) - n^{\frac{1}{2}}\rho(1-\rho^2)^{\frac{1}{2}})$ is normal with mean 0

and variance equal to E_3 and E_4 for $t_0 = t_1 = \infty$ and t_0, $t_1 < \infty$, respectively, where t_i is defined by (2.4.1),

$$E_3 = (1-\rho^2)^{-1},$$

$$E_4 = E_3 + \frac{1}{8}\,\frac{(2 + \tau_0(1+\psi))^2(t_0^{-1} + t_1^{-1})}{(1 + \tau_0(1+\psi))} \tag{2.4.12}$$

and

$$\rho = \frac{(\psi-1)\tau_0}{\sqrt{(\psi-1)^2\tau_0^2 + 4(\psi+1)\tau_0+4}}. \tag{2.4.13}$$

Since we know that ρ is increasing in ψ and also that $(1-\rho^2)^2$ is increasing in ρ for $\rho \leq 0$ ($\psi \leq 1$), and since it is easy to show that the second term of E_4 is also increasing in ψ, it follows that E_3 and E_4 are increasing in ψ. We therefore know that for a large n,

$$\max_{\psi \leq 1} P(T(\hat{R}) > c_5) = P(T(\hat{R}) > c_5 | \psi = 1).$$

Note that $\psi = 1$ gives $\rho = 0$.

<u>Theorem 2.4.4</u>. For fixed τ_0 and a large n, if c_5 satisfies (2.4.14) and (2.4.15) for $t_0 = t_1 = \infty$ and t_0, $t_1 < \infty$, respectively,

$$c_5 = z_\nu, \tag{2.4.14}$$

$$c_5 = z_\nu\left[1 + \frac{1}{2}\,\frac{(1+\tau_0)^2}{(1+2\tau_0)}\right]^{\frac{1}{2}}, \tag{2.4.15}$$

then the test (2.4.11) has asymptotic size ν.

From (2.4.15), we can see that when t_0, $t_1 < \infty$, the value of c_5 depends on the unknown parameter τ_0. As mentioned before, if we substitute a consistent estimator of τ_0 for τ_0 in (2.4.15), then the result in Theorem 2.4.3 still holds.

Since the asymptotic distribution of $\hat{\psi}$ is known in Theorem 2.4.1, using a standard technique, we can obtain an asymptotic $1-\nu$ confidence interval for ψ. A $1-\nu$ asymptotic confidence interval for ψ is as follows:

$$\{\psi: \ \hat{\psi}-n^{-\frac{1}{2}}z_{\nu/2}\hat{E}_1 \leq \psi \leq \hat{\psi} + n^{-\frac{1}{2}}z_{\nu/2}\hat{E}_1\}, \text{ for } t_0 = t_1 = \infty,$$

$$\{\psi: \ \hat{\psi}-n^{-\frac{1}{2}}z_{\nu/2}\hat{E}_2 \leq \psi \leq \hat{\psi} + n^{-\frac{1}{2}}z_{\nu/2}\hat{E}_2\}, \text{ for } t_0, t_1 < \infty,$$

where

$$\hat{E}_1 = \frac{4\hat{\psi}(1 + \tilde{\tau}_0^{-1} + \hat{\psi})}{\tilde{\tau}_0},$$

$$\hat{E}_2 = \hat{E}_1 + \frac{2\hat{\psi}(1 + 2\tilde{\tau}_0^{-1} + \hat{\psi})^2}{(1+\hat{\psi})^2}\left(\frac{1}{t_0} + \frac{1}{t_1}\right),$$

and $\tilde{\tau}_0$ is another consistent estimator of τ_0 except $\hat{\tau}_0$ defined by (2.4.9). For example, we can choose

$$\tilde{\tau}_0 = \hat{\sigma}_u^2 \, \hat{\sigma}_0^{-2},$$

where $\hat{\sigma}_u^2$ is defined in (2.4.2), then $\tilde{\tau}_0$ is a consistent estimator of τ_0.

CHAPTER 3

ESTIMATION AND SELECTION PROCEDURES FOR THE PRECISION OF P($\geq$ 3) INSTRUMENTS

3.0 Introduction

Let our model be the same model we considered in Chapter 1:

$$\underset{\sim}{y}_j = \begin{pmatrix} y_{0j} \\ y_{1j} \\ \vdots \\ y_{p-1,j} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_1 \\ \vdots \\ \alpha_{p-1} \end{pmatrix} + \begin{pmatrix} 1 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} u_j + \begin{pmatrix} e_{0j} \\ e_{1j} \\ \vdots \\ e_{p-1,j} \end{pmatrix} = \begin{pmatrix} 0 \\ \underset{\sim}{\alpha} \end{pmatrix} + \begin{pmatrix} 1 \\ \underset{\sim}{\beta} \end{pmatrix} u_j + \underset{\sim}{e}_j, \quad (3.0.1)$$

$$j = 1,\ldots,n,$$

where each $\underset{\sim}{y}_j$ is a p-dimensional vector of observations, $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are unknown (p-1)-dimensional vector parameters, u_j is a random variable, $\underset{\sim}{e}_j$ is p-dimensional vector of errors, and p is the number of instruments.

We assume that $u_1,\ldots,u_n$ is a random sample from a normal distribution with mean μ and variance σ_u^2, and that the e_j's are an independent random sample from a p-variate normal distribution with mean vector $\underset{\sim}{0}$ and unknown covariance matrix $\Sigma_e = \text{Diag}(\sigma_0^2,\sigma_1^2,\ldots,\sigma_{p-1}^2)$. We refer to instrument 0 as the "control".

As mentioned in Chapter 1, if p = 2, the model (3.0.1) is not identifiable without an extra constraint on the parameters. In this chapter, assuming that $p \geq 3$, we consider the problem of comparing the precisions, $\pi_0, \pi_1, \ldots, \pi_{p-1}$, of p instruments, where $\pi_0 = \sigma_0^{-2}$ and $\pi_i = \beta_i^2\sigma_i^{-2}$, $i = 1,\ldots,p-1$. Because the model (3.0.1) is identifiable when $p \geq 3$, no parameter constraints are required.

This model was used by Mandel (1959), for the analysis of inter-laboratory round robins, by Mosteller (see Cochran, 1968), when a number of individuals are rated by different judges, and in a slightly different notation by Smith (1959). Since the u_i are not known in these examples, both Mandel and Mosteller suggest using $\bar{y}_{.j} = p^{-1} \sum_{i=0}^{p-1} y_{ij}$, the average over all instruments, in place of u_j as the independent variable in a classical regression model. Having obtained estimators of the parameters using least squares, Mosteller compares the relative precisions of different judges based on the resulting estimators of $\sigma_i^2\beta_i^{-2}$. Note that $(\sigma_i^2\beta_i^{-2})^{-1}$ is our definition for the precision of the ith instrument. Because $\bar{y}_{.j} \neq u_j$, bias is introduced into the estimators through this approach, requiring <u>ad hoc</u> adjustments to be made to produce asymptotic consistency. The properties of the resulting estimators do not seem to have been explored.

For the model (3.0.1) with $p = 3$, the maximum likelihood estimators of the parameters have a closed form. However, for $p > 3$, the maximum likelihood solutions for the parameters are not explicit. Barnett (1969) therefore presents some consistent moment estimators of the parameters. Alternatively, Theobald and Mallinson (1978) reparameterize the model (3.0.1) as a factor analysis model with one factor; in this case the maximum likelihood solutions may be found by using a computer algorithm to carry out factor analysis.

The asymptotic properties of the maximum likelihood estimators of the parameters have been widely studied. It has been shown that the maximum likelihood estimators are consistent and that the asymptotic joint distribution of these estimators is normal. For references, see Lawley (1953), Anderson and Rubin (1956), Jennrich and Thayer (1973), Jöreskog (1969), and Lawley (1976). Fuller, Amemiya and Pantula (1983) have given an explicit expression for the covariance matrix of the limiting joint distribution of the maximum likelihood estimators of the parameters.

In Section 3.1, we discuss the estimation of the parameters. Based on the results of Fuller et al (1983), in Section 3.2, we derive the asymptotic joint distributions of the maximum likelihood estimators of the precisions $\pi_0,\ldots,\pi_{p-1}$, and of the ratios of the precisions $\psi_1,\ldots,\psi_{p-1}$, respectively. We also find joint confidence regions for the π_i's and for the ψ_i's, respectively. In Section 3.3, we attempt to use a rule originally suggested by Paulson (1952) for choosing the largest mean among the means of p independent normal populations to here select the most precise instrument among p instruments in large samples. However, difficulties arise with the asymptotic joint distribution of the statistics used, so that the rule is not applicable. To overcome these difficulties, imposing some constraints on the parameter space seems to be necessary.

3.1 Estimation of the Parameters

It is known (see Barnett, 1969) that when p = 3, the maximum likelihood estimators have the following explicit form:

$$\hat{\mu} = \bar{y}_0, \qquad \hat{\sigma}_u^2 = S_{12}^{-1} S_{01} S_{02},$$

$$\hat{\alpha}_1 = \bar{y}_1 - \hat{\beta}_1 \bar{y}_0, \; \hat{\alpha}_2 = \bar{y}_2 - \hat{\beta}_2 \bar{y}_0,$$

$$\hat{\beta}_1 = S_{02}^{-1} S_{12}, \qquad \hat{\beta}_2 = S_{01}^{-1} S_{12},$$

$$\hat{\sigma}_0^2 = S_{00} - \hat{\sigma}_u^2, \quad \hat{\sigma}_1^2 = S_{11} - \hat{\beta}_1^2 \hat{\sigma}_u^2, \; \hat{\sigma}_2^2 = S_{22} - \hat{\beta}_2^2 \hat{\sigma}_u^2, \qquad (3.1.1)$$

where

$$\bar{y}_i = n^{-1} \sum_{k=1}^{n} y_{ik}, \quad S_{ij} = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} (y_{ij} - \bar{y}_i)(y_{jk} - \bar{y}_j),$$
$$i,j = 0,\ldots,p-1.$$

For p > 3, the maximum likelihood estimators of μ and α_i have the usual forms

$$\hat{\mu} = \bar{y}_0, \qquad \hat{\alpha}_i = \bar{y}_i - \hat{\beta}_i \bar{y}_0.$$

The maximum likelihood estimators of the remaining parameters satisfy the following equations:

$$\hat{\Sigma}_y^{-1}(S - \hat{\Sigma}_y)\hat{\Sigma}_y^{-1}\begin{pmatrix} 1 \\ \hat{\beta} \end{pmatrix} = \underset{\sim}{0}$$

$$\hat{\sigma}_0^2 = S_{00} - \hat{\sigma}_u^2, \; \hat{\sigma}_i^2 = S_{ii} - \hat{\beta}_i^2 \hat{\sigma}_u^2, \; i = 1,\ldots,p-1, \qquad (3.1.2)$$

where

$$\hat{\Sigma}_y = \begin{pmatrix} 1 \\ \hat{\beta} \end{pmatrix} \begin{pmatrix} 1 \\ \hat{\beta} \end{pmatrix}' \hat{\sigma}_u^2 + \mathrm{Diag}(\hat{\sigma}_0^2,\ldots,\hat{\sigma}_{p-1}^2).$$

It appears from (3.1.2) that there is little hope of obtaining explicit closed form expressions for the maximum likelihood estimators when p > 3.

As mentioned in Chapter 1, Theobald and Mallinson (1978) reparameterize the model (3.0.1) such that $\sigma_u^2 = 1$, β_0 is not equal to 1 but equal to $\lambda_0 = \sigma_u$ and β_i corresponds to $\lambda_i = \beta_i \sigma_u$, $i = 1,\ldots,p-1$. That is, their model is

$$\begin{pmatrix} y_{0j} \\ y_{1j} \\ \vdots \\ y_{p-1,j} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_0 \\ \vdots \\ \alpha_1 \end{pmatrix} + \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{p-1} \end{pmatrix} f_j + \begin{pmatrix} e_{0j} \\ \vdots \\ e_{p-1,j} \end{pmatrix}, \quad j = 1,\ldots,r, \tag{3.1.3}$$

where $f_1,\ldots,f_n$ are an independent random sample from $N(\eta, 1)$ where $\eta = \sigma_u^{-1}\mu$. The model (3.1.3) is known to be a factor analysis model with one factor, and the maximum likelihood estimators may be obtained by a computer algorithm (for references, see Jöreskog, 1969, Lawley and Maxwell, 1971).

Instead of finding the maximum likelihood estimators of the parameters of the model (3.0.1), Barnett (1969) presents a set of p-2 alternative estimators of β_i, namely

$$\tilde{\beta}_{i(j)} = S_{ij}S_{0j}^{-1}, \quad i = 1,\ldots,p-1;\ j \neq 0,\ i,$$

and a set of $\frac{1}{2}$ (p-1)(p-2) alternative estimators of σ_u^2, namely

$$(\tilde{\sigma}_u^2)_{jk} = S_{0j}S_{0k}S_{jk}^{-1}, \quad j, k = 1,\ldots,p-1;\ j \neq k.$$

Correspondingly, there is a set of p-2 estimators of each α_i, each having the general form $\tilde{\alpha}_{ij} = \bar{y}_i - \tilde{\beta}_{i(j)}\bar{y}_0$. There will also be a set of estimators of each σ_i^2, each having the general form

$$\tilde{\sigma}^2_{i(j)} = S_{ii} - \tilde{\beta}^2_{i(j)}(\tilde{\sigma}^2_u)_{ij}, \quad i = 1,\ldots,p-1.$$

Corresponding to each $(\tilde{\sigma}^2_u)_{jk}$, there is one estimator for σ^2_0, that is

$$(\tilde{\sigma}^2_0)_{jk} = S_{00} - (\tilde{\sigma}^2_u)_{jk}, \quad j = 1,\ldots,p-1, \; j \neq k.$$

In his paper, Barnett also considers the possibility of combining the alternative estimators of each parameter, that is,

$$\bar{\beta}_i = \sum_{j\neq 0,i} \lambda_j \tilde{\beta}_{ij}, \quad \bar{\sigma}^2_u = \sum_{j\neq k} \mu_{jk}(\tilde{\sigma}^2_u)_{jk},$$

where the λ_j and μ_{jk} are chosen to minimize the asymptotic variances. However, the complicated dependence of variances and covariances of the sets of β_i-estimators and σ^2_u-estimators on the unknown parameters $\beta_1,\ldots,\beta_{p-1}$, $\sigma^2_0,\ldots,\sigma^2_{p-1}$, and σ^2_u precludes the possibility of optimal use of complete sets in general.

3.2 Asymptotic Joint Distribution of the Maximum Likelihood Estimators of the Precisions

Let

$$\hat{\beta} = (\hat{\beta}_1,\ldots,\hat{\beta}_{p-1})', \quad \hat{\gamma} = (\hat{\sigma}^2_1,\ldots,\hat{\sigma}^2_{p-1}, \hat{\sigma}^2_0)'$$

be the (vectors of) maximum likelihood estimators for

$$\beta = (\beta_1,\ldots,\beta_{p-1})', \quad \gamma = (\sigma^2_1,\ldots,\sigma^2_{p-1}, \sigma^2_0)',$$

respectively. Also, let

$$\sigma^2 = (\sigma^2_1,\ldots,\sigma^2_{p-1})'.$$

For notational convenience, we adopt the convention that for any vector $t = (t_1,\ldots,t_r)'$,

$$D_t = \mathrm{diag}(t_1,\ldots,t_r) = \begin{pmatrix} t_1 & & 0 \\ & t_2 & \\ & & \ddots \\ 0 & & & t_r \end{pmatrix}.$$

Thus, $D_\beta = \mathrm{diag}(\beta_1,\ldots,\beta_{p-1})$, $D_{\sigma^2} = \mathrm{diag}(\sigma_1^2,\ldots,\sigma_{p-1}^2)$, $D_\gamma = \mathrm{diag}(\sigma_1^2,\ldots,\sigma_{p-1}^2, \sigma_0^2)$, etc. The symbol 0 will be used to represent a scalar, a zero vector, or a matrix of zeroes; the dimensions of such a vector or matrix will always be clear from the context.

Fuller et al (1983) have considered a generalization of the model (3.0.1), in which u_i is a k-dimensional vector of normal variables and β is a $r \times k$ matrix of unknown parameters. They have shown that the asymptotic joint distribution of the maximum likelihood estimators is multivariate normal, and have given an explicit expression for the covariance matrix of this asymptotic distribution. When $k = 1$, $r = p-1$, their results can be simplified into the following form.

Theorem 3.2.1. Under the model (3.0.1),

$$\sqrt{n}\begin{pmatrix} \hat{\beta} - \beta \\ \hat{\gamma} - \gamma \end{pmatrix} \xrightarrow{L} \mathrm{MVN}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} V_{\beta\beta} & V_{\beta\gamma} \\ V'_{\beta\gamma} & V_{\gamma\gamma} \end{pmatrix}\right),$$

where

$$V_{\beta\beta} = W + PV_{\gamma\gamma}P', \qquad V_{\beta\gamma} = -PV_{\gamma\gamma},$$

$$W = \frac{1}{\sigma_u^2}\left(1 + \frac{1}{H\tau_0}\right)(D_{\sigma^2} + \sigma_0^2\beta\beta'),$$

$$P = \frac{1}{\tau_0 H} D^{-1}_{\sigma^2}(D_\beta, -\beta),$$

$$H = \sum_{i=1}^{p} \psi_i, \ \psi_p \equiv 1, \quad \tau_0 = \sigma_0^{-2} \sigma_u^2,$$

$$V_{\gamma\gamma} = 2 D_\gamma Q^{-1} D_\gamma ,$$

and $Q = ((Q_{ij}))$ is defined by

$$H^2 Q_{ij} = \begin{cases} \psi_i \psi_j, & i \neq j, \\ (H-\psi_i)^2, & i = j, \end{cases}$$

for $i,j = 1,2,\ldots,p$.

We here are interested in the maximum likelihood estimators of

$$\pi = (\pi_1,\ldots,\pi_{p-1},\pi_0)', \quad \psi = (\psi_1,\ldots,\psi_{p-1})',$$

where $\pi_0 = \sigma_0^{-2}$, $\pi_i = \sigma_i^{-2}\beta_i^2$, $\psi_i = \pi_0^{-1}\pi_i$, $i = 1,\ldots,p-1$. By the invariance property of maximum likelihood, the maximum likelihood estimators of π and ψ are

$$\begin{aligned} \hat{\pi} &= (\hat{\sigma}_1^{-2}\hat{\beta}_1^2,\ldots,\hat{\sigma}_{p-1}^{-2}\ \hat{\beta}_{p-1}^2,\ \hat{\sigma}_0^{-2})', \\ \hat{\psi} &= (\hat{\sigma}_1^{-2}\hat{\sigma}_0^2\hat{\beta}_1^2,\ldots,\hat{\sigma}_{p-1}^{-2}\hat{\sigma}_0^2\ \hat{\beta}_{p-1}^2)', \end{aligned} \tag{3.2.1}$$

respectively.

It follows from (3.2.1) that

$$\begin{aligned} \hat{\pi} - \pi &= \begin{pmatrix} D_{\hat{\beta}} + D_\beta \\ 0 \end{pmatrix} D^{-1}_{\hat{\sigma}^2}(\hat{\beta}-\beta) - \begin{pmatrix} D_\beta & 0 \\ 0 & 1 \end{pmatrix}^2 D_\gamma^{-1} D_{\hat{\gamma}}^{-1}(\hat{\gamma}-\gamma), \\ \hat{\psi} - \psi &= \hat{\sigma}_0^2(D_{\hat{\beta}} + D_\beta)D^{-1}_{\hat{\sigma}^2}(\hat{\beta}-\beta) - \sigma_0^2 D^{-1}_{\hat{\sigma}^2} D_\beta (D_\beta,-\beta) D_\gamma^{-1}(\hat{\gamma}-\gamma). \end{aligned} \tag{3.2.2}$$

Since $\hat{\beta}$ and $\hat{\gamma}$ are consistent estimators of β and γ, respectively, the following lemma is a direct consequence of Theorem 3.2.1 and (3.2.2).

Lemma 1. Under the model (3.0.1),

$$\sqrt{n}(\hat{\pi}-\pi) \xrightarrow{L} MVN(0,\ A_1WA_1' + (A_1P-B_1)V_{\gamma\gamma}(A_1P-B_1)'),$$

and

$$\sqrt{n}(\hat{\psi}-\psi) \xrightarrow{L} MVN(0,\ A_2WA_2' + (A_2P-B_2)V_{\gamma\gamma}(A_2P-B_2)'),$$

where

$$A_1 = (2D^{-1}_{\sigma^2}D_\beta,0)', \quad A_2 = 2\sigma_0^2D^{-1}_{\sigma^2}D_\beta,$$

$$B_1 = -(D^{-2}_{\sigma^2}D^2_\beta,0)', \quad B_2 = -\sigma_0^2D^{-1}_{\sigma^2}D_\beta(D_\beta,-\beta)D_\gamma^{-1}.$$

Note that

$$A_1WA_1' = 4\sigma_u^{-2}(1+(H\tau_0)^{-1})(D_\pi + \pi\pi'),\ A_2WA_2' = 4\tau_0^{-1}(1+H\tau_0)^{-1})(D_\psi+\psi\psi'). \quad (3.2.3)$$

$$A_1P-B_1 = LD_\gamma^{-1},\ L = (1 + \frac{2}{H\tau_0})D_\pi - \frac{2}{H\tau_0}(0,\pi), \quad (3.2.4)$$

$$A_2P-B_2 = (1 + \frac{2}{H\tau_0})D_\psi,-\psi)D_\gamma^{-1}. \quad (3.2.5)$$

Theorem 3.2.2. Under the model (3.0.1),

$$\sqrt{n}(\hat{\pi}-\pi) \xrightarrow{L} MVN(0,\ \frac{4}{\sigma_u^2}(1 + \frac{1}{H\tau_0})(D_\pi + \pi\pi') + 2LQ^{-1}L'),$$

and

$$\sqrt{n}(\hat{\psi}-\psi) \xrightarrow{L} MVN(0,\ \frac{4}{\tau_0}(1 + \frac{1}{H\tau_0})(D_\psi + \psi\psi') + 2(1 + \frac{2}{H\tau_0})^2(D_\psi,\ -\psi)Q^{-1}(D_\psi,\ -\psi)').$$

Proof: The stated results follow directly from Lemma 1 and (3.2.3) through (3.2.5). □

The expressions for the asymptotic covariance matrices, $\mathrm{cov}(\hat{\pi})$ and $\mathrm{cov}(\hat{\psi})$, for $\hat{\pi}$ and $\hat{\psi}$ are rather complicated. The principal complication is the dependence of these matrices on Q^{-1}, which is difficult to write in explicit form. One exception is the case $p = 3$, where

$$Q^{-1} = \frac{H}{2\psi_1\psi_2}\begin{pmatrix}\psi_1^2 & 0 & 0\\ 0 & \psi_2^2 & 0\\ 0 & 0 & H\end{pmatrix} + 1_3 1_3', \tag{3.2.6}$$

where $1_3 = (1, 1, 1)'$. When $p > 3$, inversion of Q is best left to a computer.

Let

$$\mathrm{cov}(\hat{\psi}) = \begin{pmatrix}\mathrm{var}(\hat{\psi}_1) & & \mathrm{cov}(\hat{\psi}_i,\hat{\psi}_j)\\ & \ddots & \\ \mathrm{cov}(\hat{\psi}_i,\hat{\psi}_j) & & \mathrm{var}(\hat{\psi}_{p-1})\end{pmatrix};$$

that is, $\mathrm{var}(\hat{\psi}_i)$ is the variance of the asymptotic (normal) distribution of $\hat{\psi}_i$, and $\mathrm{cov}(\hat{\psi}_i,\hat{\psi}_j)$ is the covariance between $\hat{\psi}_i$ and $\hat{\psi}_j$ in the asymptotic (bivariate normal) distribution of $\hat{\psi}_i$ and $\hat{\psi}_j$. When $p = 3$, it follows from Theorem 3.2.2 and (3.2.6) that

$$\mathrm{var}(\hat{\psi}_i) = \frac{4}{\tau_0}\left(1 + \frac{1}{\tau_0(1+\psi_1+\psi_2)}\right)\psi_i(1+\psi_i)$$
$$+ \left(1 + \frac{2}{\tau_0(1+\psi_1+\psi_2)}\right)^2(1+\psi_1+\psi_2)\,\frac{\psi_i^2(1+\psi_i)^2}{\psi_1\psi_2}, \tag{3.2.7}$$

and

$$\text{cov}(\hat{\psi}_1,\hat{\psi}_2) = \frac{4}{\tau_0} (1 + \frac{1}{\tau_0(1 + \psi_1 + \psi_2)})\psi_1\psi_2$$

$$+ (1 + \frac{2}{\tau_0(1 + \psi_1 + \psi_2)})^2(1 + \psi_1 + \psi_2)(1 + \psi_1+\psi_2 - \psi_1\psi_2). \tag{3.2.8}$$

Note that $\text{var}(\hat{\psi}_1)$ not only depends upon ψ_1, but also upon ψ_2 and τ_0, and that $\text{var}(\hat{\psi}_1)$ can be arbitrarily large when either $\psi_2 \approx 0$ (instrument 2 has small precision relative to the control) or $\tau_0 \approx 0$ (the relative precision of the control is small). Similar remarks hold for $\text{var}(\hat{\psi}_2)$. Also note that $\text{cov}(\hat{\psi}_1,\hat{\psi}_2) \geq 0$ when $\psi_1\psi_2 \leq 1$.

Similar remarks can be made for $\text{var}(\hat{\psi}_i)$ when $p > 3$, and $\text{var}(\hat{\pi}_i)$ when $p \geq 3$. That is, the asymptotic variances of the maximum likelihood estimators of the precision π_i, or of the ratio $\psi_i = \pi_i\pi_0^{-1}$, is enlarged by inclusion of any imprecise instrument in the experiment.

To construct joint confidence regions for the π_i's or the ψ_i's, respectively, we need the following theorem.

Theorem 3.2.3. (Šidák, 1967). Let $X = (X_1,\ldots,X_k)$ be a vector of random variables having a k-dimensional normal distribution with zero means, arbitrary variances $\sigma_1^2,\ldots,\sigma_k^2$, and an arbitrary correlation matrix $R = \{\rho_{ij}\}$. Then, for any positive numbers $c_1,\ldots,c_k$,

$$P(|X_1| \leq c_1,\ldots,|X_k| < c_k) \geq \prod_{i=1}^{k} P(|X_i| \leq c_i).$$

Using Theorems 3.2.2 and 3.2.3, we have

$$P\left(\bigcap_{i=0}^{p-1} \left\{\frac{n^{\frac{1}{2}}|\hat{\pi}_i-\pi_i|}{[\mathrm{var}(\hat{\pi}_i)]^{\frac{1}{2}}} \le h_i\right\}\right) \ge \prod_{i=0}^{p-1} P\left(\frac{n^{\frac{1}{2}}|\hat{\pi}_i-\pi_i|}{[\mathrm{var}(\hat{\pi}_i)]^{\frac{1}{2}}} \le h_i\right),$$

$$P\left(\bigcap_{i=1}^{p-1} \left\{\frac{n^{\frac{1}{2}}|\hat{\psi}_i-\psi_i|}{[\mathrm{var}(\hat{\psi}_i)]^{\frac{1}{2}}} \le g_i\right\}\right) \ge \prod_{i=1}^{p-1} P\left(\frac{n^{\frac{1}{2}}|\hat{\psi}_i-\psi_i|}{[\mathrm{var}(\hat{\psi}_i)]^{\frac{1}{2}}} \le g_i\right).$$

Let

$$h_i = Z_{\nu_1}, \qquad i = 0,\ldots,p-1,$$

$$g_i = Z_{\nu_2}, \qquad i = 1,\ldots,p-1,$$

where

$$\begin{aligned} 2\nu_1 &= 1-(1-\nu)^{1/p}, \\ 2\nu_2 &= 1-(1-\nu)^{1/p-1}, \end{aligned} \tag{3.2.9}$$

and Z_α is the 100(1-α) percentile of a standard normal distribution. Then

$$\begin{aligned} &P\left(\bigcap_{i=0}^{p-1} \{\hat{\pi}_i-n^{-\frac{1}{2}}z_{\nu_1} [\hat{\mathrm{var}}(\hat{\pi}_i)]^{\frac{1}{2}} \le \pi_i \le \hat{\pi}_i + n^{-\frac{1}{2}}z_{\nu_1} [\hat{\mathrm{var}}(\hat{\pi}_i)]^{\frac{1}{2}}\}\right) \ge 1-\nu, \\ &P\left(\bigcap_{i=1}^{p-1} \{\hat{\psi}_i-n^{-\frac{1}{2}}z_{\nu_2}\ \hat{\mathrm{var}}(\hat{\psi}_i)]^{\frac{1}{2}} \le \psi_i \le \hat{\psi}_i + n^{-\frac{1}{2}}z_{\nu_2} [\hat{\mathrm{var}}(\hat{\psi}_i)]^{\frac{1}{2}}\}\right) \ge 1-\nu, \end{aligned} \tag{3.2.10}$$

where $\hat{\mathrm{var}}(\hat{\pi}_i)$ and $\hat{\mathrm{var}}(\hat{\psi}_i)$ are obtained by substituting $\hat{\underset{\sim}{\pi}}$ for $\underset{\sim}{\pi}$, $\hat{\underset{\sim}{\psi}}$ for $\underset{\sim}{\psi}$ and $\hat{\sigma}_u^2\ \hat{\sigma}_0^{-2}$ for τ_0 in the formulas for the asymptotic covariance matrices of $\hat{\pi}$ and $\hat{\psi}$, respectively, in Theorem 3.2.2. The resulting 100(1-ν)% simultaneous confidence regions

$$\hat{\pi}_i - n^{-\frac{1}{2}} z_{\nu_1} [\hat{\text{var}}(\hat{\pi}_i)]^{\frac{1}{2}} \leq \pi_i \leq \hat{\pi}_i + n^{-\frac{1}{2}} z_{\nu_1} [\hat{\text{var}}(\hat{\pi}_i)]^{\frac{1}{2}},$$
$$i = 0,1,\ldots,p-1, \qquad (3.2.11)$$

and

$$\hat{\psi}_i - n^{-\frac{1}{2}} z_{\nu_2} [\hat{\text{var}}(\hat{\psi}_i)]^{\frac{1}{2}} \leq \psi_i \leq \hat{\psi}_i + n^{-\frac{1}{2}} z_{\nu_2} [\hat{\text{var}}(\hat{\psi}_i)]^{\frac{1}{2}},$$
$$i = 1,\ldots,p-1, \qquad (3.2.12)$$

are rectangles centered at the maximum likelihood estimators.

The regions (3.2.11), (3.2.12) are based on the maximum modulus method of forming simultaneous confidence regions (see Seber, 1977, Chapter 5). Alternatively, we could use the Scheffé method to construct confidence ellipsoids

$$n(\hat{\pi}-\pi)'[\hat{\text{cov}}(\hat{\pi})]^{-1}(\hat{\pi}-\pi) \leq \chi^2_{p,\nu}, \qquad (3.2.13)$$

$$n(\hat{\psi}-\psi)'[\hat{\text{cov}}(\hat{\psi})]^{-1}(\hat{\psi}-\psi) \leq \chi^2_{p-1,\nu}, \qquad (3.2.14)$$

for π and ψ, respectively, where $\chi^2_{r,\nu}$ is the 100(1-ν) percentile of the χ^2_r distribution. Although the Scheffé method permits construction of simultaneous confidence intervals for arbitrary linear combinations $\sum_{i=0}^{p} a_i\pi_i$, $\sum_{i=1}^{p} a_i\psi_i$, of the elements of π, ψ, respectively, the intervals which they give for the individual π_i's or ψ_i's are wider than the corresponding intervals obtained from (3.2.11), (3.2.12), respectively. Since, only the individual π_i or ψ_i are usually of interest in applications, the maximum modulus regions (3.2.11), (3.2.12) are likely to have greater practical usefulness than the Scheffé regions (3.2.13), (3.2.14).

3.3 Selecting the Most Precise Instrument

In this section, we discuss a procedure for selecting the most precise instrument of p instruments, giving instrument 0 the special (preferred) role of the control instrument. We attempt to use a method originally proposed by Paulson (1952) for choosing the normal population (with known variance σ^2) with the largest mean μ_i, when one population (population 0) is given favored treatment. Paulson's approach has two components:

(I) The choice of a region A in the sample space such that $P_\mu(A) \geq P_0^*$ for all μ for which H_0: $\mu_0 \geq \max\limits_{1\leq i\leq p-1} \mu_i$ holds. If the data falls in the region A, we state that the control population is "at least as good" as the other populations;

(II) The choice of a (permutation invariant) partition $B_1,\ldots,B_{p-1}$ of the complement A^c of A, and the minimal sample size n such that

$$P_\mu(B_i) \geq P_1^*, \text{ for all } \mu \text{ such that } \mu_i \geq \max_{j\neq i} \mu_j+\Delta$$

for all i = 1,2,...,p-1. If the data falls in B_i, we say that population i has the largest mean.

The constants P_0^*, P_1^*, $0 < P_0^*, P_1^* < 1$, and $\Delta > 0$ are specified in advance. Since population 0 plays a favored role, P_0^* is usually chosen to be large (e.g. $P_0^* = 0.95, 0.99$).

Here, instead of comparing p population means μ_i, $i = 0,\ldots,p-1$, we wish to compare the precisions $\pi_0,\pi_1,\ldots,\pi_{p-1}$ of p instruments. To compare means, Paulson let the region A defined by

$\{\bar{X}: \max_{1\le i\le p-1} (\bar{X}_i-\bar{X}_0) \le \lambda\sigma(2/n)^{\frac{1}{2}}\}$, and defined the regions B_i as $A^c \cap \{\bar{X}_i-\bar{X}_0 > \max_{j\ne i} (\bar{X}_j-\bar{X}_0)\}$, where $\bar{X} = (\bar{X}_0,\bar{X}_1,\ldots,\bar{X}_{p-1})$ is the vector of sample means, and λ is a specified constant. When comparing precisions, it seems more appropriate to use the ratios $\hat{\psi}_i = \hat{\pi}_i\hat{\pi}_0^{-1}$ in place of the differences $\bar{X}_i - \bar{X}_0$ in defining the regions $A,B_1,\ldots,B_{p-1}$. Alternatively, we can use

$$\ell n\ \hat{\psi}_i = \ell n\ \hat{\pi}_i - \ell n\ \hat{\pi}_0.$$

Since our conclusions are the same for both approaches, we will illustrate them by using the former approach. Hence, the rule we will use is the following:

(I) Let $A = \{t: t = (t_1,\ldots,t_{p-1}), \max_{1\le i\le p-1} t_i \le \lambda\}$.

If $\hat{\psi} \in A$, then say that the control instrument (instrument 0) is at least as precise as the other instruments.

(3.3.1)

(II) Let $B_i = A^c \cap \{t: t_i > \max_{j\ne i} t_j\}$. If $\hat{\psi} \in B_i$, then say that instrument i is the most precise instrument, $i = 1,\ldots,p-1$.

The constant λ defining the region A must be chosen to satisfy

$$P_\theta(\hat{\psi} \in A) \ge P_0^*, \text{ for all } \theta = (\beta', \gamma', \sigma_u^2)' \text{ such that } \max_{1\le i\le p-1} \psi_i \le 1, \tag{3.3.2}$$

where P_0^*, $0 < P_0^* < 1$, is a specified probability. Recall that the model (3.0.1) is parameterized by $\theta' = (\beta_1,\ldots,\beta_{p-1},\sigma_1^2,\ldots,\sigma_{p-1}^2,\sigma_0^2,\sigma_u^2)$.

Because the exact joint distribution of $\hat{\psi}$ is intractable, we will use large sample approximations. Note that from Theorem 3.2.2, (3.3.1) and (3.3.2), we have

$$P\ (\hat{\psi} \in A) \approx \Phi_R\left(\frac{n^{\frac{1}{2}}(\lambda-\psi_1)}{[\mathrm{var}_\theta(\hat{\psi}_1)]^{\frac{1}{2}}}, \ldots, \frac{n^{\frac{1}{2}}(\lambda-\psi_{p-1})}{[\mathrm{var}_\theta(\hat{\psi}_{p-1})]^{\frac{1}{2}}}\right), \qquad (3.3.3)$$

where $\Phi_\Sigma(z)$ is the joint c.d.f. of a MVN(0, Σ) distribution, $\mathrm{var}_\theta(\hat{\psi}_i)$ is the variance of the asymptotic (normal) distribution of $\hat{\psi}_i$, $1 \leq i \leq p-1$, and $R = R(\theta)$ is the correlation matrix of the asymptotic (p-1)-variate normal distribution of $\hat{\psi}$. When $\psi_1 = \ldots = \psi_{p-1} = 1$, and $\lambda < 1$, it is easily seen from (3.3.3) that

$$P_\theta(\hat{\psi} \in A) < \Phi_R(0,0,\ldots,0) \leq \frac{1}{2},$$

and that $\lim_{n\to\infty} P_\theta(\hat{\psi} \in A) = 0$. Consequently, if we wish $P_0^* \geq \frac{1}{2}$ (or, for very large n, if we wish $P_0^* > 0$), we must require that $\lambda \geq 1$.

Now fix n large enough so that the approximation in (3.3.3) holds. Recall from Section 3.2 that the covariance matrix $\mathrm{cov}_\theta(\hat{\psi})$ of the asymptotic distribution of $n^{\frac{1}{2}}(\hat{\psi}-\psi)$ depends upon τ_0 as well as $\psi_1,\ldots,\psi_{p-1}$. Indeed

$$\mathrm{cov}_\theta(\hat{\psi}) = \frac{4}{\tau_0}\,(1 + \frac{1}{H\tau_0})(D_\psi + \psi\psi') + 2(1 + \frac{2}{H\tau_0^2})^2(D_\psi, -\psi)Q^{-1}(D_\psi, -\psi)', \qquad (3.3.4)$$

where Q depends only on ψ. Fix ψ such that $\max_{1\leq i\leq p-1} \psi_i \leq 1$ and take $\tau_0 \to 0$. Note from (3.3.4) that

$$\lim_{\tau_0\to 0} \tau_0^2 \mathrm{cov}_\theta(\hat{\psi}) = \frac{1}{H}\,(D_\psi + \psi\psi') + \frac{8}{H^2}\,(D_\psi, -\psi)Q^{-1}(D_\psi, -\psi)'.$$

Consequently as $\tau_0 \to 0$, the asymptotic correlation matrix R of $\hat{\psi}$ converges to a fixed correlation matrix R*, while since $\lambda \geq 1 \geq \psi_i$ and $\text{var}_\theta(\hat{\psi}_i) \propto \tau_0^{-2} \to \infty$,

$$\lim_{\tau_0 \to 0} \frac{\lambda - \psi_i}{[\text{var}_\theta(\hat{\psi}_i)]^{\frac{1}{2}}} = 0, \qquad i = 1,\ldots,p-1.$$

Thus, for fixed $\lambda \geq 1$, fixed $\psi_1,\ldots,\psi_{p-1} \leq 1$,

$$\lim_{\tau_0 \to 0} P_\theta\{\hat{\psi} \in A\} = \Phi_{R^*}(0,\ldots,0) \leq \frac{1}{2}. \qquad (3.3.4)$$

Therefore, unless τ_0 is known to be bounded from below, it is not possible to find a region A of the form defined in (3.3.1) which satisfies (3.3.2).

Even if we know the value of τ_0 (but use the maximum likelihood estimator $\hat{\psi}$ of ψ for the case where τ_0 is unknown), it is still not possible to find λ to satisfy (3.3.2) when $P_0^* > \frac{1}{2}$. To see this, consider the case p = 3. Here, it can be shown from equations (3.2.7) and (3.2.7) that if we fix τ_0 and let $\psi_1 \to 0$, $\psi_2 = 1$, then $\text{var}_\theta(\hat{\psi}_1) \to 0$, $\text{var}_\theta(\hat{\psi}_2) \to \infty$, and $\text{correl}_\theta(\hat{\psi}_1,\hat{\psi}_2) \to \rho^*$, where

$$\rho^* = \frac{1 + \tau_0}{[\tau_0^2 + 4\tau_0 + 2]^{\frac{1}{2}}} > 0.$$

Consequently, for $\lambda \geq 1$

$$\lim_{\substack{\psi_1 \to 0 \\ \psi_2 \to 1}} P_\theta(\hat{\psi} \in A) = \Phi_{R^*}(\infty, 0) = \frac{1}{2},$$

where

$$R^* = \begin{pmatrix} 1 & \rho^* \\ \rho^* & 1 \end{pmatrix}.$$

The same result holds (by symmetry in the indices) when $\psi_1 = 1$, $\psi_2 \to 0$. For $p > 3$, setting any $\psi_i = 1$ and taking $\psi_j \to 0$, $j \neq i$, we have $\text{var}_\theta(\hat{\psi}_i) \to \infty$, $\text{var}_\theta(\hat{\psi}_j) \to 0$, $j \neq i$, and

$$\lim P_\theta(\hat{\psi} \in A) = \Phi_{R^*}(\infty,\ldots,\infty,0,\infty,\ldots,\infty) = \frac{1}{2}.$$

The above results demonstrate that if either the relative precision τ_0 of the control, or the precisions relative to the control of one or more of instruments $1,\ldots,p-1$, are not known to be bounded below by positive numbers, then it is impossible to find a region A of the form in (3.3.1) which satisfies (3.3.2) for $P_0^* > \frac{1}{2}$. Since a region of the form A seems to be the most intuitively reasonable way of implementing Step I of Paulson's method using the maximum likelihood estimators $\hat{\psi}_1,\ldots,\hat{\psi}_{p-1}$ of $\psi_1,\ldots,\psi_{p-1}$, the problem seems to lie with the properties of these estimators. If one looks at the form of the covariance matrix of the asymptotic distribution of the maximum likelihood estimators $\hat{\beta}$, $\hat{\gamma}$ in Theorem 3.2.1, it can be seen that τ_0 affects the variances of the elements of $\hat{\beta}$, but not the variances of the elements of $\hat{\gamma}$. If τ_0 is small (all other parameters being fixed), the variances of $\hat{\beta}_1,\ldots,\hat{\beta}_{p-1}$ are large. On the other hand, the variances of $\hat{\sigma}_1^2,\ldots,\hat{\sigma}_{p-1}^2,\hat{\sigma}_0^2$ are affected by $\psi_1,\ldots,\psi_{p-1}$. If $\psi_i \to 1$, $\psi_j \to 0$, $j \neq i$, the variances of $\hat{\sigma}_i^2$ and $\hat{\sigma}_0^2$ can be arbitrary large.

It is thus apparent that in order to find a region A of tne form in (3.3.1) which satisfies (3.3.2) for $P_0^* > \frac{1}{2}$, we should try to avoid including poor instruments in our comparison. However, we often do not know in advance whether any poor instrument has been included in our experiment. In this case, one might decide to utilize as few instruments as possible to estimate the precisions of each pair of instruments. As discussed already, the minimum number of instruments needed to make the model (3.0.1) identifiable is 3. Barnett (1969) suggested estimating the parameters for p instruments by using method-of-moments estimators obtained from arbitrary groups of three instruments. However, Barnett's estimators for the parameters are the maximum likelihood estimators for p = 3. As we just pointed out, if any poor instrument has been chosen, the region A in (3.3.1) cannot satisfy (3.3.2) for $P_0^* > \frac{1}{2}$. Hence, without any prior knowledge about the instruments, using Barnett's estimators for ψ in the procedure (3.3.1) in place of the maximum likelihood estimators based on all p instruments still fails to achieve the P_0^* requirement.

If we compare two instruments at a time, using only the data from these two instruments, then the parameters are not identifiable. Consequently, restrictions are needed on the parameters. However, some extra information can be obtained from the full experiment to estimate a few key parameters (for example, τ_0). Provided both the sample size n and the number p of instruments are large enough, we might be willing to assume that the key parameters are known. In Chapter 4, we will discuss use of Paulson's type of selection procedure in three special cases: (1) when the ratios $R_1,\ldots,R_{p-1}$ of measurement error variances are known; (2) when the slopes $\beta_1,\ldots,\beta_{p-1}$ are all assumed to be equal to one; and (3) when τ_0 is known.

CHAPTER 4

ESTIMATION AND SELECTION PROCEDURES FOR THE PRECISIONS OF P ($\geq$ 3) INSTRUMENTS IN SPECIAL CASES

4.0. Introduction

The model we consider in this chapter is the same as that of Chapter 3:

$$\underset{\sim}{y}_j = \begin{pmatrix} y_{0j} \\ y_{1j} \\ \vdots \\ y_{p-1,j} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_1 \\ \vdots \\ \alpha_{p-1} \end{pmatrix} + \begin{pmatrix} 1 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} u_j + \begin{pmatrix} e_{0j} \\ e_{1j} \\ \vdots \\ e_{p-1,j} \end{pmatrix} \tag{4.0.1}$$

$$= \begin{pmatrix} 0 \\ \underset{\sim}{\alpha} \end{pmatrix} + \begin{pmatrix} 1 \\ \underset{\sim}{\beta} \end{pmatrix} u_j + \underset{\sim}{e}_j, \qquad j = 1,\ldots,n,$$

where $\underset{\sim}{y}_j$ is a px1 vector of observations, $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are unknown (p-1)x1 parameter vectors, and $\underset{\sim}{e}_j$ is a px1 vector of errors. We assume that $(u_1,\ldots,u_n)$ is a random sample of size n from a normal distribution with mean μ and variance σ_u^2, and that the $\underset{\sim}{e}_j$'s are an independent random sample from a p-variate normal distribution with mean vector 0 and unknown covariance matrix $\Sigma_e = \text{diag}(\sigma_0^2,\ldots,\sigma_{p-1}^2)$. We refer to instrument 0 as "the control".

In Chapter 3, we have considered the problem of comparing the precisions $\pi_0,\ldots,\pi_{p-1}$ of p instruments to choose the most precise instrument when there are no extra constraints on the parameters. However, difficulties arose with the asymptotic joint distribution of the statistics used, so that the rule originally suggested by Paulson (1952) was not applicable. In this chapter, we will consider some special cases:

(1) $R_1,\ldots,R_{p-1}$ are known, where $R_i = \sigma_0^2\sigma_i^{-2}$,

(2) $\beta_1,\ldots,\beta_{p-1}$ are equal to 1,

(3) $\tau_0 = \sigma_u^2\sigma_0^{-2}$, the relative precision of the control, is known.

In each of cases (1) to (3), we discuss the estimation of the precisions $\pi_0,\ldots,\pi_{p-1}$ or $\psi_1,\ldots,\psi_{p-1}$, and use the statistics derived in Chapter 2 for comparing each instrument with the control in procedure (3.3.1) to choose the most precise instrument. All such procedures satisfy the P_0^* requirement. We also attempt to evaluate the lower bound for the probability of correct selection for these procedures. However, the problem is too complicated to be solved at present.

4.1 The Case Where $R_1,\ldots,R_{p-1}$ Are Known

This case has been extensively studied by many authors, especially with all the R_i equal to 1 ($\sigma_0^2 = \sigma_1^2 = \ldots = \sigma_{p-1}^2$). As noted in Section 2.1, for each i, i = 1,...,p-1,

$$\frac{\pi_i}{\pi_0} = \beta_i^2 R_i .$$

Thus, comparing the precisions, $\pi_0,\ldots,\pi_{p-1}$, is equivalent to comparing 1, $\beta_1^2R_1,\ldots,\beta_{p-1}^2R_{p-1}$.

It is known (Lawley, 1953, Theobald, 1975) that the maximum likelihood estimators of the parameters $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, μ, σ_0^2 and σ_u^2 can be expressed in the following form. Let $\bar{y}_i = n^{-1}\sum_{j=1}^{n} y_{ij}$, $i = 0,1,\ldots,p-1$, and

$$S = ((S_{ij})) = \frac{1}{n}\sum_{j=1}^{n}\begin{pmatrix} y_{0j} & - \bar{y}_0 \\ y_{1j} & - \bar{y}_1 \\ \vdots & \\ y_{p-1,j} & - \bar{y}_{p-1}\end{pmatrix}\begin{pmatrix} y_{0j} & - \bar{y}_0 \\ y_{1j} & - \bar{y}_1 \\ \vdots & \\ y_{p-1,j} & - \bar{y}_{p-1}\end{pmatrix}'$$

$i,j = 0,\ldots,p-1.$

Further, let $d_1,\ldots,d_p$, $d_1 \geq d_2 \geq \ldots \geq d_p$, be the eigenvalues of

$$S\begin{pmatrix} 1 & 0 \\ 0 & D_R\end{pmatrix}$$

where $D_R = \text{Diag}(R_1,\ldots,R_{p-1})$. Then $\hat{\mu} = \bar{y}_0$,

$$\hat{\alpha}_i = \bar{y}_i - \hat{\beta}_i\bar{y}_0, \quad i = 1,\ldots,p-1,$$

$$\hat{\sigma}_u^2 = \frac{(p-1)d_1 - \sum_{i=2}^{p} d_i}{(p-1)(1+\sum_{i=1}^{p-1}\beta_i^2 R_i)}, \quad \hat{\sigma}_0^2 = \frac{\sum_{i=2}^{p} d_i}{p-1}, \tag{4.1.1}$$

and $(1, \hat{\beta}_1 R_1^{\frac{1}{2}},\ldots,\hat{\beta}_{p-1}R_{p-1}^{\frac{1}{2}})'$ is the eigenvector corresponding to the eigenvalue d_1 of $S\begin{pmatrix} 1 & 0 \\ 0 & D_R\end{pmatrix}$. By the invariance property of maximum likelihood, the maximum likelihood estimators of $\pi_0,\ldots,\pi_{p-1}$ and $\pi_1\pi_0^{-1},\ldots,\pi_{p-1}\pi_0^{-1}$ are

$$\hat{\pi}_0 = (p-1)(\sum_{i=2}^{p} d_i)^{-1}, \ \hat{\pi}_i = R\hat{\beta}^2(p-1)(\sum_{i=2}^{p} d_i)^{-1}, \ \hat{\pi}_i\hat{\pi}_0^{-1} = R\hat{\beta}^2,$$

$$i = 1,\ldots,p-1. \qquad (4.1.2)$$

Although these estimators are known to be consistent, their exact distributions are intractable. However, if a large sample is available, then from the known asymptotic joint distribution of these estimators, asymptotic joint confidence regions for any collection of these parameters and also for $(\pi_0,\ldots,\pi_{p-1})$, or for $(\tau_0,\ldots,\tau_{p-1})$, can be obtained by standard techniques. All such regions for the parameters have ellipsoidal form with centers equal to maximum likelihood estimators and shape determined by a consistent estimator of the asymptotic covariance matrix of the estimators. In the rest of this section, we concern ourselves with the procedures to select the most precise instrument.

As mentioned in Section 3.3, if we use the maximum likelihood estimator $\hat{\psi}_i = \hat{\pi}_i\hat{\pi}_0^{-1}$ derived in Section 3.1 as the statistic for comparing π_i and π_0 in procedure (3.3.1), we cannot determine the value of λ such that the procedure (3.3.1) satisfies the P_0^* requirement when the parameter space is unrestricted. Here, assuming that $R_1,\ldots,R_{p-1}$ are known, the maximum likelihood estimators $\hat{\psi}_i$ of ψ_i are expressed explicitly in (4.1.2) and the covariance matrix of the limiting distribution of these ψ_i's is much simpler. The procedure we use here is almost identical to the procedure (3.3.1) except $\hat{\psi}_i = \hat{\pi}_i\hat{\pi}_0^{-1}$ now are defined in (4.1.2).

When the R_i are known, it is known (Amemiya & Fuller, 1984) that the limiting distribution of $n^{\frac{1}{2}}(R_1^{\frac{1}{2}}(\hat{\beta}_1-\beta),\ldots,R_{p-1}^{\frac{1}{2}}(\hat{\beta}_{p-1}-\beta_{p-1}))'$

is multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix given by

$$\left[\frac{1}{(1+\sum_{i=1}^{p-1} R_i\beta_i^2)\tau_0^2}\frac{1}{\tau_0^2}+\frac{1}{\tau_0}\right](I_{p-1} + D_R^{\frac{1}{2}}\beta\beta' D_R^{\frac{1}{2}})$$

where

$$D_R = \mathrm{diag}(R_1,\ldots,R_{p-1}).$$

The limiting joint distribution of the ψ_i's can be obtained directly from this result and the result 6a.2(iv) in Rao (1973), since $\hat{\psi}_i = R_i\hat{\beta}_i^2$, $i = 1,\ldots,p-1$.

Theorem 4.1.1. Assume that R_i are known. Then, the limiting distribution of $n^{\frac{1}{2}}(\hat{\psi}_1-\psi_1,\ldots,\hat{\psi}_{p-1}-\psi_{p-1})'$ is multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix equal to

$$4\left(\frac{1}{H\tau_0^2} + \frac{1}{\tau_0}\right)(D_\psi + \underset{\sim}{\psi}\underset{\sim}{\psi}'),$$

where

$$H = 1 + \sum_{i=1}^{p-1} \psi_i.$$

We need one more theorem to determine the value of λ.

Theorem 4.1.2. (Slepian, 1962) Let $X = (X_1,\ldots,X_p)$ be distributed according to $N(0, \Sigma)$, where Σ is a correlation matrix. Let $\Sigma = ((\sigma_{ij}))$, $T = ((\tau_{ij}))$ be two positive definite matrices with $\sigma_{ii} = \tau_{ii} = 1$ and $\sigma_{ij} \geq \tau_{ij}$ for all $i \neq j$. Then

$$P_\Sigma\{\bigcap_{i=1}^{p}(x_i \le a_i)\} \ge P_T\{\bigcap_{i=1}^{p}(x_i \le a_i)\}$$

holds for all $a = (a_1,\dots,a_p)'$.

From Theorem 4.1.1, we can see that for n large,

$$P\{\max_{1\le i\le p-1} \hat{\psi}_i \le \lambda | \psi_i \le 1,\ 1 \le i \le p-1\} \approx P\{\bigcap_{i=1}^{p-1}(M_i \le b_i)|\psi_i \le 1,\ 1 \le i \le p-1\}, \tag{4.1.3}$$

where

$$M_i = \frac{n^{\frac{1}{2}}(\hat{\psi}_i - \psi_i)}{2[(\frac{1}{H\tau_0^2} + \frac{1}{\tau_0})\psi_i(1+\psi_i)]^{\frac{1}{2}}}, \qquad i = 1,\dots,p-1,$$

have a multivariate normal distribution with zero means, variances equal to 1 and correlations

$$\rho(M_i,M_j) = \left[\frac{\psi_i\psi_j}{(1+\psi_i)(1+\psi_j)}\right]^{\frac{1}{2}}, \quad i \ne j, \quad 1 \le i,j \le p-1,$$

and where

$$b_i = \frac{n^{\frac{1}{2}}(\lambda - \psi_i)}{2[(\frac{1}{H\tau_0^2} + \frac{1}{\tau_0})\psi_i(1+\psi_i)]^{\frac{1}{2}}}, \quad i = 1,\dots,p-1. \tag{4.1.4}$$

When $\psi_1 = \dots = \psi_{p-1} = 1$ and $\lambda < 1$, it is easily seen from (4.1.3) and (4.1.4) that

$$P\{\bigcap_{i=1}^{p-1}(M_i \le b_i)|\psi_i = 1,\ 1 \le i \le p-1\} < P\{\bigcap_{i=1}^{p-1}(M_i \le 0)\} < \frac{1}{2},$$

and that $\lim_{n\to\infty} P\{\bigcap_{i=1}^{p-1}(M_i \le b_i)|\psi_i = 1,\ 1 \le i \le p-1\} = 0$. Consequently, if we wish $P_0^* \ge \frac{1}{2}$, we must require that $\lambda \ge 1$.

Note that $\rho(M_i, M_j)$ is independent of τ_0 for any i,j. Thus, the correlation matrix of the M_i's is independent of τ_0. Since $\lambda \geq 1 \geq \psi_i$ and $\text{Var}(\hat{\psi}_i) \to \infty$ as $\tau_0 \to 0$,

$$\lim_{\tau_0 \to \infty} b_i = 0, \qquad i = 1, \ldots, p-1.$$

Thus, for fixed $\lambda \geq 1$, fixed $\psi_1, \ldots, \psi_{p-1} \leq 1$,

$$\lim_{\tau_0 \to 0} P\{\bigcap_{i=1}^{p-1} (M_i \leq b_i) | \psi_i \leq 1, 1 \leq i \leq p-1\} = P\{\bigcap_{i=1}^{p-1} (M_i \leq 0)\} \leq \frac{1}{2}.$$

Therefore, unless τ_0 is known to be bounded below, it is not possible to find a λ such that the procedure satisfies the P_0^* requirement.

Lemma 1. Assume that $\tau_0 \geq c_0 (> 0)$, where c_0 is a known constant. For $\lambda > 1$ and $\psi_1, \ldots, \psi_{p-1} \leq 1$, the minimum of b_i is achieved when $\psi_i = 1$, $\psi_j = 0$ for $j \neq i$ and $\tau_0 = c_0$.

Proof. It is easily seen that from (4.1.4) that b_i is increasing in τ_0 for fixed $\psi_1, \ldots, \psi_{p-1}$ and increasing in ψ_j, $j \neq i$, for fixed τ_0 and ψ_k, $k \neq j$. While for fixed τ_0 and ψ_j, $j \neq i$, b_i is decreasing in ψ_i. To see this, taking the derivative of b_i with respect to ψ_i, we have

$$\frac{\partial b_i}{\partial \psi_i} = \frac{n^{\frac{1}{2}}}{2} \left\{ \frac{-1}{[(\frac{1}{H\tau_0^2} + \frac{1}{\tau_0})\psi_i(1 + \psi_i)]^{\frac{1}{2}}} - \frac{(\lambda - \psi_i)[(1+2\psi_i)(\frac{1}{H\tau_0^2} + \frac{1}{\tau_0}) - \psi_i(1+\psi_i)\frac{1}{H^2\tau_0^2}]}{2[(\frac{1}{H\tau_0^2} + \frac{1}{\tau_0})\psi_i(1 + \psi_i)]^{3/2}} \right\}.$$

Since

$$\frac{1+2\psi_i}{H\tau_0^2} - \frac{\psi_i(1+\psi_i)}{H^2\tau_0^2} = \frac{(1+2\psi_i)(1+\sum_{j=1}^{p-1}\psi_j) - \psi_i(1+\psi_i)}{H^2\tau_0^2} > 0$$

and $\lambda > 1$, $\psi_i \leq 1$, thus, $\frac{\partial b_i}{\partial \psi_i} < 0$ and b_i is decreasing in ψ_i. The Lemma now follows. □

From Lemma 1, we can see that for each i

$$\inf_{\substack{\psi_1,\ldots,\psi_{p-1}\leq 1 \\ \tau_0 \geq c_0}} b_i \geq \frac{n^{\frac{1}{2}}(\lambda-1)}{2(c_0^{-2} + 2c_0^{-1})^{\frac{1}{2}}} . \tag{4.1.5}$$

Since $\rho(M_i, M_j)$ is nonnegative for any i,j, applying Theorem 4.1.2, we have

$$P\{\bigcap_{i=1}^{p-1}(M_i \leq b_i)|\psi_j \leq 1,\ 1 \leq j \leq p-1\} \geq \prod_{i=1}^{p-1} P\{M_i \leq b_i|\psi_j \leq 1,\ 1 \leq j \leq p-1\}. \tag{4.1.6}$$

Under the assumption that $\tau_0 \geq c_0 (> 0)$, combining (4.1.3) (4.1.5) and (4.1.6), we therefore get

$$P\{\max_{1\leq i\leq p-1} \hat{\psi}_i \leq \lambda\} \geq \prod_{i=1}^{p-1} P\{M_i \leq 2^{-1}n^{\frac{1}{2}}(\lambda-1)(c_0^{-2} + 2c_0^{-1})^{-\frac{1}{2}}\}.$$

Thus, if $\tau_0 \geq c_0 (> 0)$ and

$$\lambda = 1 + 2n^{-\frac{1}{2}}Z_{\nu*}[c_0^{-2} + 2c_0^{-1}]^{\frac{1}{2}},$$

where

$$\nu^* = 1 - (P_0^*)^{\frac{1}{p-1}},$$

then the procedure satisfies the P_0^* requirement.

From the above discussion, we see that when the R_i are known, and we use the maximum likelihood estimators $\hat{\psi}_i$ defined in (4.1.2) as the statistics for comparisons, we are unable to determine the value of λ to insure the procedure satisfying the P_0^* requirement, $P_0^* \geq \frac{1}{2}$, unless a lower bound to the relative precision τ_0 of the control instrument is known.

Instead of using the MLE $\hat{\psi}_i = \hat{\pi}_i\hat{\pi}_0^{-1}$ as the statistic for comparing π_i and π_0, we have another statistic T_{0i} for comparing π_i and π_0, where

$$T_{0i} = \frac{(n-2)^{\frac{1}{2}}(R_iS_{ii}-S_{00})}{2R_i^{\frac{1}{2}}(S_{ii}S_{00}-S_{0i}^2)^{\frac{1}{2}}}. \tag{4.1.7}$$

From the discussion in Section 2.1, we know that for comparing the precisions π_i and π_0 of instrument i and the control, if T_{0i} is sufficiently large, we would agree that instrument i is more precise than the control. This suggests modifying the procedure (3.3.1) as follows:

Procedure: If $\max_{1\leq i\leq p-1} T_{0i} \leq \lambda_1$, then select the control as the best, (4.1.8)

If $\max_{1\leq i\leq p-1} T_{0i} > \lambda_1$, then select the instrument i which has the largest T_{0i} value as the best,

where λ_1 is chosen so that the procedure (4.1.8) satisfies the P_0^* requirement. To determine the value of λ_1, we need to know the asymptotic joint distribution of T_{0i}.

Theorem 4.1.3. Let T_{0i} be defined by (4.1.7). Then the asymptotic joint distribution of $(T_{01} - (n-2)^{\frac{1}{2}}\rho_1(1-\rho_1^2)^{-\frac{1}{2}}, \ldots, T_{0p-1} - (n-2)^{\frac{1}{2}}\rho_{p-1}(1-\rho_{p-1}^2)^{-\frac{1}{2}})'$ is multivariate normal with mean vector 0 and covariance matrix $H = ((h_{ij}))$, where

$$\rho_i = \frac{(\psi_i-1)\tau_0}{[(1-\psi_i)^2\tau_0^2 + 4(1+\psi_i)\tau_0+4]^{\frac{1}{2}}}, \tag{4.1.9}$$

$$h_{ii} = (1-\rho_i^2)^{-1},$$

$$\begin{aligned} h_{ij} = {} & 8^{-1}[1 + (1+\psi_i)\tau_0]^{-3/2}[1 + (1+\psi_j)\tau_0]^{-3/2}\{\tau_0^4[(1-\psi_i^2)(1-\psi_j^2) \\ & + (\psi_i-\psi_i^2)(\psi_j-\psi_j^2)] + \tau_0^3[5\psi_i\psi_j(\psi_i+\psi_j) + \psi_i(1-\psi_i) \\ & + (\psi_j(1-\psi_j) + 2(\psi_i+\psi_j) + 12\psi_i\psi_j + 6] + \tau_0^2[(\psi_i^2 + \psi_j^2) \\ & + 9(\psi_i+\psi_j) + 13\psi_i\psi_j + 13] + 6\tau\alpha\psi_i + \psi_j + 2) + 4\}, \end{aligned}$$

and

$$\psi_i = \beta_i^2 R_i = \pi_i\pi_0^{-1}, \qquad i \neq j,\ i,j = 1,\ldots,p-1.$$

Proof: Note that for every i, $i = 1,\ldots,p-1$,

$$T_{0i} = \frac{(n-2)^{\frac{1}{2}} r_i}{\sqrt{1-r_i^2}},$$

where

$$r_i = \frac{R_i S_{ii} - S_{00}}{[(R_i S_{ii} - S_{00})^2 + 4R_i(S_{ii}S_{00} - S_{0i}^2)]^{\frac{1}{2}}}.$$

Using a Taylor series expansion, for every $i = 1,\ldots,p-1$,

$$T_{0i}-(n-2)^{\frac{1}{2}}\rho_i(1-\rho_i^2)^{-\frac{1}{2}} \approx \frac{1}{(1-\rho_i^2)^{3/2}}\,[a_{0i}\sqrt{n}(S_{00}-\sigma_{00}) + a_{1i}\sqrt{n}(S_{0i}-\sigma_{0i})$$

$$+ a_{ii}\sqrt{n}\,(S_{ii}-\sigma_{ii})],$$

where

$$a_{0i} = \left.\frac{\partial r_i}{\partial S_{00}}\right|_{S=\Sigma} = \frac{-2\{[(1+\psi_i)\tau_0+2](1+\psi_i\tau_0)-2\psi_i\tau_0^2\}}{\sigma_0^2[(1-\psi_i)^2\tau_0^2 + 4(1+\psi_i)\tau_0+4]^{3/2}},$$

$$a_{1i} = \left.\frac{\partial r_i}{\partial S_{0i}}\right|_{S=\Sigma_y} = \frac{4\beta_i R_i^{\frac{1}{2}}(\psi_i-1)\tau_0^2}{\sigma_i\sigma_0[(1-\psi_i)^2\tau_0^2 + 4(1+\psi_i)\tau_0+4]^{3/2}}, \tag{4.1.10}$$

$$a_{2i} = \left.\frac{\partial r_i}{\partial S_{ii}}\right|_{S=\Sigma_y} = \frac{2\{[(1+\psi_i)\tau_0+2](1+\tau_0)-2\psi_i\tau_0^2\}}{\sigma_i^2[(1-\psi_i)^2\tau_0^2 + 4(1+\psi_i)\tau_0+4]^{3/2}}.$$

From Theorem 4.2.4 in Anderson (1958), we know that the limiting distribution of $n^{\frac{1}{2}}(S_{00}-\sigma_{00}, S_{0i}-\sigma_{0i}, S_{ii}-\sigma_{ii}, S_{0j}-\sigma_{0j}, S_{jj}-\sigma_{jj})'$ is multivariate normal with mean vector $\underset{\sim}{0}$ and covariance matrix

$\Sigma_{ij} =$

$$\sigma_0^4\begin{bmatrix} 2(1+\tau_0)^2 & 2\beta_i\tau_0(1+\tau_0) & 2R_i^{-1}\psi_i\tau_0^2 & 2\beta_j\tau_0(1+\tau_0) & 2R_j^{-1}\psi_j\tau_0^2 \\ & \omega_{1i} & \omega_{2i} & \beta_i\beta_j\tau_0(1+2\tau_0) & 2\beta_i\beta_j^2\tau_0^2 \\ & & 2R_i^{-2}(1+\psi_i\tau_0)^2 & 2\beta_i^2\beta_j\tau_0^2 & 2\beta_i^2\beta_j^2\tau_0^2 \\ & & & \omega_{1j} & \omega_{2j} \\ \text{symmetric} & & & & 2R_j^{-2}(1+\psi_j\tau_0)^2 \end{bmatrix}, \tag{4.1.11}$$

where

$$\omega_{1k} = R_k^{-1}[1 + (1+\psi_k)\tau_0 + 2\psi_k\tau_0^2],$$

$$\omega_{2k} = \beta_k R_k^{-1}\tau_0(1+\psi_k\tau_0), \qquad k = i,j.$$

Thus, for $i \neq j$, $i,j = 1,\ldots,p-1$,

$$h_{ij} = \mathrm{cov}(T_{0i}, T_{0j}) = (1-\rho_i^2)^{-3/2}(1-\rho_j^2)^{-3/2} \begin{pmatrix} a_{0i} \\ a_{1i} \\ a_{2i} \\ 0 \\ 0 \end{pmatrix}' \Sigma_{ij} \begin{pmatrix} a_{0j} \\ 0 \\ 0 \\ a_{1j} \\ a_{2j} \end{pmatrix}. \quad (4.1.12)$$

The h_{ij} shown in the theorem is obtained by plugging in a_{0i}, a_{1i}, a_{2i}, a_{0j}, a_{1j}, a_{2j} and Σ_{ij} defined in (4.1.10) and (4.1.11) into (4.1.12). That the asymptotic variance h_{ii} of T_{0i} equals $(1-\rho_i^2)^{-1}$ is a direct consequence of Theorem 4.2.6 in Anderson (1958). □

From Theorem 4.1.3, we know that for a large n,

$$P\{\max_{1\le i\le p-1} T_{0i} \le \lambda_1 | \psi_i \le 1, \quad i = 1,\ldots,p-1\}$$
$$\approx P\{\bigcap_{i=1}^{p-1} (X_i \le \ell_i) | \psi_i \le 1, \ i = 1,\ldots,p-1\}, \qquad (4.1.13)$$

where

$$X_i = \frac{T_{0i}-(n-2)^{\frac{1}{2}}\rho_i(1-\rho_i^2)^{-\frac{1}{2}}}{(1-\rho_i^2)^{-\frac{1}{2}}}, \qquad i = 1,\ldots,p-1,$$

are standard jointly normal random variables with correlations

$$\rho(X_i,X_j) = h_{ii}^{-\frac{1}{2}}\, h_{jj}^{-\frac{1}{2}}\, h_{ij}, \qquad i \neq j,\ i,j = 1,\ldots,p-1,$$

and where

$$\ell_i = \lambda_1(1-\rho_i^2)^{\frac{1}{2}} - (n-2)^{\frac{1}{2}}\rho_i, \quad i = 1,\ldots,p-1. \tag{4.1.14}$$

Observing h_{ij}, we can see that when ψ_i, $\psi_j \leq 1$, h_{ij} is positive for any i,j. Applying Theorem 4.1.2, we have

$$P\{\bigcap_{i=1}^{p-1}(x_i \leq \ell_i)|\psi_i \leq 1,\ 1 \leq i \leq p-1\} \geq \prod_{i=1}^{p-1} P\{x_i \leq \ell_i|\psi_i \leq 1\}. \tag{4.1.15}$$

Lemma 2. For fixed τ_0, if $0 < \lambda_1 < (n-2)^{\frac{1}{2}}$, then for each i,

$$\inf_{\psi_i \leq 1} \ell_i = \lambda_1.$$

Proof: Note that $\psi_i \leq 1$ gives $\rho_i \leq 0$. Since

$$\frac{\partial \ell_i}{\partial \rho_i} = \frac{-\lambda_1\rho_i}{[1-\rho_i^2]^{\frac{1}{2}}} - (n-2)^{\frac{1}{2}},$$

$$\frac{\partial^2 \ell_i}{\partial^2 \rho_i} = \frac{-\lambda_1}{[1-\rho_i^2]^{3/2}} < 0,$$

thus, the minimum of ℓ_i should occur either at $\rho_i = -1$ or $\rho_i = 0$. However, ρ_i equal to -1 or 0 gives ℓ_i equal to $(n-2)^{\frac{1}{2}}$ and λ_1, respectively. Because $\lambda_1 < (n-2)^{\frac{1}{2}}$, the minimum of ℓ_i is achieved when $\rho_i = 0$ (or $\psi_i = 1$). □

Combining (4.1.13), (4.1.15) and the result of Lemma 2, we thus have the following theorem.

Theorem 4.1.4. For a large n, if $\lambda_1 = z_{\nu^*}$, where $\nu^* = 1-(p^*_0)^{\frac{1}{p-1}}$, then the procedure (4.1.8) satisfies the P_0^* requirement.

To evaluate the probability of correct selection (CS) for the procedure (4.1.8), that is the probability of choosing one of the p-1 instruments as the best when that instrument is actually more precise than the others (including the control), let p_1 and p_2 denote the true numbers of instruments with $\psi_i \leq 1$ and $\psi_i \geq 1$, respectively, so that $p_1 + p_2 = p-1$. For convenience, assume that $\psi_1 \leq \psi_2 \leq \cdots \leq \psi_{p-1}$. For a fixed $\Delta > 0$, assume that

$$\psi_{p-1} = \max(1, \psi_{p-2})+\Delta,$$

that is, instrument (p-1) is more precise than the others including the control. Thus, for fixed Δ and γ_1, the probability of correct selection (CS) is given by

$$P(CS) = P\{(T_{0p-1} > \lambda_1) \cap (\bigcap_{i=1}^{p-2} (T_{0p-1} - T_{0i} \geq 0)) | \psi_1 \leq \cdots \leq \psi_{p_1} \leq 1 \leq \psi_{p_1+1} \leq \cdots \leq \psi_{p-1}, \text{ and } \psi_{p-1} = \max(1, \psi_{p-2}) + \Delta\}.$$

Note that the probability of correct selection depends on the unknown integer p_1 and ψ_{p-2}. For simplicity, we assume that $p_1 = p-2$, that is only one instrument (instrument p-1) is more precise than the control. Then we have

$$P(CS) = P\{(T_{0p-1} > \lambda_1) \cap (\bigcap_{i=1}^{p-2} (T_{0p-1} - T_{0i} \geq 0)) | \psi_1 \leq \cdots \leq \psi_{p-2} \leq 1, \psi_{p-1} = 1 + \Delta\}.$$

Let

$$Z_i = T_{0p-1} - T_{0i}, \quad i = 1,\ldots,p-2,$$
$$Z_{p-1} = T_{0p-1} \qquad (4.1.16)$$

Thus,

$$P(CS) = P\{Z_1 \geq 0,\ldots,Z_{p-2} \geq 0,\ Z_{p-1} \geq \lambda_1 | \psi_1 \leq \cdots \leq \psi_{p-2} \leq 1,\ \psi_{p-1} = 1+\Delta\}. \qquad (4.1.17)$$

If all the Z_i's are mutually positively correlated, then $P(CS) \geq \prod_{i=1}^{p-1} P\{Z_i \geq 0\}P\{Z_{p-1} \geq \gamma_1\}$ by Theorem 4.1.2, and each probability in the product can be evaluated separately by its own distribution. Unfortunately, at present we can only show that Z_{p-1} is positively correlated with each Z_i, $i = 1,\ldots,p-2$, but not that Z_i and Z_j are positively correlated, $i \neq j$, $1 \leq i, j \leq p-2$. The following lemma shows that Z_{p-1} is positively correlated with Z_i, $i = 1,\ldots,p-2$.

Lemma 3. Let Z_i be defined in (4.1.16). For a large n, if $\psi_i \leq 1$, $1 \leq i \leq p-2$, $\psi_{p-1} = 1+\Delta(> 1)$, then $Cov(Z_{p-1}, Z_i) \geq 0$, $1 \leq i \leq p-2$.

Proof: From (4.1.16) and Theorem 4.1.3, we have

$$Cov(Z_{p-1},Z_i) = Var(T_{0p-1}) - Cov(T_{0p-1},T_{0i}) = h_{p-1,p-1}-h_{i,p-1}.$$

From the definition of $h_{p-1,p-1}$ and $h_{1,p-1}$,

$$h_{p-1,p-1}-h_{i,p-1} = 8^{-1}(1+(1+\psi_{p-1})\tau_0)^{-3/2}(1+(1+\psi_i)\tau_0)^{-3/2}\{2(1 + (1+\psi_{p-1})\tau_0)^{\frac{1}{2}}(1+(1+\psi_i)\tau_0)^{3/2}[(1-\psi_{p-1})^2\tau_0^2+4(1+\psi_{p-1})\tau_0+4] - k_{i,p-1}\},$$

where

$$k_{i,p-1} = \tau_0^4[(1-\psi_{p-1}^2)(1-\psi_i^2) + (\psi_{p-1}-\psi_{p-1}^2)(\psi_i-\psi_i^2)] + \tau_0^3[5\psi_{p-1}\psi_i(\psi_{p-1}+\psi_i)$$

$$+ \psi_{p-1}(1-\psi_{p-1}) + \psi_i(1-\psi_i) + 2(\psi_i+\psi_{p-1}) + 12\psi_i\psi_{p-1} + 6] + \tau_0^2[2(\psi_i^2+\psi_{p-1}^2)$$

$$+ 9(\psi_i+\psi_{p-1}) + 13\psi_i\psi_{p-1} + 13] + 6\tau_0(\psi_i+\psi_{p-1}+2) + 4.$$

Since $\psi_{p-1} = 1 + \Delta > 1 > \psi_i$,

$$h_{p-1,p-1}-h_{i,p-1} \geq 8^{-1}(1+(1+\psi_{p-1})\tau_0)^{-3/2}(1+(1+\psi_i)\tau_0)^{-3/2}\{2(1 +$$

$$(1+\psi_i)\tau_0)^2[(1-\psi_{p-1})^2\tau_0^2 + 4(1+\psi_{p-1})\tau_0+4] - k_{i,p-1}\}.$$

After simplification, we obtain

$$2(1+(1+\psi_i)\tau_0)^2[(1-\psi_{p-1})^2\tau_0^2+4(1+\psi_{p-1})\tau_0+4] - k_{i,p-1}$$

$$= a_4\tau_0^4 + a_3\tau_0^3 + a_2\tau_0^2 + a_1\tau_0 + 4,$$

where

$$a_1 = 10\psi_i + 2\psi_{p-1} + 12,$$

$$a_2 = 6\psi_i^2 + 3\psi_i\psi_{p-1} + 23\psi_i + 3\psi_{p-1} + 13,$$

$$a_3 = \psi_{p-1}^2(5-\psi_i) + \psi_{p-1}(3\psi_i^2 - 4\psi_i - 3) + 9\psi_i^2 + 17\psi_i + 6,$$

$$a_4 = \psi_{p-1}^2(5\psi_i+3) + \psi_{p-1}(-3\psi_i^2 - 9\psi_i - 4) + 3\psi_i^2 + 4\psi_i + 1.$$

It is clear that a_1 and a_2 are positive. In the following, we show that a_3 and a_4 are nonnegative.

Since

$$(3\psi_i^2-4\psi_i-3)^2 - 4(5-\psi_i)(9\psi_i^2 + 17\psi_i + 6) = 9\psi_i^4 + 12\psi_i^3 + 21\psi_i^2$$

$$- 37\psi_i-21 < 0$$

and $5-\psi_i > 0$ for $\psi_i < 1$, thus $a_3 \geq 0$. Taking the derivative of a_4 with respect to ψ_i, we have

$$\frac{\partial a_4}{\partial \psi_i} = -6\psi_i(\psi_{p-1}-1) + 5\psi_{p-1}^2 - 9\psi_{p-1} + 4.$$

We can show that $5\psi_{p-1}^2 - 9\psi_p + 4 > 0$ for $\psi_{p-1} > 1$ and

$$6^{-1}(5\psi_{p-1}^2 - 9\psi_{p-1}+4)(\psi_{p-1}-1)^{-1} = \begin{cases} < 1 & \text{if } 1 < \psi_{p-1} < 2 \\ > 1 & \text{if } \psi_{p-1} > 2 \end{cases}.$$

Thus, if $\psi_{p-1} > 2$, a_4 is increasing in ψ_i, that is, the minimum of a_4 occurs when $\psi_i = 0$. $a_4 = (3\psi_{p-1}-1)(\psi_{p-1}-1) > 0$ when $\psi_i = 0$. If $1<\psi_{p-1} \leq 2$, a_4 is increasing in ψ_i when $\psi_i \leq 6^{-1}(5\psi_{p-1}^2 - 9\psi_{p-1} + 4)$ $(\psi_{p-1}-1)$, and is decreasing in ψ_i when $\psi_i > 6^{-1}(5\psi_{p-1}^2-9\psi_{p-1}+4)(4_{p-1}-1)$. Hence, the minimum of a_4 occurs when $\psi_i = 0$ or $\psi_i = 1$. When $\psi_i = 1$, $a_4 = 8(\psi_{p-1}-1)^2 > 0$. The Lemma now follows. □

When p = 3, the lower bound to P(CS) in (4.1.17) can be obtained using Lemma 3 and Theorem 4.1.2, that is,

$$P(CS) = P\{Z_2 \geq \lambda_1 \text{ and } Z_1 \geq 0 | \psi_1 \leq 1, \psi_2 = 1+\Delta\}$$

$$\geq P\{Z_2 \geq \lambda_1 | \psi_2 = 1+\Delta\} P\{Z_1 \geq 0 | \psi_1 \leq 1, \psi_2 = 1+\Delta\}. \quad (4.1.18)$$

Given $\psi_2 = 1+\Delta$, the asymptotic distribution of Z_2 is normal with mean $(n-2)^{\frac{1}{2}}\rho_2(1-\rho_2^2)^{-\frac{1}{2}}$ and variance $(1-\rho_2^2)^{-1}$, where $\rho_2 = \Delta\tau_0[\Delta^2\tau_0^2 + 4(2+\Delta)\tau_0+4]^{-\frac{1}{2}}$. Further the asymptotic distribution of Z_1 is normal with mean $(n-2)^{\frac{1}{2}}[\rho_2(1-\rho_2^2)^{-\frac{1}{2}} - \rho_1(1-\rho_1^2)^{-\frac{1}{2}}]$ and variance

$h_{22} + h_{11} - 2h_{12}$, where h_{11}, h_{22}, h_{12} are defined in Theorem 4.1.3. Note that the two probabilities in (4.1.18) still depend on the unknown parameter τ_0.

4.2 The Case Where $\beta_1,\ldots,\beta_{p-1}$ Are Equal to 1

As mentioned already in Section 2.2, when all the slopes β_i are equal to 1, the model (4.0.1) is known to be a variance component model. The precision π_i of instrument i is equal to σ_i^{-2}. Thus, choosing the most precise instrument is equivalent to choosing the instrument with the smallest measurement error variance.

Grubbs (1948) takes the average of $S_{ij}(i \neq j,\ i,j = 0,\ldots,p-1)$ to estimate σ_u^2, namely $\tilde{\sigma}_u^2 = 2p^{-1}(p-1)^{-1} \sum_{i=0}^{p-1} \sum_{j<i}^{p-1} S_{ij}$, and lets $\tilde{\sigma}_i^2 = S_{ii} - \tilde{\sigma}_u^2$, $i = 0,\ldots,p-1$. As is usual with estimated components of variances, the resulting estimators sometimes take negative values. For p = 2, these estimators are maximum likelihood estimators if they are positive. However, when $p \geq 3$ the maximum likelihood estimators of $\sigma_0^2,\ldots,\sigma_{p-1}^2$ and σ_u^2 are more complicated than $\tilde{\sigma}_i^2$, and have no closed form.

When the covariance matrix of a multivariate normal distribution is an unknown linear combination of given matrices, Anderson (1968) provides an iterative algorithm for finding the maximum likelihood estimators of the parameters and discusses the asymptotic properties of these estimators. For the model (4.0.1) with the β_i all equal to 1, the covariance matrix Σ_y of $\underset{\sim}{y}_i$ is equal to

$$\Sigma_y = \sum_{i=0}^{p} \sigma_i^2 \, G_i ,$$

where G_i $(i = 0,\ldots,p-1)$ has 1 in the $(i+1)$st diagonal position and zeros elsewhere, $G_p = 1_p 1_p'$ and $\sigma_p^2 = \sigma_u^2$. Hence, applying the algorithm suggested by Anderson, we can calculate the maximum likelihood estimators $\hat{\sigma}_0^2,\ldots,\hat{\sigma}_{p-1}^2$ and $\hat{\sigma}_u^2$ of $\sigma_0^2,\ldots,\sigma_{p-1}^2$ and σ_u^2. It has been shown by Anderson (1968) that the asymptotic joint distribution of $n^{\frac{1}{2}}(\hat{\sigma}_0^2 - \sigma_0^2,\ldots,\hat{\sigma}_{p-1}^2-\sigma_{p-1}^2,\ \hat{\sigma}_u^2-\sigma_u^2)'$ is multivariate normal with mean vector 0 and a covariance matrix C whose inverse C^{-1} has $\frac{1}{2}$ tr $\Sigma_y^{-1}G_i\Sigma_y^{-1}G_j$ as its $(i+1,\ j+1)$th element, $0 \le i,j \le p-1$. Thus, joint confidence regions for any collection of these parameters can be obtained by standard large sample techniques.

We now consider, for this special case of model (4.0.1), the problem of choosing the most precise instrument. For $p = 3$ with $\beta_1 = \beta_2 = 1$, Grubbs (1973) treats two of the instruments as standard instruments and the third instrument as a new instrument, and applies the test statistic, that is T^* defined in (2.2.8), suggested by Maloney and Rastogi (1970) for the two-instrument case to compare the measurement error variance σ_2^2 for the third instrument with the average measurement error variance, $2^{-1}(\sigma_0^2 + \sigma_1^2)$, of the two standards.

In Section 3.3, it was shown that the procedure (3.3.1) was not applicable without some extra constraint on the parameter space; in Section 4.1 we showed that even assuming that $R_1,\ldots,R_{p-1}$ are known, the procedure (3.3.1) based on the appropriate maximum likelihood estimators of $\psi_1,\ldots,\psi_{p-1}$ can be applied only when a lower bound to τ_0 is known. Here, assuming that the slopes β_i are all equal to 1, the asymptotic joint distribution of the maximum likelihood estimators for $\psi_1,\ldots,\psi_{p-1}$ can be obtained from the known asymptotic joint distribution of $\hat{\sigma}_0^2,\ldots,\hat{\sigma}_{p-1}^2$, but the covariance matrix of this asymptotic distribution is too complicated to be of much help. Consequently, as in Section 4.1, we use another statistic T_{0i}^* for comparing π_i and π_0, namely, the statistic

$$T_{0i}^* = \frac{(n-2)^{\frac{1}{2}}(S_{ii}-S_{00})}{2(S_{ii}S_{00}-S_{0i}^2)^{\frac{1}{2}}} \tag{4.2.1}$$

introduced in Section 2.2.

From the discussion in Section 2.2, we know that for comparing the precisions, π_i and π_0, of instrument i and the control, if T_{0i}^* is sufficiently small, we would decide that the instrument i is more precise than the control. Thus, we modify procedure (3.3.1) as follows:

Procedure: If $\min_{1\le i\le p-1} T_{0i}^* \ge \lambda_2$, then select the control as the best;

If $\min_{1\le i\le p-1} T_{0i}^* < \lambda_2$, then select the instrument which has the smallest T value as the best, (4.2.2)

where λ_2 is chosen so that the procedure (4.2.2) satisfies the P_0^* requirement. Similar to the proof of Theorem 4.1.3, we obtain the asymptotic joint distribution for the statistics T_{0i}^* in the following theorem.

Theorem 4.2.1. Let T^*_{0i} be defined by (4.2.1). Then the asymptotic joint distribution of

$$(T^*_{01} - (n-2)^{\frac{1}{2}} \rho^*_1(1-(\rho^*_1)^2)^{-\frac{1}{2}}, \ldots, T^*_{0p-1} - (n-2)^{\frac{1}{2}}\rho^*_{p-1}(1-(\rho^*_{p-1})^2)^{-\frac{1}{2}})'$$

is multivariate normal with mean vector 0 and covariance matrix $V = ((v_{ij}))$, where

$$\rho^*_i = \frac{1-\psi_i}{[(1+\psi_i)^2 + 4\psi_i(1+\psi_i)\tau_0]^{\frac{1}{2}}}, \tag{4.2.3}$$

$$V_{ii} = (1-(\rho^*_i)^2)^{-1},$$

$$V_{ij} = 8^{-1}\psi_i^{-\frac{1}{2}}\psi_j^{-\frac{1}{2}}(1+(1+\psi_i)\tau_0)^{-3/2}(1+(1+\psi_j)\tau_0)^{-3/2}L_{ij},$$

$$\begin{aligned} L_{ij} = {} & 8\tau_0^3(\psi_i^2\psi_j^2 + \psi_i\psi_j^2 + \psi_i^2\psi_j + \psi_i\psi_j) + \tau_0^2(2\psi_i^2\psi_j^2 + \\ & 7(\psi_i\psi_j^2 + \psi_i^2\psi_j) + 3(\psi_i^2+\psi_j^2) + 17\psi_i\psi_j + 4(\psi_i+\psi_j)+1) + \\ & \tau_0(\psi_i\psi_j^2 + \psi_i^2\psi_j + \psi_i^2 + \psi_j^2 + 8\psi_i\psi_j + 5(\psi_i+\psi_j)+2) \\ & + \psi_i\psi_j + \psi_i + \psi_j + 1 \end{aligned}$$

and

$$\psi_i = \sigma_0^2\,\sigma_i^{-2} = \pi_i\pi_0^{-1}, \quad i \neq j,\ i,j = 1,\ldots,p-1.$$

From the above theorem, we know that for large n,

$$\begin{aligned} & P\{\min_{1\le i\le p-1} T^*_{0i} \ge \lambda_2 | \psi_i \le 1,\ i = 1,\ldots,p-1\} \\ & \quad \approx P\{\bigcap_{i=1}^{p-1} (X^*_i \le a_i) | \psi_i \le 1, \quad i = 1,\ldots,p-1\}, \end{aligned} \tag{4.2.4}$$

where

$$X^*_i = \frac{-T^*_{0i} + (n-2)^{\frac{1}{2}}\rho^*_i(1-(\rho^*_i)^2)^{-\frac{1}{2}}}{(1-(\rho^*_i)^2)^{-\frac{1}{2}}}, \quad i = 1,\ldots,p-1,$$

are standard jointly normal random variables, and

$$a_i = -\lambda_2(1-(\rho_i^*)^2)^{\frac{1}{2}} + (n-2)^{\frac{1}{2}}\rho_i^*, \quad i = 1,\ldots,p-1. \tag{4.2.5}$$

It can be seen from Theorem 4.2.1 that the correlation coefficient $\rho(X_i^*, X_j^*)$ between X_i^* and X_j^* is nonnegative for any i,j. Applying Theorem 4.1.2, we obtain

$$P\{\bigcap_{i=1}^{p-1}(X_i^* \le a_i) \mid \psi_i \le 1,\ 1 \le i \le p-1\} \ge \prod_{i=1}^{p-1} P\{X_i^* \le a_i \mid \psi_i \le 1\}. \tag{4.2.6}$$

Lemma 4. For fixed τ_0, if $-(n-2)^{\frac{1}{2}} < \lambda_2 < 0$, then for each i

$$\inf_{\psi_i \le 1} a_i = -\lambda_2.$$

Proof: Note that when $\psi_i \le 1$, $\rho_i^* \ge 0$. As in the proof of Lemma 2, we can show that the minimum of a_i should occur either at $\rho_i^* = 0$ or $\rho_i^* = 1$. When ρ_i^* is equal to 0 and 1, a_i is equal to $-\lambda_2$ and $(n-2)^{\frac{1}{2}}$, respectively. Since $-\lambda_2 < (n-2)^{\frac{1}{2}}$, the minimum of a_i is equal to $-\lambda_2$.□

Combining (4.2.4), (4.2.6) and the result of Lemma 4, we obtain the following theorem.

Theorem 4.2.2. For a large n, if $\lambda_2 = -Z_{\nu^*}$, where $\nu^* = 1-(P_0^*)^{\frac{1}{p-1}}$, then the procedure (4.2.2) satisfies the P_0^* requirement. □

To evaluate the probability of correct selection (CS) for the procedure (4.2.2), for simplicity, we assume that only one instrument is more precise than the control. For convenience, assume that

$\psi_{p-1} = \max_{1 \le i \le p-1} \psi_i$. For a fixed $\Delta > 0$, we assume that

$$\psi_{p-1} = 1+\Delta; \; \psi_i \le 1, \quad i = 1,\dots,p-2.$$

In this case, the probability of correct selection is given by

$$P(CS) = P\{(T^*_{0p-1} < \lambda_2) \cap (\bigcap_{i=1}^{p-2} [T^*_{0p-1} - T^*_{0i} \le 0]) \mid \psi_i \le 1, \; 1 \le i \le p-2, \; \psi_{p-1} = 1+\Delta\}. \qquad (4.2.7)$$

However, when $\psi_i = 0$, $1 \le i \le p-2$, T_{0p-1} and $T_{0p-1}-T_{0i}$ are negatively correlated. To see this, note that

$$\rho(T^*_{0p-1}, T^*_{0p-1}-T^*_{0i}) = \frac{V_{p-1,p-1}-V_{i,p-1}}{[V_{p-1,p-1}(V_{p-1,p-1} + V_{ii} - 2V_{i,p-1})]^{\frac{1}{2}}}$$

$$= \frac{8^{-1}(1+(1+\psi_{p-1})\tau_0]^{-3/2}(1+(1+\psi_i)\tau_0]^{-3/2}N}{[\psi_{p-1}V_{p-1,p-1}(V_{p-1,p-1} + V_{ii} - 2V_{i,p-1})\psi_i\psi_{p-1}]^{\frac{1}{2}}}, \qquad (4.2.8)$$

where

$$N = 2\psi_i^{\frac{1}{2}}(1+(1+\psi_{p-1})\tau_0)^{\frac{1}{2}}(1+(1+\psi_i)\tau_0)^{3/2}((1+\psi_{p-1})^2 + 4\psi_{p-1}(1+\psi_{p-1})\tau_0) - \psi_{p-1}^{\frac{1}{2}}L_{i,p-1}.$$

It can be shown that the denominator of (4.2.8) is finite when $\psi_i = 0$ and $\psi_{p-1} = 1+\Delta$. However, when $\psi_i = 0$,

$$N = -\psi_{p-1}^{\frac{1}{2}}[(3\psi_{p-1}^2 + 4\psi_{p-1}+1)\tau_0^2 + (\psi_{p-1}^2 + 5\psi_{p-1}+2)\tau_0+\psi_{p-1} + 1] < 0.$$

Thus, for $\psi_i \le 1$, $\psi_{p-1} = 1+\Delta$, T^*_{0p-1} and $T^*_{0p-1}-T^*_{0i}$ are not positively correlated. To evaluate the lower bound of the probability of correct selection for the procedure (4.2.2), we need more work to find the lower bound of the correlation coefficients between T^*_{0p-1} and $T^*_{0p-1}-T^*_{0i}$ and $T^*_{0p-1}-T^*_{0j}$, respectively.

4.3 The Case Where the Relative Precision τ_0 of the Control Is Known

As discussed in Chapter 1 and in Section 2.3, in some situations, it is reasonable to assume that τ_0 is known.

Assuming that τ_0 is known, for the two-instrument case, the maximum likelihood estimators for the parameters have been shown in Section 2.3. However, the maximum likelihood estimators have no closed form for $p \geq 3$. It is worth noticing that the ordinary regression estimator $S_{0i}S_{00}^{-1}$ of β_i converges to $\beta_i\tau_0(1+\tau_0)^{-1}$. Therefore the quantity $\hat{\beta}_i = (1+\tau_0)\tau_0^{-1}S_{0i}S_{00}^{-1}$, $i = 1,\ldots,p-1$, is a consistent estimator of β_i.

Instead of finding the maximum likelihood estimators for the parameters, we may consider other consistent estimators motivated by the maximum likelihood estimators for the case $p = 2$. These consistent estimators are as follows:

$$\hat{\mu} = \bar{y}_0, \quad \hat{\alpha}_i = \bar{y}_i - \hat{\beta}_i\bar{y}_0, \quad \hat{\beta}_i = (1+\tau_0)\tau_0^{-1}S_{0i}S_{00}^{-1},$$

$$\hat{\sigma}_u^2 = S_{00}\tau_0(1+\tau_0)^{-1}, \quad \hat{\sigma}_0^2 = S_{00}(1+\tau_0)^{-1}, \quad \hat{\sigma}_i^2 = S_{ii}-S_{0i}^2S_{00}^{-1}(1+\tau_0)\tau_0^{-1}.$$

Hence, we estimate $\psi_i = \tau_i\tau_0^{-1} = \pi_i\pi_0^{-1}$ by $\hat{\psi}_i$, where

$$\hat{\psi}_i = \hat{\tau}_i\tau_0^{-1} = \frac{(1+\tau_0)\tau_0^{-2}}{r_i^{-2} - (1+\tau_0)\tau_0^{-1}}, \tag{4.3.1}$$

$$r_i^2 = \frac{S_{0i}^2}{S_{00}S_{ii}}, \qquad i = 1,\ldots,p-1. \tag{4.3.2}$$

To compare the precisions π_i and π_0 (or τ_i and τ_0), using $\hat{\psi}_i$ as the statistic, we would agree that instrument i is more precise than the control if $\hat{\psi}_i$ is sufficiently large. However, since $\hat{\psi}_i$ is increasing in r_i^2, an equivalent procedure is as follows:

$$\text{Procedure:}\quad \begin{cases} \text{If } \max\limits_{1\le i\le p-1} r_i^2 \le \lambda_3, \text{ then select the control as the best,} \\ \text{If } \max\limits_{1\le i\le p-1} r_i^2 > \lambda_3, \text{ then select the instrument which has the largest } r_i \text{ as the best, } i \ne 0, \end{cases} \tag{4.3.3}$$

where λ_3 (> 0) is chosen so that the procedure (4.3.3) satisfies the P_0^* requirement. Using a proof similar to that of Theorem 4.1.3, we obtain the asymptotic joint distribution for $r_1^2,\ldots,r_{p-1}^2$.

<u>Theorem 4.3.1</u>. Let r_i^2 be defined by (4.3.2). Then the asymptotic joint distribution of $n^{\frac{1}{2}}(r_1^2-\delta_1^2,\ldots,r_{p-1}^2-\delta_{p-1}^2)$ is multivariate normal with mean vector 0 and covariance matrix $C = ((C_{ij}))$, where

$$\begin{aligned} \delta_i^2 &= \frac{\psi_i\tau_0^2}{(1+\tau_0)(1+\psi_i\tau_0)}, \\ C_{ii} &= 4\delta_i^2(1-\delta_i^2)^2, \\ C_{ij} &= \frac{\tau_0^3\psi_i\psi_j\ E_{ij}}{(1+\tau_0)^4(1+\psi_i\tau_0)^2(1+\psi_j\tau_0)^2}, \end{aligned} \tag{4.3.4}$$

$$E_{ij} = \tau_0^3(4\psi_i\psi_j+2(\psi_i+\psi_j)+2) + \tau_0^2(4\psi_i\psi_j + 6(\psi_i+\psi_j) + 8) + \tau_0(4(\psi_i+\psi_j) + 10) + 4.$$

As a consequence of the above theorem, for a large n, we have

$$P\{\max_{1\le i\le p-1} r_i^2 \le \lambda_3 | \psi_i \le 1, \quad i = 1,\ldots,p-1\}$$
$$\approx P\{\bigcap_{i=1}^{p-1} (N_i \le f_i) | \psi_i \le 1, \quad i = 1,\ldots,p-1\}, \tag{4.3.5}$$

where

$$N_i = \frac{n^{\frac{1}{2}}(r_i^2 - \delta_i^2)}{[4\delta_i^2(1-\delta_i^2)^2]^{\frac{1}{2}}}, \quad i = 1,\ldots,p-1$$

are standard jointly normal random variables, and

$$f_i = \frac{n^{\frac{1}{2}}(\lambda_3-\delta_i^2)}{[4\delta_i^2(1-\delta_i^2)^2]^{\frac{1}{2}}}, \quad i = 1,\ldots,p-1. \tag{4.3.6}$$

It can be shown that the (asymptotic) correlation coefficient $\rho(N_i,N_j)$ between N_i and N_j is equal to

$$\rho(N_i,N_j) = \frac{\tau_0\psi_i^{\frac{1}{2}}\ \psi_j^{\frac{1}{2}}\ E_{ij}}{4(1+\tau_0)[(1+\psi_i\tau_0)(1+\psi_j\tau_0)(1+(1+\psi_i)\tau_0)(1+(1+\psi_j)\tau_0)]^{\frac{1}{2}}}. \tag{4.3.7}$$

Note that $\rho(N_i,N_j)$ is nonnegative for any i,j, $1 \le i,j \le p-1$. Applying Theorem 4.1.2, we have

$$P\{\bigcap_{i=1}^{p-1}(N_i \le f_i)|\psi_i \le 1,\ 1 \le i \le p-1\} \ge \prod_{i=1}^{p-1} P\{N_i \le f_i|\psi_i \le 1\}. \tag{4.3.8}$$

Note that δ_i^2 defined in (4.3.4) is increasing in ψ_i. Thus, $0 \le \psi_i \le 1$ gives $0 \le \delta_i^2 \le \tau_0^2(1+\tau_0)^{-2}$. When $\delta_1^2 = \ldots = \delta_{p-1}^2 = \tau_0^2(1+\tau_0)^{-2}$, and $\lambda_3 < \tau_0^2(1+\tau_0)^{-2}$, it is easy to see from (4.3.5) and (4.3.6) that

$$P\{\bigcap_{i=1}^{p-1}(N_i \le f_i)|\psi_i = 1,\ 1 \le i \le p-1\} < P\{\bigcap_{i=1}^{p-1}(N_i \le 0)\} \le \frac{1}{2}.$$

Consequently, if we wish $P_0^* \geq \frac{1}{2}$, we must require that $\lambda_3 > \tau_0^2(1+\tau_0)^{-2}$. Since the probability that $\max_{1 \leq i \leq p-1} r_i^2 \leq 1$ is equal to 1, if $\lambda_3 > 1$, then

$$P\{\max_{1 \leq i \leq p-1} r_i^2 > \lambda_3\} = 0.$$

Thus, if we wish the probability of correct selection to be greater than 0, λ_3 is required to be less than 1. We therefore require that $\tau_0^2(1+\tau_0)^{-2} < \lambda_3 < 1$.

Lemma 5. Assume that τ_0 (> 0.5) is a known constant. For $1 > \lambda_3 > \tau_0^2(1+\tau_0)^{-2}$, $P\{N_i \leq f_i | 0 < \psi_i \leq 1\}$ is decreasing in ψ_i.

Proof: If $\tau_0 > 0.5$ and $\lambda_3 > \tau_0^2(1+\tau_0)^{-2}$, then $\lambda_3 > \frac{1}{9}$. Taking the derivative of f_1 defined in (4.3.6) with respect to δ_1^2, we have

$$\frac{\partial f_i}{\partial \delta_i^2} = \frac{-1}{4} (\delta_i^2)^{-3/2}(1-\delta_i^2)^{-2}[(\delta_i^2)^2 + (1-3\lambda_3)\delta_i^2 + \lambda_3].$$

Since $\frac{1}{9} < \lambda_3 < 1$, $(1-3\lambda_3)^2 - 4\lambda_3 = (9\lambda_3-1)(\lambda_3-1) < 0$, thus, $(\delta_i^2)^2 + (1-3\lambda_3)\delta_i^2 + \lambda_3 > 0$ and $\frac{\partial f_i}{\partial \delta_i^2} < 0$. Because δ_i^2 is increasing in ψ_i, f_i is decreasing in ψ_i. The Lemma now follows. □

Note that $\psi_i = 1$ gives $f_i = n^{\frac{1}{2}}[\lambda_3-\tau_0^2(1+\tau_0)^{-2}][2\tau_0(1+2\tau_0)(1+\tau_0)^{-3}]^{-1}$. We summarize the results in the following theorem.

Theorem 4.3.2. Assume that τ_0 (> 0.5) is a known constant. For a large n, if $\lambda_3 = \tau_0^2(1+\tau_0)^{-2} + n^{-\frac{1}{2}} z_{\nu*}[2\tau_0(1+2\tau_0)(1+\tau_0)^{-3}]$, where

$\nu^* = 1-(P_0^*)^{\frac{1}{p-1}}$, then the procedure (4.3.3) satisfies the P_0^* requirement.

Proof: Directly from (4.3.5), (4.3.8) and Lemma 5. □

In practice, the instrument chosen as the control usually is known to be a reasonably good instrument from previous experience. A rule of thumb mentioned by Thompson (1963) suggests that if the instrumentation of an experiment is to be effective, τ_0 should be ≥ 100. Thus, assuming that $\tau_0 > 0.5$ seems reasonable.

To evaluate the probability of correct selection for the procedure (4.3.3), for simplicity, we assume that only one instrument is more precise than the control. For convenience, assume that $\psi_{p-1} = \max_{1 \leq i \leq p-1} \psi_i$. Thus, for a fixed $\Delta > 0$, assume that

$$\psi_{p-1} = 1+\Delta; \; \psi_i \leq 1, \quad i = 1,\ldots,p-2.$$

In this case, the probability of correct selection is given by

$$P(CS) = P\{(r_{p-1}^2 > \lambda_3) \cap (\bigcap_{i=1}^{p-2} [r_{p-1}^2 - r_i^2 \geq 0]) | \psi_i \leq 1, \; 1 \leq i \leq p-2, \; \psi_{p-1} = 1+\Delta\}. \qquad (4.3.9)$$

From Theorem 4.3.1, the covariance of r_{p-1}^2 and $r_{p-1}^2 - r_i^2$ is given by

$$\mathrm{Cov}(r_{p-1}^2, r_{p-1}^2 - r_i^2) = \mathrm{Var}(r_{p-1}^2) - \mathrm{Cov}(r_{p-1}^2, r_i^2) = c_{p-1,p-1} - c_{i,p-1}$$

$$= \frac{\psi_{p-1}\tau_0^2\{4(1+\tau_0)(1+\psi_i\tau_0)^2(1+(1+\psi_{p-1})\tau_0)^2 - \psi_i\tau_0(1+\psi_{p-1}\tau_0)E_{i,p-1}\}}{(1+\tau_0)^4(1+\psi_{p-1}\tau_0)^3(1+\psi_i\tau_0)^2}.$$

We find that when $\psi_i = 0.5$, $\psi_{p-1} = 1 + \Delta \geq 3$ and $\tau_0 \geq 10$, the covariance is negative. Thus, we know that r_{p-1}^2 and $r_{p-1}^2 - r_i^2$, $1 \leq i \leq p-2$, are not positively correlated. The problem of finding a lower bound for the probability of correct selection for the procedure (4.3.3) appears to be very complicated.

BIBLIOGRAPHY

1. Amemiya, Y. and Fuller, W. A. (1984). Estimation for the Multivariate Errors-In-Variables Model with Estimated Error Covariance Matrix. Ann. of Statist. 12, 497-509.

2. Anderson, T. W. and Rubin, H. (1956). Statistical Inference in Factor Analysis. Proceedings of Third Berkeley Symposium. University of California Press, Berkeley.

3. Anderson, T. W. (1958). An Introduction to Multivariate Analysis. John Wiley and Sons Inc., New York.

4. Anderson, T. W. (1968). Estimation of Covariance Matrices Which Are Linear Combinations or Whose Inverse Are Linear Combinations of Given Matrices. Essays in Probability and Statistics. University of North Carolina Press, Chapel Hill.

5. Anderson, T. W. (1984). Estimating Linear Statistical Relationships. Ann. of Statist. 12, 1-45.

6. Barnett, V. D. (1969). Simultaneous Pairwise Structural Relationships. Biometrika 25, 129-142.

7. Cochran, W. G. (1968). Errors of Measurement in Statistics. Technometrics 10, 637-666.

8. Creasy, M. A. (1957). Confidence Limits for the Gradient in the Linear Structural Relationship. J. of Roy. Statist. Soc., Ser. B, 18, 65-69.

9. Fuller, W. A., Amemiya, Y. and Pantula, S. G. (1983). The Covariance Matrix of Estimators for the Factor Model. Unpublished Technical Report, Iowa State University.

10. Gleser, L. J. and Watson, G. S. (1973). Estimation of a Linear Transformation. Biometrika 60, 525-534.

11. Gleser, L. J. (1981). Estimation in a Multivariate "Errors-In-Variables" Regression Model: Large Sample Results. Ann. of Statist. 9, 24-44.

12. Graybill, F. A. (1976). Theory and Application of the Linear Model. Duxbury Press, Massachusetts.

13. Grubbs, F. E. (1948). On Estimating Precision of Measuring Instruments and Product Variability. J. Amer. Statist. Assoc. 43, 243-264.

14. Grubbs, F. E. (1973). Errors of Measurement, Precision, Accuracy, and the Statistical Comparison of Measuring Instruments. Technometrics 15, 53-66.

15. Jennrich, R. I. and Thayer, D. T. (1973). A Note on Lawley's Formulas for Standard Errors in Maximum Likelihood Factor Analysis. Psychometrika 38, 571-580.

16. Jolicoeur, P. and Mosimann, J. E. (1968). Intervalles de confiance pour la pente de l'axe majeur d'une distribution normale bidimension nella. Biom. Prax. IX, 121-140.

17. Jöreskog, K. G. (1969). A General Approach to Confirmatory Maximum Likelihood Factor Analysis. Psychometrika 34, 183-202.

18. Kendall, M. G. and Stuart, A. (1979). The Advanced Theory of Statistics (Vol. II, Fourth Edition), New York: Macmillan.

19. Lawley, D. N. (1953). A Modified Method of Estimation in Factor Analysis and Some Large Sample Results. Uppsala Symposium on Psychological Factor Analysis. Nordisk Psykologi's Monograph Series No. 3, 35-42. Copenhagen: Ejnar Mundsgaards: Sotockholm: Almqvist and Wiksell.

20. Lawley, D. N. and Maxwell, A. E. (1971). Factor Analysis As a Statistical Method. American Elsevier, New York.

21. Lord, F. M. and Novick, M. R. (1968). Statistical Theories of Mental Test Scores. Addison-Wesley Publishing Company.

22. Maloney, C. J. and Rastogi, S. C. (1970). Significance Test for Grubbs' Estimators. Biometrics 26, 671-676.

23. Mandel, J. (1959). The Measuring Process. Technometrics 1, 251-267.

24. Miller, R. G. Jr. (1980). Kanamycin Levels in Premature Babies. Technical Report No. 57. Stanford University.

25. Moran, P. A. P. (1971). Estimating Structural and Functional Relationships. J. Multivariate Anal. 1, 232-255.

26. Morgan, W. A. (1939). A Test for the Significance of the Difference between Two Variances in a Sample from a Normal Bivariate Population. Biometrika 31, 13-19.

27. Murphy, R. B. (1969). On the Measuring of Precision and Accuracy. Precision Measurement and Calibration. National Bureau of Standards, special publication 300, Vol. 1.

28. Pitman, E. J. G. (1939). A Note on Normal Correlation. Biometrika 31, 9-12.

29. Paulson, E. (1952). On the Comparison of Several Experimental Categories with a Control. Ann. Math. Statist. 23, 239-246.

30. Rao, C. R. (1973). Linear Statistical Inference and Its Applications. John Wiley and Sons Inc., New York.

31. Seber, G. A. F. (1977). Linear Regression Analysis. John Wiley and Sons Inc., New York.

32. Šidák, A. (1967). Rectangular Confidence Regions for the Means of Multivariate Normal Distributions. J. Amer. Statist. Assoc. 62, 626-633.

33. Slepian, D. (1962). The One-Sided Barrier Problem for Gaussian Noise. Bell System Tech. J. 41, 463-501.

34. Smith, H. F. (1950). Estimating Precision of Measuring Instruments. J. Amer. Statist. Assoc. 45, 447-451.

35. Theobald, C. M. (1975). An Inequality with Application to Multivariate Analysis. Biometrika 62, 461-466.

36. Theobald, C. M. and Mallinson, J. R. (1978). Comparative Calibration, Linear Structural Relationships and Congeneric Measurements. Biometrics 34, 39-45.

37. Thompson, W. A. Jr. (1962). Estimation of Dispersion Parameters. Journal of Research of the National Bureau of Standards-B, 66B, 161-164.

38. Thompson, W. A. Jr. (1963). Precision of Simultaneous Measurement Procedures. J. Amer. Statist. Assoc. 58, 474-479.

39. Williams, E. J. (1969). A Note on Regression Methods in Calibration. Technometrics 11, 189-192.