Zeroes of Infinitely Differentiable Characteristic Functions

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Abstract

Let A° and B° be subsets of $\mathbb{R}^n$. Necessary and sufficient conditions on the pair (A°, B°) are found for the existence of an n-dimensional characteristic function whose real part is zero precisely on A° and whose imaginary part is zero precisely on B°. The necessary and sufficient conditions continue to apply if the characteristic function is required to be infinitely differentiable. A corollary of the main result is the existence of infinitely differentiable characteristic functions with compact support.

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1. Introduction and Summary

Let $f: \mathbb{R}^n \to \mathbb{C}$ be the characteristic function of a probability distribution on $\mathbb{R}^n$. Let $A^+ \subset \mathbb{R}^n$ be the set on which $\mathrm{Re}\{f(.)\}$ is strictly positive, and let A^- be the set on which $\mathrm{Re}\{f(.)\}$ is strictly negative. Let B^+ be the set on which $\mathrm{Im}\{f(.)\}$ is strictly positive. What can we say about the sets A^+, A^-, and B^+? Since f is continuous, A^+, A^-, and B^+ are open sets. Since $f(t) = \overline{f(-t)}$ for all $t \in \mathbb{R}^n$, we have $A^+ = -A^+$, $A^- = -A^-$, and $B^+ \cap (-B^+) = \emptyset$. Clearly, $A^+ \cap A^- = \emptyset$. Finally, it follows from $f(0) = 1$ that $0 \in A^+$ and $0 \notin B^+$.

This paper will show that these obviously necessary conditions on the triple (A^+, A^-, B^+) are also sufficient to insure the existence of an n-dimensional characteristic function whose real part is positive precisely on A^+ and negative precisely on A^-, and whose imaginary part is positive precisely on B^+. Furthermore, this characteristic function may be taken to be infinitely differentiable.

Let $A^0 \subset \mathbb{R}^n$ be a closed set satisfying $0 \notin A^0$ and $A^0 = -A^0$. Let $B^0 \subset \mathbb{R}^n$ be a closed set containing 0 whose complement $(B^0)^c$ can be expressed as $(B^0)^c = B^+ \cup (-B^+)$, where B^+ is an open set satisfying $B^+ \cap (-B^+) = \emptyset$. It follows immediately from the main result that there exists an n-dimensional C^∞ characteristic function whose real part is zero precisely on A^0 and whose imaginary part is zero precisely on B^0. These sufficient conditions on A^0 and B^0 are obviously necessary.

Examples of one-dimensional characteristic functions with compact support are well known and are usually mentioned in graduate courses in probability theory. However, the usual examples, and all those obtainable from the famous sufficient condition of Polya (see Theorem 6.5.3 of Chung (1974)) are not differentiable at zero, and the authors are not aware of any previously published examples of C^∞ characteristic functions with compact support.

We now summarize the method of proof. Let f_1 and f_2 be one-dimensional characteristic functions corresponding to strictly positive bounded densities r_1 and r_2. Then the convolution $f_1 * f_2$ (call it f^*) is the Fourier transform of the product $r_1 r_2$. Since $r_1 r_2$ is the density of a finite positive measure with total mass $f^*(0)$, it follows that h defined by

$$h(t) = f^*(t)/f^*(0)$$

is the characteristic function of the probability density $r_1 r_2 / f^*(0)$. Now suppose further that f_1 and f_2 are real-valued, nonnegative C^k functions with compact support, and that r_1 and r_2 are unimodal densities. Then h will be a nonnegative C^{2k} function with compact support, and the corresponding probability density will be unimodal. Section 2 uses this trick (in a slightly disguised form with the roles of characteristic function and probability density being switched at first) to construct nonnegative, one-dimensional characteristic functions g_1 and g_2 which are C^∞ and positive precisely on (-1,1). Furthermore, the corresponding densities p_1 and p_2 are unimodal, and the tails of p_1 are thin compared to those of p_2. Section 2 proceeds by using g_1 and g_2 to construct n-dimensional characteristic functions $g_{1,n}$ and $g_{2,n}$ which are nonnegative, C^∞, and positive precisely on the open unit ball centered at the origin. The corresponding densities remain unimodal, with $p_{1,n}$ having thinner tails than $p_{2,n}$. The main theorem is proved in Section 2. A function f is obtained by adding C^∞ perturbations defined in terms of $g_{1,n}$ to the characteristic function $g_{2,n}$. The perturbations make the real and imaginary parts of f positive and negative on the proper sets, and they do not destroy the property of infinite differentiability. Finally, the relative tail behavior of $p_{1,n}$ and $p_{2,n}$ causes the Fourier transforms of the perturbations to be uniformly small compared to $p_{2,n}$ (the Fourier transform of $g_{2,n}$), so that the Fourier transform of f is everywhere positive. It follows that f is the characteristic function of a probability density, and this completes the proof of the theorem.

2. Construction of the Characteristic Functions $g_{1,n}$ and $g_{2,n}$

For $x \in \mathbb{R}$, $x \neq 0$, define

$$r(x) = \frac{6}{x^2}\left(1 - \frac{\sin x}{x}\right).$$

Let $r(0) = 1$, so that r is continuous.

Lemma 1 The characteristic function of the probability density $(3/2)\ \{(1-|t|)^+\}^2$ is r.

Proof: Direct calculation.

Lemma 2 The function r is unimodal and positive.

Proof: Since r is symmetric and since $r(0) = 1$ and $\lim_{x\to\infty} r(x) = 0$, it will suffice to prove that the first derivative $r'(.)$ has no zeroes for $x \in (0,\infty)$. But

$$r'(x) = -\frac{6}{x^4}\left[(2+\cos x)x - 3\sin x\right],$$

so that it will suffice to prove that w(.) defined by

$$w(x) = (2+\cos x)\, x - 3\sin x$$

has no zeroes on $(0,\infty)$. It is easy to see that $w(x)$ is positive for $x \geq \pi$. To take care of $x \in (0,\pi)$, note that

$$w'(x) = 2 - 2\cos x - x \sin x$$

$$w''(x) = \sin x - x \cos x$$

$$w'''(x) = x \sin x$$

The third derivative $w'''(x)$ is positive for $x \in (0,\pi)$. Since $w''(0) = w'(0) = w(0) = 0$, it follows that $w(x)$ is positive for $x \in (0,\pi)$, and we are done. □

Let $X_1, X_2, \ldots$ be i.i.d. random variables with density $(3/2)\,\{(1-|t|)^+\}^2$. Define

$$S_1 = \sum_{k=1}^{\infty} X_k/k^2 \quad \text{and} \quad S_2 = \sum_{k=1}^{\infty} X_k/k^4.$$

Let h_1 be the density of S_1, and let h_2 be the density of S_2. Since $\Sigma_{k=1}^{\infty} k^{-2} = \pi^2/6$, the density h_1 is positive precisely on the interval $(-\pi^2/6, \pi^2/6)$. Likewise, since $\Sigma_{k=1}^{\infty} k^{-4} = \pi^4/90$, h_2 is positive precisely on $(-\pi^4/90, \pi^4/90)$.

It follows from Lemma 1 that the characteristic functions of S_1 and S_2 are given by

$$q_1(x) = \prod_{k=1}^{\infty} r(x/k^2)$$

and

$$q_2(x) = \prod_{k=1}^{\infty} r(x/k^4),$$

respectively.

By the Fourier inversion theorem (see the corollary on page 155 of Chung (1974)),

$$h_j(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} q_j(x)dx,$$

for j = 1,2. Setting t = 0 yields

$$2\pi h_j(0) = \int_{-\infty}^{\infty} q_j(x)dx.$$

Thus, $\tilde{p}_j(.)$ defined by

$$\tilde{p}_j(.) = q_j(x)/(2\pi h_j(0))$$

is a probability density with characteristic function given by

$$\tilde{g}_j(t) = h_j(t)/h_j(0), \quad j = 1,2.$$

Obviously, $\tilde{g}_1$ and $\tilde{g}_2$ are positive precisely on $(-\pi^2/6, \pi^2/6)$ and $(-\pi^4/90, \pi^4/90)$, respectively. Since $r(.)$ is symmetric about 0 and unimodal, $\tilde{p}_1$ and $\tilde{p}_2$ are also symmetric and unimodal. From the definitions of $r(.)$ and $q_j(.)$ above, it is easy to see that

$$\lim_{x\to\infty} x^m \tilde{p}_j(x) = 0$$

for $j = 1,2$ and for all m 0. Thus, the densities $\tilde{p}_1$ and $\tilde{p}_2$ have all moments. It follows that $\tilde{g}_1$ and $\tilde{g}_2$ are C^∞. (See Theorem 6.4.1 of Chung (1974)). Finally, we need to show that, for each $a>0$, there is a positive constant $K(a)$ such that

$$\tilde{p}_1(ax) < K(a)\, \tilde{p}_2(x) \tag{2.1}$$

for all $x \in \mathbb{R}$. (This is the result concerning relative tail behavior which was alluded to in the introduction.) To do this, it will suffice to show that the ratio

$$\frac{q_1(ax)}{q_2(x)} = \prod_1^\infty \frac{r(ax/k^2)}{r(x/k^4)} \tag{2.2}$$

is bounded in x for each fixed $a>0$. The kth factor $\frac{r(ax/k^2)}{r(x/k^4)}$ in (2.2) is continuous in x and converges to $(a^2k^4)^{-1}$ at $\pm\infty$ and is therefore bounded in x. If $b>c>0$, then $0 < \frac{r(bx)}{r(cx)} \leq 1$ for all $x \in \mathbb{R}$, by Lemma 2. Since $\frac{a}{k^2} > \frac{1}{k^4}$ for $k > a^{-\frac{1}{2}}$, all but finitely many factors in (2.2) are bounded above by 1 for all x. Since all factors in (2.2) are bounded in x, and all but finitely many factors are bounded by 1, it follows that (2.2) is bounded in x.

Define g_1, g_2, p_1, and p_2 by rescaling $\tilde{g}_1$, $\tilde{g}_2$, $\tilde{p}_1$, and $\tilde{p}_2$ as follows.

$$g_1(t) = \tilde{g}_1(\pi^2 t/6)$$

$$g_2(t) = \tilde{g}_2(\pi^4 t/90)$$

$$p_1(x) = (6/\pi^2)\tilde{p}_1(6x/\pi^2)$$

$$p_2(x) = (90/\pi^4)\tilde{p}_2(90x/\pi^4).$$

Our results for $\tilde{g}_1$, $\tilde{g}_2$, $\tilde{p}_1$, and $\tilde{p}_2$ imply the results for g_1, g_2, p_1, and p_2 given in the following lemma.

Lemma 3 The functions g_1 and g_2 defined above are real-valued, nonnegative, C^∞ characteristic functions which are positive precisely on (-1,1). For each a>0, there is a constant K'(a) such that the corresponding (unimodal) density functions p_1 and p_2 satisfy

$$p_1(ax) < K'(a)\, p_2(x)$$

for all $x \in \mathbb{R}$.

In order to prove our main theorem, we will need an n-dimensional analog of Lemma 3. For the remainder of this paper, t and x will denote points in $\mathbb{R}^n$ with respective coordinates t_i and x_i, $i=1,\ldots,n$.

For $j=1$ and 2, let $\underline{Y}_j$ be a random vector in $\mathbb{R}^n$ whose coordinates are i.i.d. random variables with density p_j. Then $\underline{Y}_j$ has density

$$\hat{p}_{j,n}(x) = \prod_{i=1}^{n} p_j(x_i)$$

and characteristic function

$$\hat{g}_{j,n}(t) = \prod_{i=1}^{n} g_j(t_i).$$

Let M be a random $n\times n$ orthogonal matrix (with the normalized Haar measure on the group of $n\times n$ orthogonal matrices as its probability distribution), and suppose M is independent of $\underline{Y}_j$. Then $\underline{Z}_j = M\underline{Y}_j$ is a spherically symmetric random vector in $\mathbb{R}^n$ with density

$$\tilde{p}_{j,n}(x) = \int_{S^{n-1}} \hat{p}_{j,n}(||x||u)d\nu(u),$$

where $S^{n-1} = \{t\in \mathbb{R}^n\colon ||t||=1\}$ is the unit sphere in $\mathbb{R}^n$, and ν is the rotation invariant probability measure on S^{n-1}. The characteristic function of $\underline{Z}_j$ is

$$\tilde{g}_{j,n}(t) = \int_{S^{n-1}} \hat{g}_{j,n}(||t||u)d\nu(u),$$

which is C^∞ and is positive precisely on $\{t \in \mathbb{R}^n\colon ||t||<\sqrt{n}\}$. For $j=1$ and 2, let

(2.3) $$g_{j,n}(t) = \tilde{g}_{j,n}(n^{\frac{1}{2}}t)$$

and

(2.4) $$p_{j,n}(x) = n^{-\frac{1}{2}}\,\tilde{p}_{j,n}(n^{-\frac{1}{2}}x).$$

If we let $L(a) = \{K'(a)\}^n$ for $a>0$, then Lemma 3 and the definitions of $p_{1,n}$ and $p_{2,n}$ imply

$$p_{1,n}(ax) < L(a)\; p_{2,n}(x)$$

for all $x \in \mathbb{R}^n$. This completes the proof of the following lemma.

Lemma 4 The functions $g_{1,n}$ and $g_{2,n}$ defined above are real-valued, nonnegative, C^∞ characteristic functions which are positive precisely on $\{t \in \mathbb{R}^n\colon ||t||<1\}$. For each $a>0$, there is a constant $L(a)$ such that the corresponding densities functions $p_{1,n}$ and $p_{2,n}$ satisfy

$$p_{1,n}(ax) < L(a)p_{2,n}(x)$$

for all $x \in \mathbb{R}^n$.

Remark The spherically symmetric densities $p_{1,n}$ and $p_{2,n}$ are obviously unimodal, but this will not be needed in what follows.

3. The Main Theorem

Theorem Let A^+, A^-, and B^+ be open subsets of $\mathbb{R}^n$ satisfying $A^+ = -A^+$, $A^- = -A^-$, $B^+ \cap (-B^+) = \emptyset$, $A^+ \cap A^- = \emptyset$, $0 \in A^+$, and $0 \notin B^+$. Then there exists an infinitely differentiable characteristic function f on $\mathbb{R}^n$ satisfying

$$A^+ = \{t \in \mathbb{R}^n: \ \mathrm{Re}(f(t)) > 0\}$$

$$A^- = \{t \in \mathbb{R}^n: \ \mathrm{Re}(f(t)) < 0\}$$

and
$$B^+ = \{t \in \mathbb{R}^n: \ \mathrm{Im}(f(t)) > 0\}.$$

Proof For $c \in \mathbb{R}^n$ and r a positive constant, let

$$B_r(c) = \{t \in \mathbb{R}^n: \ ||t-c|| < r\}$$

be the open ball in $\mathbb{R}^n$ with center c and radius r. We may assume without loss of generality that $B_1(0) \subset A^+$. Define

$$\tilde{A}^+ = A^+ \cap \{t \in \mathbb{R}^n: \ ||t|| > 1/2\}.$$

Since $\tilde{A}^+$ is open, it is the union of a countable set $\{B_{r_j}(c_j)\}_{i=1}^{\infty}$ of open balls. Since $\tilde{A}^+ = - \tilde{A}^+$, we have $B_{r_j}(-c_j) \subset \tilde{A}^+$ for all j. Define

$$f_j^+(t) = g_{1,n}\{(t-c_j)/r_j\} + g_{1,n}\{(t+c_j)/r_j\}.$$

By Lemma 4, f_j^+ is positive precisely on $B_{r_j}(c_j) \cup B_{r_j}(-c_j)$. Taking a Fourier transform yields

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i(x\cdot t)} f_j^+(t)dt = \{e^{-i(x\cdot c_j)} + e^{i(x\cdot c_j)}\} \, r_j p_{1,n}(r_j x)$$

$$= 2r_j \cos(x\cdot c_j) p_{1,n}(r_j x)$$

(Part (c) of the Fourier inversion Theorem 7.7 of Rudin (1973) implies that $p_{1,n}$ and $p_{2,n}$ are the Fourier transforms of $g_{1,n}$ and $g_{2,n}$, respectively.) Let $\{\alpha_j\}_{j=1}^{\infty}$ be a sequence of positive constants satisfying $\alpha_j < 2^{-j-2}\{2r_j L(r_j)\}^{-1}$. Then

$$|(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i(x\cdot t)} \sum_{i=1}^{\infty} \alpha_j f_j^+(t)dt|$$

$$< \sum_{j=1}^{\infty} 2^{-j-2}\{L(r_j)\}^{-1} p_{1,n}(r_j s)$$

$$< \frac{1}{4} p_{2,n}(x).$$

Furthermore, by choosing the α_j's to converge to zero sufficiently fast, we can insure that $f^+(.)$ defined by

$$f^+(t) = \sum_{j=1}^{\infty} \alpha_j f_j^+(t)$$

is C^∞ and in $L^1(\mathbb{R}^n)$. Note that the real-valued, nonnegative function $f^+(.)$ is nonzero precisely on A^+.

Let $\{B_{r_j'}(c_j')\}_{j=1}^{\infty}$ be a sequence of open balls whose union is A^-, and let

$$f_j^-(t) = -g_{1,n}\{(t-c_j')/r_j'\} - g_{1,n}\{(t+c_j')/r_j'\}.$$

The same argument used above shows that we can choose a sequence of positive constants $\{\beta_j\}_{j=1}^{\infty}$ such that $f^-(.)$ defined by

$$f^-(t) = \sum_{j=1}^{\infty} \beta_j f_j^-(t)$$

is C^∞, in $L^1(\mathbb{R}^n)$, and satisfies

$$|(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-(x\cdot t)} f^-(t)dt| < \frac{1}{4} p_{2,n}(x).$$

Note that the real-valued, nonpositive function $f^-(.)$ is nonzero precisely on A^-.

Let $\{B_{r_j''}(c_j'')\}_{j=1}^{\infty}$ be a sequence of open balls whose union is B^+. Let

$$f_j^{im}(t) = i[g_{1,n}\{(t-c_j'')/r_j''\} - g_{1,n}\{(t+c_j'')/r_j''\}].$$

Then

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i(x\cdot t)} f_j^{im}(t)dt =$$

$$= \{e^{-i(x\cdot c_j'')} - e^{i(x\cdot c_j'')}\}\ r_j'' p_{1,n}(r_j'' x)$$

$$= -2\ r_j''\ \sin(x\cdot c_j'')\ p_{1,n}(r_j'' x)$$

Again, we can choose a sequence of positive constants $\{\gamma_j\}_{j=1}$ so that $f^{im}(.)$ defined by

$$f^{im}(t) = \sum_{j=1}^{\infty} \gamma_j f_j^{im}(t)$$

is C^∞, in $L^1(\mathbb{R}^n)$, and satisfies

$$|(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i(x \cdot t)} f^{im}(t) dt| < \frac{1}{4} p_{2,n}(x).$$

Note that the function $f^{im}(.)$ is pure imaginary, and that its imaginary part is positive precisely on B^+.

Now let

$$f(t) = g_{2,n}(t) + f^+(t) + f^-(t) + f^{im}(t).$$

Clearly the real and imaginary parts of f are positive and negative on the proper sets. The function f is C^∞, and in $L^1(\mathbb{R}^n)$. Define

$$p(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-(x \cdot t)} f(t) dt.$$

The function p(.) is real-valued, since $f(t) = \overline{f(-t)}$ for all $t \in \mathbb{R}^n$. Since

$$|(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-(x \cdot t)} \{f^+(t) + f^-(t) + f^{im}(t)\} dt| < \frac{3}{4} p_{2,n}(x)$$

and

$$(2\pi)^{-n} \int_{\mathbb{R}^n} e^{-(x \cdot t)} g_{2,n}(t) dt = p_{2,n}(x),$$

we have

$$\frac{1}{4}\, p_{2,n}(x) < p(x) < 2p_{2,n}(x).$$

By the Fourier inversion theorem (Again, see Theorem 7.7(c) of Rudin (1973).),

$$f(t) = \int_{\mathbb{R}^n} e^{i(x\cdot t)} p(x)\,dx.$$

Also, since $f(0) = g_{2,n}(0) = 1$, we have

$$\int_{\mathbb{R}^n} p(x)\,dx = f(0) = 1.$$

Thus, f is the characteristic function of the probability density p, and f satisfies all the requirements of the theorem.

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References

Chung, K. L. (1974). A Course in Probability Theory, Academic Press, New York.

Rudin, W. (1973). Functional Analysis, McGraw-Hill, New York.