Admissibility and Minimaxity Results in the Estimation Problem of Exponential Quantiles

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SUMMARY

Admissibility and Minimaxity Results in the Estimation Problem of Exponential Quantiles.

The estimation problem of the quantiles $\xi + b\sigma$ of an exponential distribution with unknown location-scale parameter (ξ,σ) is considered. We establish the admissibility of the traditional (best equivariant) estimator for quadratic loss when $n^{-1} \leqslant b \leqslant 1 + n^{-1}$ where n is the sample size. For $b > 1 + n^{-1}$ a class of minimax procedures is found. This class contains generalized Bayes rules and one of them is shown to be admissible within the class of scale-equivariant procedures.

1. Introduction and Summary

Let $x_1,\ldots,x_n$, $n \geq 2$, be a random sample from an exponential distribution with unknown location-scale parameter (ξ,σ). We consider the statistical estimation problem of a quantile $\theta = \xi + b\sigma$. Clearly if p, $0 < p < 1$, is the order of this quantile, then $b = -\log p$. It is assumed that the loss is quadratic, $(\delta - \theta)^2 \sigma^{-2}$, where δ is an estimate of θ. This loss function is invariant under location and scale transformations.

The estimation problem of exponential quantiles is of importance in reliability theory, life testing and related subjects. Many papers have been dedicated to practical aspects of this problem (see, for instance, Epstein and Sobel (1954), Epstein (1962), Ali, Umbach and Hassanein (1981), Saleh (1981), Ali, Umbach and Saleh (1982)).

From a theoretical point of view this problem is interesting since the best equivariant estimator of a quantile which is linear function of unknown location and scale parameters, is always minimax but typically inadmissible. In the case of a normal sample, for example, this estimator is inadmissible if $b \neq 0$ and admissible if $b = 0$ (see Zidek (1971) and Rukhin (1983)).

Rukhin and Strawderman (1982) established the inadmissibility of the best equivariant estimator δ_0 of an exponential quantile for which either $b > 1 + n^{-1}$ or $0 \leq b < n^{-1}$, where n is the sample size. They exhibited a class of procedures which have risk uniformly smaller than that of δ_0. However all these procedures coincide with δ_0 with positive probability and none of them can be admissible. A similar result was obtained by Rukhin and Zidek (1984) in the case of several independent exponential samples.

In this paper we show that in the case $n^{-1} \leq b \leq 1 + n^{-1}$ the estimator δ_0 is admissible. In the case when $b > 1 + n^{-1}$ we construct a class of minimax procedures which are different from δ_0 with probability one. Some of these are generalized Bayes rules. An explicit formula is given for a minimax generalized Bayes estimator which is admissible within the class of all scale-equivariant procedures.

Heuristically the admissibility result for δ_0 in the case $n^{-1} \leq b \leq 1 + n^{-1}$ is due to the fact that it is generalized Bayes not only with respect to the uniform (right Haar) measure over the group of linear transformations of real line, but also with respect to many other prior distributions. Some of these distributions are considerably less "flat" than the uniform one, so that they can be "better approximated" by probability measures. The latter fact is known to be responsible for admissibility (see Stein (1965), Brown (1979)). Thus our admissibility proof is just a slight modification of the standard one for the one-dimensional location parameter (see Blyth (1951), Stein (1959) and Farrell (1964)). The non uniqueness of the uniform distribution as a prior happens also in many other problems involving location-scale parameter. This phenomenon leads to many surprising admissibility results for traditional estimators of functions of ξ and σ for normal and exponential samples.

The above mentioned admissibility result along with the admissibility within the class of scale-equivariant procedures is given in Section 4. In Section 3 we study the form of generalized Bayes estimators and prior distributions which generate the estimator δ_0. In Section 2 a class of minimax procedures when $b > 1 + n^{-1}$ is constructed. The paper is concluded by a discussion of some open problems in Section 5.

2. A Class of Minimax Quantile Estimators

Let $x = \min_{1 \leqslant j \leqslant n} x_j$, $y = n^{-1} \Sigma x_j - x$. Then (x,y) is a version of a minimal sufficient statistic, and its distribution has density $\sigma^{-2}p((x-\xi)/\sigma,y/\sigma)$ where

$$p(x,y) = n^n \exp\{-n(x+y)\}y^{n-2}/\Gamma(n-1) \qquad (2.1)$$

for $x,y \geqslant 0$ and $p(x,y) = 0$ otherwise.

Any equivariant estimator δ which depends only on x and y must be of the form $\delta(x,y) = x + cy$ for some constant c. It is easy to see that the best choice of c, which minimizes the risk of such estimator, is $c = a = b - n^{-1}$. This best equivariant estimator $\delta_0(x,y) = x + ay$ is known to be minimax, but inadmissible for $a > 1$ or $-n^{-1} \leqslant a < 0$. Due to the structure of the exponential distribution it is convenient to study the minimaxity for $\xi \geqslant 0$ separately. Thus we call an estimator δ to be minimax (admissible) for $\xi \geqslant 0$ if it is minimax (admissible) when the parameter space is restricted to $\{(\xi,\sigma),\ \xi \geqslant 0,\ \sigma > 0\}$. It was noticed by Rukhin and Strawderman (1982) that the minimaxity of the estimator $\delta(x,y)$, $x \geqslant 0$ for $\xi \geqslant 0$ implies the minimaxity of $\delta_1(x,y)$; $\delta_1(x,y) = \delta(x,y)$, $x \geqslant 0$; $= \delta_0(x,y)$, $x < 0$. In this Section we obtain a class of procedures which are minimax for $\xi \geqslant 0$ when $a > 1$. Hence a class of minimax rules is obtained for all ξ.

We study scale-equivariant estimators δ of the form

$$\delta(x,y) = x+ay - 2ay\ f(y/x),\ x > 0 \qquad (2.2)$$

where f is a positive measurable function.

Theorem 1. For $a > 1$ an estimator δ of the form (2.2) is minimax for $\xi \geqslant 0$ if for some positive p , $f(z)(1 + z^{-1})^p$ is a nondecreasing function of z, $z \geqslant 0$,

$$0 \leqslant f(z)\ (1+ z^{-1})^p \leqslant \bar{f}, \qquad (2.3)$$

and

$$a\bar{f} \leq (a-1)\min[1,\ p(n+2p+1)(n+p)^{-1}(n+p+1)^{-1}]/(n+1). \tag{2.4}$$

Proof. Let $\eta = n\xi/\sigma$. Then the risk function $R(\xi,\sigma;\delta)$ of any procedure (2.2) depends only on η, so that one can put $\sigma = 1$. For $\eta \geq 0$

$$\Delta_\delta(\eta) = R(\eta,\delta_0) - R(\eta,\delta)$$

$$= 4a\,[n^2\Gamma(n-1)]^{-1} \iint_{x>\eta,\,y>0} f(y/x)[x-\eta + ay(1-f(y/x)) - bn]$$

$$y^{n-1} \exp\{-(x+y-\eta)\}\,dxdy$$

$$= 4a[n^2\,\Gamma(n-1)]^{-1} e^{\eta} \int_0^\infty f(z)\, z^{n-1}\,(1+z)^{-n-2}$$

$$[(1+az(1-f(z)) \int_{\eta(1+z)}^\infty e^{-u}\,u^{n+1}\,du$$

$$- (bn+\eta)(1+z) \int_{\eta(1+z)}^\infty e^{-u}\,u^n\,du]\,dz. \tag{2.5}$$

Because of (2.3)

$$\Delta_\delta(\eta) \geq 4a\,[n^2\Gamma(n-1)]^{-1} e^{\eta} \int_0^\infty f(z) z^{n-1}\,(1+z)^{-n-2}$$

$$[(1+az(1-\bar{f}\,(1+z^{-1})^{-p}) \int_{\eta(1+z)}^\infty e^{-u}\,u^{n+1}\,du$$

$$- (bn+\eta)(1+z) \int_{\eta(1+z)}^\infty e^{-u}\,u^n\,du]dz.$$

To establish the inequality $\Delta_\delta(\eta) \geq 0$ or to prove the minimaxity of δ we use Lemma 1 from the Appendix which was used earlier in similar problems by Baranchik (1970) and Strawderman (1974).

To apply Lemma 1 write the integral in the preceding expression as $\int g(z)h(z)dz$ where $g(z) = f(z)\,(1+z^{-1})^p$. We need first to show that h changes sign at most once from negative to positive.

Denote $P_n(t) = 1 + t + \ldots + t^n/n! = e^t \int_t^\infty e^{-u} u^n \, du/n!$, and

$\ell(z) = h(z)\,(1+z)^{n+p-2} z^{1-n-p} e^{n(1+z)} / P_{n+1}(n(1+z))$.

It suffices to show that $\ell'(z) \geq 0$ if $\ell(z) > 0$. The latter inequality means that

$$(bn+n)\,(1+z)P_n(n(1+z))/[(n+1)P_{n+1}(n(1+z))] \leq 1 + az(1 - \bar{f}(1+z^{-1})^{-p}). \quad (2.6)$$

Since for all t, $P_{n-1}(t)\,P_{n+1}(t) \leq P_n^2(t)$,

one has $\ell\,(z) > 0$ if

$$(bn+n)\,P_n(n(1+z)/[(n+1)\,P_{n+1}(n(1+z))] \leq a[1-\bar{f}(1+z^{-1})^{-p}\,(1+p(1+z)^{-1})]. \quad (2.7)$$

Inequality (2.6) implies (2.7) if

$$1 \leq a[1-\bar{f}(p+1)\,(1+z^{-1})^{-p}]$$

or

$$a\bar{f}(p+1) \leq a-1. \quad (2.8)$$

This inequality is met because of (2.4) and Lemma 1 is applicable. Thus it remains to be shown that

$$\int_0^\infty h(z)dz \geq 0.$$

Using the easily verifiable formula

$$(m+1)^{-1}\int_{\eta}^{\infty} e^{-u} u^{n-m}(u-\eta)^{m+1}\, du$$

$$= \int_0^{\infty} z^m(1+z)^{-m-2} \int_{\eta(1+z)}^{\infty} e^{-u} u^{n+1}\, du\, dz$$

one sees that (2.8) means that

$$(a-1)\ (n+p+1)^{-1} \int_{\eta}^{\infty} e^{-u} u^{-p}(u-\eta)^{n+p+1}\, du$$

$$+ (n+p)^{-1}\int_{\eta}^{\infty} e^{-u} u^{-p+1}(u-\eta)^{n+p} du$$

$$- a\ \bar{f}(n+2p+1)^{-1} \int_{\eta}^{\infty} e^{-u} u^{-2p}(u-\eta)^{n+2p+1} du$$

$$-(an+t+1)(n+p)^{-1} \int_{\eta}^{\infty} e^{-u} u^{-p}(u-\eta)^{n+p}\, du \geq 0,$$

which follows from Corollary to Lemma 2 and (2.4).

Corollary 1. The estimator suggested by Rukhin and Strawderman (1982) with $f(z)=\max[0,a-1 - anz]/[(n+1)a]$ is minimax for $\xi \geq 0$ if

$$a \geq 1+n/(2(n+1)) + n(1/4+(n+1))^{1/2}/(n+1).$$

Indeed it is easy to check that $f(z)(1+z^{-1})^p$ is increasing if $p \leq an/(a-1)=p_0$. Also $a\bar{f} = (a-1)/(n+1)$, and
$\min[1, p_0(n+2p_0+1)(n+p_0)^{-1} (n+p_0+1)^{-1}] = 1$ if $p_0^2 - np_0 - n(n+1) \leq 0$. This inequality follows from the condition of Corollary 1

Corollary 2. Assume that for positive z, $af(z) = (a-1)v^m P(v)/Q(v)$, $v = z(1+z)^{-1}$, for a positive m, $m \leq n+1$ and some positive functions P and Q. The corresponding estimator (2.2) is minimax for $\xi \geq 0$ if

$$0 < p = m - \max_{0 \leq v \leq 1} v\,[Q'(v)/Q(v) - P'(v)/P(v)] \tag{2.8}$$

and

$$P(1)/Q(1) \leq 3p/[2(2n+1)(n+1)]. \tag{2.9}$$

Indeed condition (2.8) guarantees that $v^p f(z)$ is a nondecreasing function of z, and condition (2.9) implies (2.4).

3. Generalized Bayes Estimators of Exponential Quantiles.

Let $\lambda(\xi,\sigma)$ be the density of a (generalized) prior distribution over (ξ,σ) with respect to right Haar measure $d\xi\, d\sigma/\sigma$. Also let $\eta = n\xi/\sigma$ and $t = n\sigma^{-1}$. We shall denote by $\lambda(\eta,t)$ the density corresponding to this transformation. The Bayes estimator $\delta_B(x,y)$ has the form

$$\delta_B(x,y) = \iint [\xi + b\sigma]\sigma^{-4}\, p((x-\xi)/\sigma, y/\sigma)\lambda(\xi,\sigma)d\xi d\sigma/\sigma$$

$$/\iint \sigma^{-4}\, p((x-\xi)/\sigma,\, y/\sigma)\ \lambda(\xi,\sigma)\ d\xi\ d\sigma/\sigma$$

$$= \iint_{\eta<tx} (\eta + bn)\ e^{-t(x+y)+\eta}\ \lambda(\eta,t)\ t^{n-1}\ dt\ d\eta$$

$$/\iint_{\eta<tx} e^{-t(x+y)+\eta}\ \lambda(\eta,t)\ t^{n}\ dt\ d\eta. \tag{3.1}$$

Let $K(u,t) = e^{-u}\int_{-\infty}^{u} e^{\eta}\ \lambda(\eta,t)\ d\eta$. Then $\int_{-\infty}^{tx} (\eta-tx)\ e^{\eta}\ \lambda(\eta,t)d\eta = -\int_{-\infty}^{tx} e^{u}K(u,t)du$,

so that

$$\delta_B(x,y) = x + \int_0^{\infty} e^{-ty}\ t^{\,n-1}[bnK(tx,t) - e^{-tx}\int_{-\infty}^{tx} e^{u}\ K(u,t)du]dt$$

$$/\int_0^{\infty} e^{-ty}\ t^{n}\ K(tx,t)\ dt. \tag{3.2}$$

Of special interest to us will be prior densities of the form

$$\lambda(\eta,t) = \ell(\eta)\ t^{\alpha}\ ,\ t > 0\ . \tag{3.3}$$

In this case

$$K(u,t) = e^{-u}\int_{-\infty}^{u} e^{\eta}\ \ell(\eta)d\eta\ t^{\alpha} = K(u)t^{\alpha},$$

and with $z = g/x$

$$\delta_B(x,y) = x+y \int_0^\infty e^{-t}\, t^{n+\alpha-1}[bnK(t/z) - e^{-t/z} \int_{-\infty}^{t/z} e^u K(u)\, du]\, dt$$

$$/ \int_0^\infty e^{-t}\, t^{n+\alpha}\, K(t/z)\, dt.$$

Thus the generalized Bayes estimator has the form (2.2) with

$$f(z) = \int_0^\infty e^{-tz}\, t^{n+\alpha-1}\, [atz\, K(t) - (an+1)\, K(t) + e^{-t} \int_{-\infty}^{t} e^u\, K(u)\, du]\, dt$$

$$/ [2az \int_0^\infty e^{-tz}\, t^{n+\alpha}\, K(t)dt].$$

An alternative formula for $f(z)$ is derived by integration by parts

$$f(z) = \int_0^\infty e^{-tz}\, t^{n+\alpha-1}[a\alpha\, K(t) + atK'(t) - e^{-t} \int_{-\infty}^{t} e^u\, K'(u)\, du]dt$$

$$/[2az \int_0^\infty e^{-tz}\, t^{n+\alpha}\, K(t)dt]. \tag{3.4}$$

Clearly the best equivariant estimator $\delta_o(x,y) = x + ay$ corresponds to the choice $\ell \equiv 1$, $\alpha = 0$, $K(u) \equiv 1$. However there are many other prior densities for which δ_o is the generalized Bayes procedure.

It follows from (3.2) that $\delta_B(x,y) = x + ay$ if and only if

$$\int_0^\infty e^{-ty}\, t^{n-1}\, [bn\, K(tx,t) - e^{-tx} \int_{-\infty}^{tx} e^u\, K(u,t)du]dt$$

$$= ay \int_0^\infty e^{-ty}\, t^n\, K(tx,t)dt$$

$$= a \int_0^\infty e^{-ty}\, t^{n-1}[nK(tx,t) + txK'_u\, (tx,t) + tK'_t\, (tx,t)]dt\ .$$

This identity holds for all positive y if and only if

$$K(u,t) - e^{-u} \int_{-\infty}^{u} e^s\, K(s,t)ds$$

$$= a[u\, K'_u + t\, K'_t]\ . \tag{3.5}$$

It is easy to see that with $d = 1/a$, $K(u,t) = e^{-u}\, u^{d-1}\, t$, $u > 0$ $K(u,t) = 0$, $u < 0$ solves (3.5) and in this case $\delta_B(x,y) = x + ay$ for $x > 0$.

Moreover if $d > 1$, δ_0 is generalized Bayes also with respect to the prior distribution with density

$$\lambda(\eta,t) = (d-1)\, e^{-\eta}\, \eta^{d-2}t\ ,\ \eta > 0,\ \lambda(\eta,t) = 0,\ \eta < 0.$$

We formulate the results as

Theorem 2. The generalized Bayes estimator δ_B of an exponential quantile $\xi + b\sigma$ corresponding to a prior density $\lambda(\eta,t)$, $\eta = n\xi/\sigma$, $t = n/\sigma$ has the form

(3.2). If $d = 1/a > 1$ and $\lambda(\eta,t) = e^{-\eta}\,\eta^{d-2}\,t,\ \eta > 0;\ \lambda(\eta,t) = 0,\ \eta < 0,$ then for $x > 0$, $\delta_B(x,y) = x + ay$.

In the remainder of this Section we consider the generalized prior density (3.3) with

$$\ell_\alpha(u) = \ell(u) = (a-1)\int_0^1 e^{-us}\, s^{\alpha-1}\, R(s)ds \quad , u > 0 , \tag{3.6}$$

where $R(s) = \sum_{k=0}^{n} r_k\, s^k\, (1-s)^{n-k}$. Then

$$K(u) = w\, e^{-u} + (a-1)\int_0^1 [e^{-us} - e^{-u}]\, s^{\alpha-1}\, R(s)\, (1-s)^{-1}\, ds,\ u > 0,\ w = \int_{-\infty}^{0} e^t \ell(t)dt,$$

and the form of the generalized Bayes estimator can be derived from (3.4) and Lemma 3 of Appendix. We define coefficients r_k so that the polynomial terms of degree less than n in (A.3) vanish,

$ak\, r_k = r_{k-1}\,(a(n-k)+1) \quad k = 1,\ldots,n, r_o = 1$, and we choose constant

$\ell = \int_{-\infty}^{0} e^t\, K'(t)dt = w + \int_{-\infty}^{0} te^t\, \ell(t)dt$, so that the polynomial term of degree n vanishes as well:

$$\ell = wa\alpha - (a-1)\,(w-r_n)(n+\alpha) + (a-1)^2\,(r_n + r_{n-1})$$

$$+ (a-1)(n(a-1) + 1-\alpha)\sum_{k=0}^{n-1} r_k\, B(\alpha + k,\ n - k)\ ,$$

$$r_k = \prod_{i=1}^{k} (n - i + a^{-1})/k!\ ,\ k = 1,\ldots,n, r_0 = 1, \tag{3.7}$$

Let $Q(v) = \int_0^\infty t^{n+\alpha} e^{-t}\, K(t/z)dt\, [v^\alpha\, \Gamma\,(n+\alpha+1)]^{-1}$,

$$\varepsilon = 1 - [r_n(1+(a-1)(n+\alpha)^{-1}) + \sum_{k=0}^{n-1} r_k\, B(\alpha+k,n-k)]/w$$
$$= 1 - \gamma/w.$$

Theorem 3. Let the generalized prior density $\lambda(\eta,t)$ be of the form (3.3) with $\ell(\eta)$ given by (3.6) and (3.7).

Then the corresponding generalized Bayes estimator δ_α has the form (2.2) with

$$2af(z) = (a-1)v^{n+1}\ (\varepsilon\ w - (a-1)r_n \log v)/Q(v), \tag{3.8}$$

$v = z/(1+z)$.

Furthermore, δ_α is minimax for $\xi \geq 0$ if

$$3(a-1)(r_n - (n+\alpha+1)\sum_{k=0}^{n} r_k\, B(\alpha+k,\ n-k+1))^2 \geq 8(2n+1)(n+1)\ \gamma\ r_n$$

and w is defined by (3.13).

Proof. Formula (3.8) follows directly from (3.4) and Lemma 3 of Appendix, and the minimaxity result follows from Corollary 2 to Theorem 1. Indeed

$$Q(v) = \int_0^\infty t^{n+\alpha}\, e^{-t}\, K(t/z)dt\ [v^\alpha\ \Gamma\ (n+\alpha+1)]^{-1},$$

$$Q(1) = K(0) = w$$

and

$$Q'(1) = -\alpha w - (n+\alpha+1)\ K'(0)$$

$$= (n+1)w - (a-1)(n+\alpha+1)\int_0^1 s^{\alpha-1}\, R(s)ds$$

$$= (n+1)w - (a-1)q.$$

Also

$$v\, Q'(v)/Q(v) = [n+1+\alpha(1-v)]/(1-v)$$

$$- v(1-v)^{-2} \int_0^\infty t^{n+\alpha+1}\, e^{-tz}\, K(t)dt / \int_0^\infty t^{n+\alpha}\, e^{-tz}\, K(t)dt\ . \qquad (3.9)$$

which is easily seen to be non decreasing in v. Therefore

$$\max_{0 \leq v \leq 1}[v\, Q'(v)/Q(v) + (a-1)\, r_n/(\varepsilon w-(a-1)r_n \log v)]$$

$$= Q'(1)/Q(1) + (a-1)\, r_n/(\varepsilon w).$$

Because of Corollary 2 the corresponding estimator is minimax for $\xi \geq 0$ if

$$0 < p = n + 1 - Q'(1)/Q(1) - (a-1)\, r_n/(\varepsilon w)$$

$$= (a-1)\ [q - r_n/\varepsilon]/w \qquad (3.10)$$

and

$$\varepsilon \leq 3p/[2(2n+1)(n+1)]. \qquad (3.11)$$

Inequality (3.10) means that

$$r_n/q < \varepsilon\ ,$$

and inequality (3.11) means that

$$\varepsilon \leq 3(a-1)(q-r_n/\varepsilon)/[2(2n+1)(n+1)w]$$

$$= 3(a-1)(q-r_n/\varepsilon)(1-\varepsilon)/[2(2n+1)(n+1)\gamma].$$

These two inequalities hold for some ε if

$$\varepsilon^2[2(2n+1)(n+1)\gamma+3(a-1)q] - 3\varepsilon(a-1)(r_n+q)+3(a-1)r_n \leq 0$$

which happens if

$$3(a-1)(r_n-q)^2 \geqslant 8(2n+1)(n+1)\gamma r_n \ . \qquad (3.12)$$

Recall that here

$$q=(n+\alpha+1) \sum_{k=0}^{n} r_k \, B(\alpha+k,\ n-k+1)$$

and

$$\gamma = r_n(1+(a-1)(n+\alpha)^{-1}) + \sum_{k=0}^{n-1} r_k \, B(\alpha+k,\ n-k).$$

If (3.12) is satisfied then the value of $\varepsilon = 1 - \gamma/w$ (or w) in (3.12) generating a minimax estimator is found from the formula

$$\varepsilon=3(a-1)(r_n+q)[2(2(2n+1)(n+1)\gamma+3(a-1)q)]^{-1} \ . \qquad (3.13)$$

Remark. It is easy to see that condition (3.12) is met for sufficiently small positive α.

4. Admissibility Results

We start here with the following result.

Theorem 4. If $0 \leq a \leq 1$ then $\delta_o(x,y) = x+ay$ is an admissible estimator of exponential quantile $\xi+b\sigma$.

Proof. If δ_o were inadmissible then there would exist δ such that for all ξ and σ

$$R(\xi,\sigma;\delta) \leq E(\xi,\sigma;\delta_o)$$

with strict inequality for some ξ_o, σ_o. Notice that we can assume that $\xi_o > 0$. Indeed if $\delta_c(x,y) = \delta(x+c,y)-c$, then

$$R(\xi,\sigma;\delta_c) = R(\xi+c,\sigma;\delta).$$

Thus δ_c also improves upon δ_o and is strictly better than δ at $\xi_o + c$, which is positive for large c.

Because of the continuity of the risk functions to prove the admissibility of δ_o it suffices to find a sequence of positive densities $\lambda_m(\xi,\sigma)$, $\xi > 0$, such that

$$\iint \lambda_m(\xi,\sigma)d\xi\ d\sigma/\sigma < \infty$$

and as $m \to \infty$

$$r_m = \iint [R(\xi,\sigma,\delta_o) - R(\xi,\sigma,\delta_m)]\lambda_m(\xi,\sigma)d\xi\ d\sigma/\sigma \to 0$$

where δ_m is Bayes estimator with respect to λ_m.

A straightforward calculation shows that with p(x,y) defined by (2.1)

$$r_m = \iint (\delta_o(x,y) - \delta_m(x,y))^2\ dxdy$$

$$\iint \sigma^{-4} p((x-\xi)/\sigma, y/\sigma)\lambda_m(\xi,\sigma)d\xi\, d\sigma/\sigma$$

$$= n^{-2}[\Gamma(n-1)]^{-1} \underset{0<\eta<tx}{\iiiint} [\delta_o(x,y)-\delta_m(x,y)]^2 e^{-(x+y)t+\eta}$$

$$x\, y^{n-2}\, t^n \lambda_m(\eta,t)dtd\eta\, dxdy .$$

Here we used our previous notation $\eta = n\xi/\sigma$, $t = n/\sigma$. The traditional condition would be $\lambda_m(\xi,\sigma) \to 1$ but in this case the desired limiting relation is false. Making use of Theorem 2 we put in the case $0 < a < 1$

$\lambda_m(\eta,t) = e^{-\eta}\, \eta^{d-2}\, t\, h_m(t)$, $\eta > 0$, $d = 1/a$, where h_m are positive differentiable functions, $\int_0^\infty h_m(t)\, t^{-1}\, dt < \infty$, $h_m(t) \to 1$ as $m \to \infty$.

One obtains from (3.2)

$$\delta_m(x,y) - x - ay = a \int_0^\infty e^{-t(z+1)}\, t^{n+d}\, h_m'(t/x)dt/\int_0^\infty e^{-t(z+1)}\, t^{n+d}\, h_m(t/x)dt,$$

Thus with a generic constant C independent of m

$$r_m = C\iint [\int_0^\infty e^{-t(z+1)}\, t^{n+d}\, h_m'(t/x)dt]^2\, y^{n-2}\, x^{-n-2}$$

$$[\int_0^\infty e^{-t(z+1)}\, t^{n+d}\, h_m(t/x)dt]^{-1}\, dy\, dx$$

$$\leq C\iint y^{n-2}\, x^{-n-2}\, dy\, dx \int_0^\infty e^{-t(z+1)}\, t^{n+d}\, [h_m'(t/x)]^2/h_m(t/x)\, dt$$

$$= C\int_0^\infty v[h_m'(v)]^2/h_m(v)dv.$$

Now we can specify the choice of functions h_m by putting

$$h_m(v) = [1+(\log v)^2/m]^{-1}.$$

Then

$$\int_0^\infty h_m(v)\ dv/v = \int_{-\infty}^{\infty} [1 + u^2/m]^{-1}\ du < \infty$$

and

$$\int_0^\infty v[h_m'(v)]^2/h_m(v)\ dv$$

$$= 4\ m^{-1/2} \int_0^\infty t^2\ (1+t^2)^{-3}\ dt \to 0 \quad \text{as } m \to \infty.$$

Thus if $0 < a < 1$, $r_m \to 0$ and Theorem 4 is proven. In the case $a = 0$ δ_o is generalized Bayes with respect to any density of the form $\lambda(\eta,t) = \lambda(t)$, and in the case $a = 1$ with respect to any density of the form $\lambda(\eta,t) = \lambda(\eta/t)$. Slight modifications of Blyth's (1951) admissibility proof prove our Theorem in these cases.

Theorem 5. For $\alpha = 1$ the estimator δ_α defined by (3.8) is admissible for $\xi > 0$ within the class of all scale-equivariant procedures.

Proof. It suffices to show that there exists a sequence of positive integrable functions $\ell_m(\eta)$, $\eta > 0$ such that

$$r_m = \int_0^\infty [R(\eta,\delta_\alpha) - R(\eta,\delta_m)]\ \ell_m(\eta)d\eta \to 0\ , \tag{4.1}$$

where δ_m is the estimator which minimizes $\int_0^\infty R(\eta,\delta)\ell_m(\eta)d\eta$. It is easy to see that

$$\delta_m(x,y) = y \iint_{\eta<zv} (\eta+bn)v^n e^{-v(1+z)+\eta} \ell_m(\eta)d\eta\, dv$$

$$/\iint_{\eta<zv} v^{n+1} e^{-v(1+z)+\eta} \ell_m(\eta)d\eta\, dv, \quad z = x/y .$$

Comparing this formula with (3.1) we notice that δ_m is the generalized Bayes estimator corresponding to the prior density $\lambda(\eta,t) = \ell_m(\eta)t$. If one defines $\ell_m(\eta)$ by formula (3.6) for $\alpha = 1 + m^{-1}$, then

$$\int_0^\infty \ell_m(\eta)d\eta = (a-1) \int_0^\infty s^{1/m-1} R(s)ds < \infty ,$$

and using a calculation similar to that done in Lemma 3 and Theorem 4 one can prove (4.1). We do not give here details since the conclusion of Theorem 5 can also be obtained by extension of Zidek's result (1973) to the case of exponential quantile estimation. According to this result the generalized density $\ell(\eta)\, t$ generates an admissible (for $\xi \geqslant 0$) procedure within the class of all equivariant estimators if

$$\int_1^\infty [\eta^2\, \ell(\eta)]^{-1}\, d\eta = \infty \quad .$$

In our case $\ell(\eta) = (a-1) \int_0^\infty e^{-\eta s} R(s)ds \sim (a-1)r_n/\eta$ as $\eta \to \infty$, so that the integral above diverges.

5. Open Problems.

An interesting unsolved problem is the question of admissibility of the estimator (3.8). Even its admissibility in the class of all scale-equivariant procedures (2.2) for arbitrary positive α seems to be difficult to establish. Indeed the relationship between Bayes rules corresponding to a prior density (3.3) and the rules which minimize $\int R(\eta,\delta)\ \ell(\eta)d\eta$ for $\alpha \neq 1$ is not evident.

Another intriguing question is the largest amount of relative improvement $r=\Delta_\delta(\eta)/R(\eta,\delta_o)$ for minimax estimators δ. Because of (2.5) for $\eta \geq 0$

$$r = 4a\iint_{zy>\eta} e^{-y(1+z)+\eta}\ f(z)[zy-\eta + ay(1-f(z))-bn]$$

$$\times\ y^n\ dzdy\ /\ [n^2\ \Gamma(n-1)(n^{-2} + a^2n^{-1})],$$

and for a fixed value $\eta = \eta_o$ the function f_o which maximizes this quantity in the class (2.2) has the form

$$2af_o(z) = \max\{0, a+z - \int_{\eta_o/z}^{\infty} (b_n+\eta_o)y^n\ e^{-y(1+z)}dy$$

$$/ \int_{\eta_o/z}^{\infty} y^{n+1}\ e^{-y(1+z)}\ dy\}.$$

In this case

$$r = 4a^2\ e^\eta \iint_{zy>\eta} f_o^2\ (z)\ y^{n+1}\ e^{-y(1+z)}\ dy\ dz\ /\ [n^2\Gamma(n-1)(n^{-2} + a^2\ n^{-1})].$$

For instance, when $\eta_o = 0, 2af_o(z) = (a-1)\max\{0,(1-nz)/(n+1)\}$, and

$$r = (a-1)^2\ \Gamma(n+2) \int_0^{n^{-1}} (1-nz)^2(1+z)^{-n-2}\ dz\ /\ [n^2(n+1)^2\ \Gamma(n-1)(n^{-2} + a^2\ n^{-1})]$$

$$= (a-1)^2[1-2(n/n+1))^n]/((n+1)(a^2+n^{-1}))\ .$$

The choice of generalized priors leading to minimax estimators of location -scale exponential parameters also remains largely an open problem. It is not difficult to show that prior densities (3.3) produce tail-minimax estimators only if $\ell(\eta) \sim C/|\eta|^{\alpha}$ as $|\eta| \to \infty$ (in which case $f(\infty) = 0$). One may conjecture that the prior densities corresponding to minimax Bayes estimators of the scale parameter σ also generate minimax estimators of quantiles $\xi + b\sigma$. However this conjecture is false. In fact, Brewster (1974) has found a minimax estimator of σ for quadratic loss. This estimator coincides for positive x with the generalized Bayes estimator for the prior density $\lambda(\xi,\sigma) = 1$, $\xi > 0$; $= 0$, $\xi < 0$. The resulting quantile estimator has the form (2.2) with

$$f(z) = (a-1)z^{n}[a((1+z)^{n} - z^{n})]^{-1} \quad , \; z > 0 \; ,$$

and it is not minimax for $\xi > 0$.

Appendix

We give here three technical Lemmas needed to prove Theorem 1 and Theorem 3.

Lemma 1. Let X be a random variable taking values in an interval I, and let g be a non-decreasing nonnegative function defined over I. Assume that h is a function which changes sign at most once from negative on I and such that $E\,h(X) \geq 0$. Then

$$E\,g(X)h(X) \geq 0.$$

Proof. Let $c = \sup\{x\colon h(x) \leq 0\}$. Then $(g(x)-g(c))h(x)$ is nonnegative for all x from I and

$$E\,g(X)h(X) \geq g(c)Eh(X) \geq 0.$$

Lemma 2. For all positive p and t

$$\int_t^\infty e^{-u} u^{-p}(u-t)^{n+p+1}\,du$$

$$- n(n+p+1)(n+p)^{-1} \int_t^\infty e^{-u} u^{-p}(u-t)^{n+p}\,du$$

$$\geq p[(n+1)(n+p)]^{-1} \int_t^\infty e^{-u} u^{-2p} (u-t)^{n+2p+1}\,du \tag{A.1}$$

and

$$(n+1+t) \int_t^\infty e^{-u} u^{-p} (u-t)^{n+p}\,du$$

$$\leq \int_t^\infty e^{-u} u^{-p+1} (u-t)^{n+p}\,du. \tag{A.2}$$

Proof. Inequality (A.1) means that $\int_0^\infty e^{-u} (u+t)^{-p} u^{n+p} [u-n(n+p+1)(n+p)^{-1}$

$- p[(n+1)(n+p)]^{-1} u^{p+1} (u+t)^{-p}] \, du \geq 0.$

It is easy to check that for a fixed t the function

$h(u) = u - n(n+p+1)(n+p)^{-1} - p[(n+1)(n+p)]^{-1} u^{p+1}(u+t)^{-p}$ changes sign only once for $u \geq 0$. Since the function $g(u) = u^p(u+t)^{-p}$ is increasing, (A.1) will follow from Lemma 1 if we show that

$$\int_0^\infty e^{-u} u^n[u - n(n+p+1)(n+p)^{-1} - p[(n+1)(n+p)]^{-1} u^{p+1} (u+t)^{-p}]du \geq 0.$$

The latter inequality is equivalent to an evident one:

$$\int_0^\infty e^{-u} u^{n+p+1} (u+t)^{-p} du \leq (n+1)! = \int_0^\infty e^{-u} u^{n+1} du.$$

Inequality (A.2) also follows from Lemma 1 with the same function g and $h(u) = u - n - 1$.

Corollary. For all positive p and t

$$(a-1)p[n+p+1)(n+1)(n+p)]^{-1} \int_t^\infty e^{-u} u^{-2p} (u-t)^{n+2p+1} du$$

$$\leq (a-1) (n+p+1)^{-1} \int_t^\infty e^{-u} u^{-p} (u-t)^{n+p+1} du$$

$$+ (n+p)^{-1} \int_t^\infty e^{-u} u^{-p+1} (u-t)^{n+p} du$$

$$- (an+1+t)(n+p)^{-1} \int_t^\infty e^{-u} u^{-p} (u-t)^{n+p} du.$$

Lemma 3. Let for $t \geq 0$

$$K(t) = we^{-t}+(a-1) \int_0^1 [e^{-ts} - e^{-t}] \; s^{\alpha-1} \; R(s) \; (1-s)^{-1} \, ds \; ,$$

where $\quad R(s) = \sum_{k=0}^{n} r_k \, s^k \, (1-s)^{n-k}$.

Then with $v = z/(1+z)$, $\ell = \int_{-\infty}^{0} e^t \, K'(t)dt$,

$$\int_0^\infty t^{n+\alpha-1} \, e^{-tz}[a \; \alpha \; K(t)+a \; t \; K'(t)-e^{-t} \int_{-\infty}^{t} e^s \, K'(s)ds]dt$$

$$= \Gamma(n+\alpha+1)z^{-n}(1+z)^{-\alpha}$$

$$\times \{v^{n+1} \; (a-1)[w-r_n - (a-1) \; r_n(n+\alpha)^{-1}$$

$$- (a-1) \sum_{k=0}^{n-1} r_k B(\alpha+k, \; n-k)]$$

$$+ v^n[(wa\alpha - \ell) \; (n+\alpha)^{-1} - (a-1) \; (w-r_n)$$

$$+ (a-1)^2 \; (n+\alpha)^{-1} \; (r_n + r_{n-1})$$

$$+ (a-1) \; (n(a-1) + 1-\alpha)(n+\alpha)^{-1} \sum_{k=0}^{n-1} r_k B(\alpha+k, \; n-k)]$$

$$- v^{n+1} \log v \; (a-1)^2 \; r_n - v^n \log v \; (a-1)(r_{n-1} - an \; r_n)(n+\alpha)^{-1}$$

$$+ (a-1) \sum_{k=1}^{n-1} v^k \; B(\alpha+k, \; n-k) \; (n+\alpha)^{-1} \; [r_{k-1}(a(n-k)+1) - ak \; r_k]\} \qquad (A.3)$$

and

$$\int_0^\infty t^{n+\alpha} e^{-tz} K(t)dt = \Gamma(n+\alpha+1)z^{-n-1} (1+z)^{-\alpha}$$

$$x \{-v^{n+1} \log v \ (a-1)\ r_n$$

$$+ v^{n+1}(w-(a-1)r_n \ (n+\alpha)^{-1} - (a-1) \sum_{k=0}^{n-1} r_k \ B(\alpha+k,\ n-k))$$

$$+ (a-1) \sum_{k=0}^{n} v^k \ B(\alpha+k,\ n-k+1)\ (r_k + r_{k-1})\}, r_{-1} = 0.$$

Proof. Using integration by parts one obtains

$$H(t) = a\,\alpha\,K(t) + a\,t\,K'(t) - e^{-t}\int_{-\infty}^{t} e^{s}\,K''(s)ds$$

$$= [w\,a\,\alpha - \ell]\,e^{-t} + [w - R(1)]\,(a-1)\,t\,e^{-t}$$

$$+ [(a\alpha - 1)(a-1) - (a-1)t]\int_0^1 [e^{-st} - e^{-t}]\,T(s)\,(1-s)^{-1}\,ds$$

$$- (a-1)\int_0^1 [e^{-st} - e^{-t}]\,T'(s)ds$$

$$+ a(a-1)\,t\int_0^1 e^{-st}\,T(s)\,ds,$$

where $T(s) = s^{\alpha-1}R(s)$. A straightforward calculation shows that

$$\int_0^\infty t^{n+\alpha-1}\,e^{-tz}\,H(t)dt = \Gamma(n+\alpha+1)$$

$$\times\{(w\,a\,\alpha-\ell)\,(1+z)^{-n-\alpha}\,(n+\alpha)^{-1} + (a-1)\,[R(1) - w]\,(1+z)^{-n-\alpha-1}$$

$$+ a(a-1)\int_0^1 T(s)\,(s+z)^{-n-\alpha-1}\,ds$$

$$+ (a\,\alpha-1)(a-1)(n+\alpha)^{-1}\int_0^1 T(s)[(s+z)^{-n-\alpha} - (1+z)^{-n-\alpha}](1-s)^{-1}\,ds$$

$$- (a-1)^2\int_0^1 T(s)[(s+z)^{-n-\alpha-1} - (1+z)^{-n-\alpha-1}]\,(1-s)^{-1}\,ds$$

$$- (a-1)(n+\alpha)^{-1}\int_0^1 T'(s)[(s+z)^{-n-\alpha} - (1+z)^{-n-\alpha}]\,(1-s)^{-1}\,ds.$$

By using integral representation of beta function one obtains for $k < n$

$$\int_0^1 s^{\alpha+k-1} (1-s)^{n-k-1}[(s+z)^{-n-\alpha} - (1+z)^{-n-\alpha}]ds$$

$$= B(\alpha+k, n-k)[z^{k-n} (1+z)^{-\alpha-k} - (1+z)^{-\alpha-n}]$$

and by letting k tend to n one obtains

$$\int_0^1 s^{\alpha+n-1}[(s+z)^{-n-\alpha} - (1+z)^{-n-\alpha}] (1-s)^{-1} ds$$

$$= [\log(1+z)-\log z] (1+z)^{-\alpha-n} .$$

Also for $k < n$, $\int_0^1 s^{\alpha+k-1} (1-s)^{-n-k-1}[(s+z)^{-n-\alpha-1} - (1+z)^{-n-\alpha-1}]ds$

$$= B(\alpha+k+1, n-k) z^{k-n} (1+z)^{-\alpha-k-1}$$

$$+ B(\alpha+k, n-k+1) z^{k-n-1} (1+z)^{-\alpha-k}$$

$$-B(\alpha+k, n-k) (1+z)^{-\alpha-n-1}$$

and

$$\int_0^1 s^{\alpha+n-1}[(s+z)^{-n-\alpha-1} - (1+z)^{-n-\alpha-1}](1-s)^{-1}\,ds$$

$$= \{\log(1+z) - \log z + [(n+\alpha)z]^{-1}\}\ (1+z)^{-\alpha-n-1}.$$

Using these formulas we deduce that

$$\int_0^\infty t^{n+\alpha-1}\, e^{-tz}\, H(t)dt = \Gamma(n+\alpha+1)$$

$$\times \{(wa\alpha-\ell)(n+\alpha)^{-1}\ (1+z)^{-n-\alpha} + (a-1)(r_n-w)(1+z)^{-n-\alpha-1}$$

$$+ a(a-1)\sum_{k=0}^{n} r_k\, B(\alpha+k,\ n-k+1)\ z^{k-n-1}\ (1+z)^{-\alpha-k}$$

$$+(a\alpha-1)(a-1)(n+\alpha)^{-1}[r_n(\log(1+z)-\log z)(1+z)^{-\alpha-n}$$

$$+\sum_{k=0}^{n-1} r_k\, B(\alpha+k,\ n-k)(z^{k-n}\ (1-z)^{-\alpha-k} - (1+z)^{-\alpha-n})]$$

$$- (a-1)^2[r_n(\log\ (1+z)-\log\ z +((n+\alpha)z)^{-1})(1+z)^{-\alpha-n-1}$$

$$+\sum_{k=0}^{n-1} r_k(B(\alpha+k+1,\ n-k)z^{k-n}\ (1+z)^{-\alpha-k-1}$$

$$+ B(\alpha+k,\ n-k+1)z^{k-n-1}\ (1+z)^{-\alpha-k}$$

$$- B(\alpha+k,\ n-k)(1+z)^{-\alpha-n-1})]$$

$$-(a-1)(n+\alpha)^{-1}[(r_n(n+\alpha-1)-r_{n-1})$$

$$\times\ (\log(1+z)-\log z+((n+\alpha-1)z)^{-1})(1+z)^{-\alpha-n}$$

$$+\sum_{k=0}^{n-1} r_k(\alpha+k-1)(B(\alpha+k,\ n-k)z^{k-n}\ (1+z)^{-\alpha-k}$$

$$+ B(\alpha+k-1,\ n-k+1)z^{k-n-1}\ (1+z)^{-\alpha-k+1}$$

$$- B(\alpha+k-1,\ n-k+1)(1+z)^{-\alpha-n})$$

$$-\sum_{k=0}^{n-2} r_k(n-k)(B(\alpha+k+1,\ n-k-1)z^{k-n+1}\ (1+z)^{-\alpha-k-1}$$

$$+ B(\alpha+k,\ n-k)\ z^{k-n}\ (1+z)^{-\alpha-k}$$

$$- B(\alpha+k,\ n-k)(1+z)^{-n-\alpha})]\}.$$

The first formula of Lemma 3 follows now after some calculation, and the second one is proved analogously. It follows also that

$$\begin{aligned} Q(v) &= \int_0^\infty t^{n+\alpha}e^{-tk(t/z)}dt[v^\alpha\Gamma(n+\alpha+1]^{-1} \\ &= -(a-1)r_n v^{n+1}\log v \\ &\quad + [w-(a-1)(n+\alpha)^{-1}\ r_n - (a-1)\sum_{k=0}^{n-1} r_k\ B(\alpha+k,\ n-k)]v^{n+1} \\ &\quad + (a-1)\sum_{k=0}^{n} B(\alpha+k,\ n-k+1)(r_k+r_{k-1})v^k, \qquad r_{-1}=0. \end{aligned}$$

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