BROWNIAN SLOW POINTS: THE CRITICAL CASE[1]

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Summary

It is known that if B_t is a standard Wiener process then $\sup_t \liminf_{h\to 0+}(B_{t+h}-B_t)h^{-\frac{1}{2}} = 1$ a.s. Here this is sharpened to

$$P(\exists t: \lim\inf_{h\to 0+}(B_{t+h}-B_t)h^{-\frac{1}{2}} = 1) = 1, \text{ and}$$

$$P(\exists t: B_{t+h}-B_t \geq h^{\frac{1}{2}} \ \forall h \in (0,\alpha) \text{ for some } \alpha > 0) = 0.$$

A number of other theorems of the same flavor are proved. Our results for the critical case for slow points are not as complete as the above.

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1. Introduction

If B_t is a one-dimensional Brownian motion defined on a complete probability space, consider the sets of one-sided and two-sided slow points defined by

$$S^{\pm}(c) = \{(t,\omega) \mid t \geq 0, \ \exists\Delta > 0 \ni |B(t \pm h)-B(t)| \leq c\sqrt{h} \quad \forall 0 \leq h \leq \Delta \}$$

$$S(c) = S^{+}(c) \cap S^{-}(c) \qquad (c > 0).$$

In Davis [5] and Greenwood and Perkins [7] it was shown that

$$(1.1) \qquad S^{+}(c)(\omega) \begin{cases} = \emptyset & \text{if } c < 1 \quad \text{a.s.} \\ \neq \emptyset & \text{if } c > 1 \end{cases}$$

(Here $A(\omega)$ denotes the ω-section of $A \subset [0,\infty)\times\Omega$.) A related result concerns the times of rapid increase of B, defined by

$$I^{+}(c) = \{(t,\omega) \mid t \geq 0, \exists\Delta > 0 \ni B(t+h)-B(t) \geq c\sqrt{h} \quad \forall 0 \leq h \leq \Delta\} (c \in \mathbb{R}).$$

In [5] one of us (Davis) shows that

$$(1.2) \qquad I^{+}(c)(\omega) \begin{cases} = \emptyset & \text{if } c > 1 \quad \text{a.s.,} \\ \neq \emptyset & \text{if } c < 1 \end{cases}$$

thus answering part of a question of Knight [8, p. 148]. In this work we attempt to refine results like (1.1) and (1.2) by studying the critical cases, c = 1. One of our theorems settles the rest of Knight's question. A number of the theorems of [5] and [7] were unified and extended in Perkins [9], where it was shown that for certain random sets $A \subset [0,\infty)\times\Omega$ (including all sets of the form $A_1\times\Omega$ where A_1 is Borel measurable),

$$(1.3) \qquad \dim(A \cap S^+(c)(\omega)) = \dim(A(\omega)) - \lambda_0(c) \quad \text{a.s.}$$

and in particular

$$(1.4) \qquad A \cap S^+(c)(\omega) \begin{cases} \neq \emptyset & \text{if } \lambda_0(c) < \dim(A(\omega)) \quad \text{a.s.} \\ = \emptyset & \text{if } \lambda_0(c) > \dim(A(\omega)) \end{cases}$$

Here dim denotes Hausdorff dimension and $\lambda_0(c)$ may be described in terms of the zeros of certain special functions. The reader is referred to Proposition 1 of [9] for a precise description of those constants. ($\lambda_0(c) = \lambda_0(-c,c)$ in the notation of [9]). For our purposes, it suffices to know that λ_0 is a continuous, strictly monotone function of c whose values are easy to approximate. If

$$Z = \{(t,\omega) \mid B(t,\omega) = 0\},$$

then (1.4) yielded (Corollaries 6 and 8 in [9])

$$(1.5) \qquad Z \cap S^+(c)(\omega) \begin{cases} = \emptyset & \text{if } c < \lambda_0^{-1}(1/2) \approx 1.3069 \quad \text{a.s.} \\ \neq \emptyset & \text{if } c > \lambda_0^{-1}(1/2) \end{cases}$$

$$(1.6) \qquad S(c)(\omega) \begin{cases} = \emptyset & \text{if } c < \lambda_0^{-1}(1/2) \qquad \text{a.s.} \\ \neq \emptyset & \text{if } c > \lambda_0^{-1}(1/2) \end{cases}$$

Originally, (1.5) appeared in [7], and the first half of (1.6) appeared in [5]. In examining (1.1) - (1.6) in the appropriate critical cases one can consider the sets $S^+(c)$ and $I^+(c)$ or the slightly larger sets

$$S^+(c^+) = \{(t,\omega) \mid \overline{\lim_{h \to 0^+}} |B_{t+h} - B_t| h^{-\frac{1}{2}} \leq c\} = \bigcap_{d>c} S^+(d)$$

$$I^+(c^-) = \{(t,\omega) \mid \underline{\lim_{h \to 0^+}} (B_{t+h} - B_t) h^{-\frac{1}{2}} \geq c\} = \bigcap_{d<c} I^+(d).$$

Leaving aside the general results (1.3) and (1.4) for the moment, the discerning reader will be able to pose at least 8 questions concerning the state of affairs in the various critical cases. We will not obtain a perfect score.

In section 2, fairly elementary arguments are used to show that $I^+(1^-) \neq \emptyset$ a.s. but $I^+(1) = \emptyset$ a.s. (Theorems 2.3 and 2.4). Knight's original question [8, p. 148] was to find a function g such that $\underline{\lim}_{h\to 0^+} (B(t+h)-B(t))g(t) \leq 1$ for all t and equals 1 for some t a.s. Evidently $g(t) = t^{-\frac{1}{2}}$ is precisely such a function.

An important tool in previous studies of slow points is the tail behaviour of the stopping time

$$T(c) = \inf\{t \geq 0 \mid |B(t)| > c(t+1)^{\frac{1}{2}}\}.$$

Breiman [4] showed that

$$\lim_{t\to\infty} t^{\lambda_0(c)} P(T(c) \geq t) = K(c).$$

To obtain finer results we study the slightly larger times

$$T_u(\alpha,c) = \inf\{t \mid |B_t| > \alpha + c(t+1)^{\frac{1}{2}}\}, \quad \alpha \geq 0,\ c \geq 0.$$

In section 3 we show that there is a $K(\alpha,c)$ such that

$$P(T_u(\alpha,c) > t) \leq K(\alpha,c)t^{-1} \quad \text{for all } t > 0.$$

This theorem plays an important role in the study of $S^+(c)$ and $S^+(c^+)$ in sections 4 and 5, respectively. Theorems 4.1 and 4.3 are improvements of the corresponding results in Perkins [9] (i.e. the precise formulations of (1.3) and (1.4)), but they are not as sharp as one might hope. They leave open a fundamental problem that we have not been able to resolve:

Question. Is $S^+(1)(\omega) \neq \emptyset$ a.s.?

We use Theorem 4.1 to show (Corollary 4.5) that $S^+(1)(\omega)$ is at most countable a.s. A direct argument similar to those in [9] proves that $S^+(\lambda_0^{-1}(1/2)) \cap Z(\omega) = \emptyset$ a.s. (Theorem 4.7), thus improving (1.5).

In the case of $S^+(c^+)$, our results are much more precise than (1.3) and (1.4). To describe an analogue of (1.4) for deterministic sets we need some notation.

Notation. $\mathcal{H} = \{\psi \mid \exists\ \varepsilon > 0 \ni \psi\colon\ [0,\varepsilon] \to [0,\infty)$ is strictly increasing, continuous and $\psi(0) = 0\}$.

If $\lambda \geq 0$, let

$$\mathcal{H}^\lambda = \{\psi \in \mathcal{H} \mid \overline{\lim_{n\to\infty}}\ \psi(2^{-n-1})\psi(2^{-n})^{-1} \leq 2^{-\lambda}\}.$$

If $\psi \in \mathcal{H}$ and $A \subset [0,\infty)$, ψ-m(A) denotes the Hausdorff ψ-measure of A (see Rogers [10, p. 50]). □

Note that $(\log 1/t)^\gamma t^\lambda \in \mathcal{H}^\lambda$ for all $\lambda \geq 0$, $\gamma \in \mathbb{R}$ but that $\psi \in \mathcal{H}^\lambda$ and $\gamma < \lambda$ implies that $\lim_{t\to 0^+} \psi(t)t^{-\gamma} = 0$.

Theorem 1.1. Let A be an analytic subset of $[0,\infty)$. Let $c \in [0,\infty)$.

$$P(A \cap S^+(c^+)(\omega) \neq \emptyset) = \begin{cases} 1 & \text{if } \exists\ \psi \in \mathcal{H}^{\lambda_0(c)} \ni \psi\text{-m}(A) > 0 \\ 0 & \text{if } \forall\ \psi \in \mathcal{H}^{\lambda_0(c)}\ \psi\text{-m}(A) = 0 \end{cases} .\ \square$$

This is obtained as a corollary to a corresponding result for random sets $A \subset [0,\infty)\times\Omega$ (Theorems 5.2 and 5.3). Several other corollaries are derived from these theorems. For example, it is shown that $S^+(1^+)(\omega)$ is a.s.

uncountable, $S^+(\lambda_0^{-1}(1/2)^+) \cap Z(\omega) \neq \emptyset$ a.s., (Corollary 5.4 (a), (b)), and $S^+(\lambda_0^{-1}(1/2)^+) \cap S^-(\lambda_0^{-1}(1/2))(\omega) \neq \emptyset$ a.s. (Theorem 5.9). (Thus improving (1.1), (1.5) and (1.6), respectively.)

Assume $(\Omega, \mathfrak{F}, \mathfrak{F}_t, B_t, \theta_t, P^x)$ is a Brownian motion process, i.e., B_t is distributed as a Wiener process and the above Markov process is a Hunt process in the sense of Blumenthal and Getoor [2, p. 48]. We write P for P^0 which will be the underlying measure unless indicated otherwise. If S and T are non-negative random variables, then

$$[[S,T]] = \{(t,\omega) \in (0,\infty)\times\Omega \mid S(\omega) \leq t \leq T(\omega)\}$$

and

$$[[S]] = \{(S(\omega),\omega) \mid \omega \in \Omega, S(\omega) < \infty\}.$$

We use $K, K_0, K_1, \ldots$ to denote constants whose value may change from line to line, and $B(X)$ denotes the Borel σ-field of X.

2. Points of Increase

The following elementary lemma will be used throughout this work.

Lemma 2.1. Let $\{T_i \mid i \geq 1\}$ be a non-decreasing sequence of stopping times and define $N = \min\{i \mid T_i > 1\}$ ($\min \emptyset = \infty$). Let $A_i \in \mathfrak{F}_{T_i}$, assume

(2.1) $$P(A_i \mid \mathfrak{F}_{T_{i-1}}) = p \text{ for some constant } p,$$

and put $T_0 \equiv 0$. Assume $\{T_i - T_{i-1} \mid i \geq 1\}$ are i.i.d. and each $T_i - T_{i-1}$ is independent of $\mathfrak{F}_{T_{i-1}}$.

Then

(a) For any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$E(T_1)p^{-1} < \delta \Rightarrow P(\bigcup_{i=1}^{N-1} A_i) > 1-\varepsilon.$$

(b) If in addition $T_i - T_{i-1} \leq K_0$ for some $K_0 \in \mathbb{R}$ then for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, K_0)$ such that

$$E(T_1)p^{-1} > \delta^{-1} \Rightarrow P(\bigcup_{i=1}^{N} A_i) < \varepsilon.$$

<u>Proof</u>. (a) $P(\bigcup_{i=1}^{N-1} A_i) \geq P(\bigcup_{j=1}^{k} A_i) - P(N \leq k) \quad (k \in \mathbb{N})$

$$\geq 1-(1-p)^k - E(T_k) \qquad \text{(by (2.1))}$$

$$\geq 1-e^{-pk} - k\, E(T_1).$$

Now let $k = [(E(T_1)p)^{-\frac{1}{2}}]$ ($[x]$ is the greatest integer of x) and note that $E(T_1)p^{-1} < \delta$ implies

$$P(\bigcup_{i=1}^{N-1} A_i) \geq 1-e^{-\delta^{-\frac{1}{2}}+1} - \delta^{\frac{1}{2}}.$$

(b) As $\{i \leq N\} \in \mathfrak{F}_{T_{i-1}}$, it follows from the bound on $T_i - T_{i-1}$ that $E(N) \leq (1+K_0)(E(T_1))^{-1}$. Therefore if $E(T_1)p^{-1} > \delta^{-1}$,

$$P(\bigcup_{i=1}^{N} A_i) \leq P(\bigcup_{i=1}^{k} A_i) + P(N > k)$$

$$\leq pk + (1+K_0)(k\, E(T_1))^{-1}$$

$$\leq pk + (1+K_0)\delta(kp)^{-1}.$$

The proof is completed by letting $k = [\delta^{\frac{1}{2}} p^{-1}]$. □

Lemma 2.2. If

$$T = \inf\{t \geq 1 | B_t < \inf_{s\leq 1} B_s - 1 + \sqrt{t}\},$$

then $\lim_{t\to\infty} tP(T > t) = K \in (0,\infty)$.

Proof. If $m_1 = \inf_{s\leq 1} B_s$ then

$$P^0(T > t|\mathfrak{F}_1) = P^{B_1-m_1+1}(\tilde{T} > t-1) \tag{2.2}$$

where

$$\tilde{T} = \inf\{t|B_t < (t+1)^{\frac{1}{2}}\}.$$

Lemma 10 (a) of Perkins [9] states that for $x \geq 1$

$$P^x(\tilde{T} > t-1) = t^{-1}(\psi(x) + r(t,x)) \quad \forall t \geq 1,$$

where $\psi \in L^2([1,\infty), e^{-z^2/2}dz)$ and for each $\varepsilon > 0$,

$$|r(t,x)| \leq K_\varepsilon \, e^{\varepsilon x^2} t^{-\lambda}$$

for some positive constant, λ, independent of ε. It follows from (2.2) that

$$P^0(T > t) = t^{-1}[\int_0^\infty \psi(x+1)(2/\pi)^{\frac{1}{2}} e^{-x^2/2} dx + \varepsilon(t)]$$

where

$$|\varepsilon(t)| \leq t^{-\lambda} \, K\int_0^\infty e^{(x+1)^2/4} e^{-x^2/2} \, dx. \quad \square$$

Theorem 2.3. $I^+(1)(\omega) = \emptyset$ a.s.

That is, for a.e.ω there is no $t \geq 0$ and $\Delta > 0$ such that

$$B(t+h)-B(t) \geq \sqrt{h} \quad \text{for } 0 \leq h \leq \Delta. \tag{2.3}$$

Proof. For each $n \in \mathbb{N}$ define a sequence of stopping times $\{T_i^n | i = 0,1,\ldots\}$ by

$$T_0^n = 0$$

$$T_{i+1}^n = \inf\{t > T_i^n + \tfrac{1}{n} | B_t < \inf_{T_i^n \leq s \leq T_i^n + \frac{1}{n}} B_s - n^{-\frac{1}{2}} + (t-T_i^n)^{\frac{1}{2}}\} \wedge (T_i^n+1).$$

If $N_n = \min\{i | T_i^n > 1\}$ we claim that

$$\text{(2.4)} \qquad \max_{i \leq N_n} T_i^n - T_{i-1}^n \overset{P}{\to} 0.$$

Fix $\Delta > 0$. A scaling argument shows that the i.i.d. random variables $\{T_i^n - T_{i-1}^n\}$ are equal in law to $(T/n) \wedge 1$, where T is as in Lemma 2.2. Therefore Lemma 2.2 gives as

$$E(T_i^n - T_{i-1}^n)(P(T_i^n - T_{i-1}^n > \Delta))^{-1} \geq n^{-1} K_0 \int_1^n t^{-1} dt (n\Delta)$$

$$= K_0 \Delta \log n$$

for some positive constant K_0. Now (2.4) is a consequence of Lemma 2.1 (b).

Assume $t \in I^+(1) \cap [0,1]$ satisfies (2.3) for some $\Delta > 0$. Then $t \in [T_i^n, T_{i+1}^n)$ for some $i < N_n$. We now show that $T_{i+1}^n - T_i^n \geq \Delta$. Let $m = \inf_{s \in [T_i^n, T_i^n + 1/n]} B_s$.

Case 1. $t \in [T_i^n + \frac{1}{n}, T_{i+1}^n)$. Then if $s \in [0,\Delta]$,

$$B(t+s) \geq B(t) + s^{\frac{1}{2}} \geq m + (t-T_i^n)^{\frac{1}{2}} - n^{-\frac{1}{2}} + s^{\frac{1}{2}} \quad (\text{since } t < T_{i+1}^n)$$

$$> m + (t+s-T_i^n)^{\frac{1}{2}} - n^{-\frac{1}{2}}.$$

Case 2. $t \in [T_i^n, T_i^n + \frac{1}{n})$. If $s \in [0,\Delta]$, then

$$B(t+s) \geq B(t) + s^{\frac{1}{2}} \geq m + (s+t-T_i^n)^{\frac{1}{2}} - n^{-\frac{1}{2}}.$$

In either case we get $T^n_{i+1} \geq t+\Delta \geq T^n_i + \Delta$. Therefore if

$I^+_\Delta(1) = \{(t,\omega) | (2.3) \text{ holds}\}$,

$$\{I^+_\Delta(1)(\omega) \cap [0,1] \neq \emptyset\} \subset \{\max_{i<N_n} T^n_i - T^n_{i-1} \geq \Delta\}.$$

The theorem now follows easily from (2.4). □

Theorem 2.4. $I^+(1^-)(\omega)$ is a.s. dense. That is, with probability one there is a dense set of times t such that $\underline{\lim}_{h\to 0^+} (B(t+h)-B(t))h^{-\frac{1}{2}} = 1$.

Proof. It suffices to show $I^+(1^-)(\omega) \cap [0,1] \neq \emptyset$ a.s., as a scaling argument would then show $I^+(1^-)(\omega)$ is a.s. dense. If Δ, $\alpha > 0$, $c \in \mathbb{R}$, let

$$P_{\Delta,c} = P(\exists t \in [\Delta\alpha,\alpha) \ni B_s \geq B_t + c(t-s)^{\frac{1}{2}} \quad \forall s \in [0,t]).$$

A scaling argument shows that $P_{\Delta,c}$ is independent of α. Fix $c < 1$. We claim that

(2.5) $$\lim_{\Delta\to 0^+} P_{\Delta,c} = 1,$$

that is

(2.6) $$P(\exists t \in (0,1) \ni B_s \geq B_t + c(t-s)^{\frac{1}{2}} \quad \forall s \in [0,t]) = 1.$$

The above probability is greater than

$$P(\exists t \in (0,1) \ni B_s \geq B_t + c(s-t)^{\frac{1}{2}} \quad \forall s \in [t,t+1])$$

by a time reversal, and this probability was shown to be positive in Davis [5, Thm 3.1]. A scaling argument shows that for any $\varepsilon > 0$ the probability in (2.6) equals

$$P(\exists t \in (0,\varepsilon) \ni B_s \geq B_t + c(t-s)^{\frac{1}{2}} \quad \forall s \in [0,t]).$$

The Blumenthal 0-1 law implies that this probability must be one.

Let $c_n = 1-1/n$ and fix $\varepsilon > 0$. By (2.5) we may choose $\Delta_n \downarrow 0$ such that $P_{\Delta_n, c_n} \geq e^{-\varepsilon^2/n^2}$ and $\Delta_1 < 1/2$. Inductively define a sequence of stopping times by $T_0 = 0$ and

$$T_{i+1} = \inf\{t \geq T_i + \Delta_{i+1}(1-T_i) \,|\, B_s \geq B_t + c_{i+1}(t-s)^{\frac{1}{2}} \quad \forall s \in [T_i, t]\} \wedge 1.$$

(It is easy to check that T_{i+1} is a stopping time because $T_i + \Delta_{i+1}(1-T_i)$ is.) Then, if $B_s^T = B_{T+s} - B_T$ for a stopping time T, we have, on $\{T_{n-1} < 1\}$,

$$\begin{aligned} P(T_n < 1 \,|\, \mathcal{F}_{T_{n-1}})(\omega) &= P(\exists t \in [\Delta_n(1-T_{n-1}(\omega)),\ 1-T_{n-1}(\omega)) \ni \\ &\qquad B_s^{T_{n-1}(\omega)} \geq B_s^{T_{n-1}(\omega)} + c_n(t-s)^{\frac{1}{2}} \quad \forall s \in [0,t]) \\ &= P_{\Delta_n, c_n} \geq e^{-\varepsilon^2/n^2}. \end{aligned}$$

This gives

$$P(T_n < 1 \quad \forall n) \geq \exp\{-\varepsilon^2 \sum_{n=1}^{\infty} \frac{1}{n^2}\} = \exp\{-\varepsilon^2 \pi^2/6\}. \tag{2.7}$$

Let $T = \lim_{i\to\infty} T_i \leq 1$. If $\omega \in \{T_n < 1 \quad \forall n\}$ and $t \in [T_i, T_{i+1})$, then

$$\begin{aligned} B(t) &\geq B(T_{i+1}) + c_{i+1}(T_{i+1}-t)^{\frac{1}{2}} \\ &\geq B(T_{i+2}) + c_{i+2}(T_{i+2}-T_{i+1})^{\frac{1}{2}} + c_{i+1}(T_{i+1}-t)^{\frac{1}{2}} \\ &\geq B(T_{i+2}) + c_{i+1}(T_{i+2}-t)^{\frac{1}{2}} \\ &\ \ \vdots \\ &\geq B(T_{i+n}) + c_{i+1}(T_{i+n}-t)^{\frac{1}{2}} \\ &\to B(T) + c_{i+1}(T-t)^{\frac{1}{2}} \qquad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore $\underline{\lim}_{h\to 0^-} (B(T+h)-B(T))h^{-\frac{1}{2}} \geq 1$ on $\{T_n < 1 \quad \forall n\}$ and so (2.7) shows that $I^-(1^-) \cap [0,1] \neq \emptyset$ a.s. By symmetry the same is true of $I^+(1^-) \cap [0,1]$.

The second statement of the theorem is now immediate from Theorem 2.3. □

3. A Hitting Time Estimate

In this section we bound the tail of the distribution of the stopping time

$$T_u(\alpha,c) = \inf\{t \mid |B_t| > \alpha + c(t+1)^{\frac{1}{2}}\}.$$

We will need a comparison lemma for reflecting (at 0) Brownian motion. Bramson [3, Lemma 5] showed that if $h(t) < g(t)$ are C^1 functions then for $x < h(0)$, and any real number y,

$$(3.1)\qquad P^x(B_T < y \mid B_s \leq h(s)\ \ \forall s \leq T) \geq P^x(B_T < y \mid B_s \leq g(s)\ \ \forall s \leq T).$$

Uchiyama [12, Lemma 2.4] proves this result by first proving a related inequality for the simple symmetric random walk (by an induction on time) and then using Donsker's invariance principle. Uchiyama's argument goes through with only minor changes to give us the following version of (3.1) for $|B|$ (Proposition 3.1 below).

Notation. If $h: [0,\infty) \to [0,\infty)$, let

$$T_h = \inf\{t \mid |B_t| > h(t)\} \quad (\inf \emptyset = \infty).$$

Proposition 3.1. Let $h,g: [0,\infty) \to (0,\infty)$ be C^1 functions such that $h(t) \leq g(t)$ for all t. If $|x| < h(0)$, then for all $y \geq 0$,

$$P^x(|B_t| \leq y \mid T_h > t) \geq P^x(|B_t| \leq y \mid T_g > t).\ \square$$

Also required are simpler comparison theorems concerning different starting points (Lemmas 3.3 and 3.4 below). We include a proof of Lemma 3.3 as it is short.

Lemma 3.2. Let $\{X_n | n = 0,1,\ldots\}$ denote a reflecting simple symmetric random walk and let P^x denote the probability measure $P(\cdot | X_0 = x)$. If h is a function from the nonnegative integers $\mathbb{N}_0$ to $[0,\infty)$, let

$$\hat{T}_h = \min\{n \geq 0 | X_n > h(n)\}.$$

Then, whenever x-x' is an even non-negative integer ($x,x' \in \mathbb{N}_0$),

$$P^x(X_n \leq y | \hat{T}_h \geq n) \leq P^{x'}(X_n \leq y | \hat{T}_h \geq n) \tag{3.2}$$

for all $y \geq 0$, provided that both conditioned on events have positive probability.

Proof. Fix n. Let $Z_0, Z_1, \ldots, Z_n$, denote simple symmetric reflecting random walk conditioned not to exceed h(k) at time k for all k = 0,1,...,n. We will show for k = 0,1,...,n, that

$$P(Z_k \leq y | Z_0 = x) \leq P(Z_k \leq y | Z_0 = x'),\ y \geq 0, \tag{3.3}$$

by induction on k. The case k = 0 is immediate. We use the fact that, since Z only makes jumps of magnitude one,

$$P(Z_{k+1} \leq y | Z_k = m) \geq P(Z_{k+1} \leq y | Z_k = m+2)$$

for all y and all m. This and summation by parts gives (3.3) for k = j+1 given (3.3) for k = j. Note that (3.3) for k = n is (3.2). □

A routine weak convergence argument now gives us

Lemma 3.3. If $h: [0,\infty) \to (0,\infty)$ is a C^1 function and $x' \leq x < h(0)$, then

$$P^x(|B_t| \leq y | T_h > t) \leq P^{x'}(|B_t| \leq y | T_h > t) \quad \forall y \geq 0. \ \square$$

A similar argument shows:

Lemma 3.4. If h is as above, then $P^x(T_h > t)$ is non-increasing in $x \in [0,\infty)$. □

Recall that $X_t = e^{-t/2}B(e^t-1)$ is an Ornstein-Uhlenbeck process starting at B_0, and with generator $\frac{1}{2}\, d^2/dx^2 - \frac{1}{2}\, xd/dx$. If $\rho(c) = \inf\{t|\ |X_t| > c\}$ then (see Greenwood-Perkins [7, Lemma 3 and Proposition 4]) for any $|x| < c$, $P^x(X_t \in dy|\rho(c) > t)$ converges weakly to a symmetric distribution η^c on $[-c,c]$ satisfying

(3.4) $$P^{\eta_c}(X_t \in dy|\rho(c) > t) = \eta_c(dy)$$

and

(3.5) $$P^{\eta_c}(\rho(c) > t) = e^{-\lambda_0(c)t}.$$

The distribution η_c is the stationary initial distribution of X conditioned to stay in $[-c,c]$.

Theorem 3.5. There is a constant $K(\alpha,c)$ such that for all α, $c \geq 0$, $P^0(T_u(\alpha,c) \geq t) \leq K(\alpha,c)t^{-\lambda_0(c)}$ for all $t \geq 0$. Moreover $K(\alpha,c)$ may be chosen so that $\sup\{K(\alpha,c)|\alpha \leq N,\ 1/N \leq c \leq N\} < \infty$, for each $N > 0$.

Proof. Define X_t as above and let

$$\rho(\alpha,c) = \inf\{t|\ |X_t| > c + \alpha e^{-t/2}\}.$$

If

$$T(c) = \inf\{t|\ |B_t| > c(t+1)^{\frac{1}{2}}\},$$

then $T(c) = e^{\rho(c)}-1$ and $T_u(\alpha,c) = e^{\rho(\alpha,c)}-1$, and so it suffices to show

(3.6) $$P^0(\rho(\alpha,c) > t) \leq K(\alpha,c)e^{-t\lambda_0(c)}.$$

We first establish the above bound for $P^{\eta_c}(\rho(\alpha,c) > t)$. Note that

$$P^{\eta_c}(|X(n)| \le y | \rho(\alpha,c) > n) = P^{\eta_c}(|B(e^n-1)| \le ye^{n/2} | T_u(\alpha,c) > e^n-1) \tag{3.7}$$

$$\le P^{\eta_c}(|B(e^n-1)| \le ye^{n/2} | T(c) > e^n-1) \text{ (Prop. 3.1)}$$

$$= P^{\eta_c}(|X(n)| \le y | \rho(c) > n)$$

$$= \eta_c([-y,y]) \qquad \text{(by (3.4))}.$$

Now $P^{\eta_c}(\rho(\alpha,c) > n+1 | \rho(\alpha,c) > n) = \int_0^\infty P^y(|X_t| \le c+\alpha e^{-\frac{1}{2}(t+n)} \quad \forall t \le 1)$

$$P^{\eta_c}(|X(n)| \in dy | \rho(\alpha,c) > n).$$

The above integrand is non-increasing in $y \in [0,\infty)$ (use Lemma 3.4) and so (3.7) and an integration by parts leads to

$$P^{\eta_c}(\rho(\alpha,c) > n+1 | \rho(\alpha,c) > n) \le P^{\eta_c}(|X_t| \le c + \alpha e^{-\frac{1}{2}(t+n)} \quad \forall t \le 1)$$

$$\le 1-P^{\eta_c}(\rho(c) \le 1-e^{-n/2})P^c(\sup_{t \le e^{-n/2}} |X_t| > c+\alpha e^{-n/2})$$

$$\le 1-(1-e^{-\lambda_0(c)(1-e^{-n/2})})(1-P^c(\sup_{t\le e^{-n/2}} |W_t| \le (c+\alpha e^{-n/2}) \times (1+\tfrac{1}{2}e^{-n/2}))), \tag{3.8}$$

where $W_t = X_t + 1/2 \int_0^t X_s ds$ is a Brownian motion and we have used (3.5). Of course P^{-c} could have been used instead of P^c above. Now

$$P^c(\sup_{t\le e^{-n/2}} |W_t| \le (c+\alpha e^{-n/2})(1+\tfrac{1}{2}e^{-n/2}))$$

$$\le P^0(\sup_{t\le 1} W_t \le e^{n/4}(2\alpha+c)e^{-n/2})$$

$$\le (2\alpha+c)e^{-n/4}.$$

Substitute the above in (3.8) to get

$$P^{\eta_c}(\rho(\alpha,c) > n+1 \mid \rho(\alpha,c) > n) \le e^{-\lambda_0(c)+\lambda_0(c)e^{-n/2}} + (4\alpha+2c)e^{-n/4}.$$

An easy computation now shows

$$P^{\eta_c}(\rho(\alpha,c) > n) = \prod_{i=0}^{n-1} P^{\eta_c}(\rho(\alpha,c) > i+1 \mid \rho(\alpha,c) > i)$$
$$\le K_0(\alpha,c)e^{-\lambda_0(c)n}$$

where $K_0(\alpha,c)$ is bounded if α, c and c^{-1} are uniformly bounded, implying $P^{\eta_c}(\rho(\alpha,c) > T) \le K_1(\alpha,c)e^{-\lambda_0(c)T}$, where $K_1(\alpha,c)$ is bounded if α, c, and c^{-1} are uniformly bounded.

Assume now that $c \in [N^{-1},N]$ and $\alpha \in [0,N]$ for N fixed. Let $T_X(0)$ be the first time X_t hits zero, and let T be a fixed positive number. Now Lemma 3.4 and the definition of X_t imply that if $0 < s < T$ and $\gamma_{r,s} = P(|X_t| \le c+\alpha e^{-t/2}, s \le t \le T \mid X_s=r)$ then $\gamma_{r,s} \le \gamma_{y,s}$ if $|y| \le |r|$. Thus $\gamma_{0,0} \le E\gamma_{X_s,s} \le \gamma_{0,s}$.

Now $P^y(T_X(0) < \rho(\alpha,c))$ is easily seen to be nonincreasing for $y \in [0,\infty)$. Using this fact and the Strong Markov Property for X, and putting $\gamma_{0,s} = 1$ if $s \ge T$, we get

$$\text{(3.9)}\qquad P^{\eta_c}(\rho(\alpha,c) > T) \ge \int_0^\infty \gamma_{0,s}P^{\eta_c}(T_X(0) \in ds,\, T_X(0) < \rho(\alpha,c))$$
$$\ge \gamma_{0,0}P^{\eta_c}(T_X(0) < \rho(\alpha,c)) \ge \gamma_{0,0}P^{N^{-1}/2}(T_X(0) < \rho(\alpha,c))\eta_c[-N^{-1}/2,N^{-1}/2].$$

Now it is known that $\eta_c(dx)$ $= K_cM(-\lambda_0(c),\, 1/2,\, x^2/2)e^{-x^2/2}\,dx$, where M is the confluent hypergeometric function and K_c is an integration constant (see, for example, the proof of Proposition 1 in [9]). Therefore $c \to \eta_c[-N^{-1}/2,N^{-1}/2]$ is continuous and positive for $c \in [N^{-1}, N]$ and in particular is bounded away from zero. Finally,

$$P^{N^{-1}/2}(T_X(0) < \rho(\alpha,c)) \geq P^{N^{-1}/2}(T_X(0) < \rho(0,N^{-1})) > 0.$$

The required estimate follows from this, the previous comment, and (3.9).

4. Slow Points

Consider now the problem of refining (1.3). More precisely we will try to improve the following results from [9].

Theorem A ([9, Theorem 12]). Assume $A \subset [0,\infty)\times\Omega$ and for some $d > 0$ there are sequences of stopping times $\{S_i^n, T_i^n | i,n \in \mathbb{N}\}$ such that $A \subset \bigcup_{i=1}^{\infty} [[S_i^n, T_i^n]]$ for all n and

$$\text{(4.1)} \qquad \lim_{n\to\infty} E\Big(\sum_{i=1}^{\infty} (T_i^n - S_i^n)^d\Big) = 0.$$

Then $\dim(A(\omega) \cap S^+(c)(\omega)) \leq d-\lambda_0(c)$ $\forall\, 0 \leq c < \infty$ a.s., where a negative dimension indicates the set is empty. □

Theorem B ([9, Theorem 15]). Assume there are optional sets $A_k \subset [0,\infty)\times\Omega$ such that $A_k(\omega)$ is a.s. closed and $A = \bigcup_{k=1}^{\infty} A_k$. Then

$$\dim(A(\omega) \cap S^+(c)(\omega)) \geq \dim(A(\omega)) - \lambda_0(c) \quad \forall c \in (0,\infty) \text{ a.s.} \quad \square$$

Actually Theorem A is slightly stronger than Theorem 12 in [9] (also see the subsequent remarks) but the same proof works with only minor changes.

If $\Delta > 0$, let $S_\Delta^+(c) = \{(t,\omega) \mid |B(t+h)-B(t)|h^{-\frac{1}{2}} \leq c \text{ for all } h \in (0,\Delta]\}$.

Notation.

We let $\emptyset_\alpha(t) = t^\alpha$, and if $\alpha \leq 0$, $\emptyset_\alpha$-m(A) denotes the cardinality of A if A is finite and ∞, otherwise. □

The following result refines Theorem A.

Theorem 4.1. (a) Assume the hypotheses of Theorem A. Then

$$\emptyset_{d-\lambda_0(c)}\text{-}m(A(\omega) \cap S^+(c)(\omega)) = 0 \quad \text{a.s.} \quad \forall 0 < c < \infty.$$

In particular if (4.1) holds with $d = \lambda_0(c)$ then $A(\omega) \cap S^+(c)(\omega) = \emptyset$ a.s.

(b) Assume the hypotheses of Theorem A but instead of (4.1) suppose only that

(4.2) $\quad \underline{\lim}_{n\to\infty} E(\sum_{i=1}^{\infty}(T_i^n - S_i^n)^d) < \infty$, and $\sup_{i\geq 1}|T_i^n - S_i^n| \leq K(n)$, where $K(n)$, $n \geq 1$, are constants approaching 0 as n approaches infinity.

Then for a.a.ω and for all $n \in \mathbb{N}$,

(4.3) $$\emptyset_{d-\lambda_0(c)}\text{-}m(A(\omega) \cap S^+_{n^{-1}}(c)(\omega)) < \infty.$$

In particular if (4.2) holds with $d = \lambda_0(c)$ then $A(\omega) \cap S^+(c)(\omega)$ is countable a.s.

Proof. (a) If $\Delta > 0$ and $S \leq T$ are stopping times, suppose $t \in [S,T] \cap S^+_\Delta(c)$. For each $u \in [T, S+\Delta]$ we have

$$|B(u) - B(T)| \leq |B(u)-B(t)|+|B(t)-B(T)| \leq c(u-t)^{\frac{1}{2}} + c(T-t)^{\frac{1}{2}}$$

$$\leq c(u-S)^{\frac{1}{2}} + c(T-S)^{\frac{1}{2}}.$$

Therefore

$$P([S,T) \cap S^+_\Delta(c) \neq \emptyset|\mathcal{F}_T)(\omega) \leq P(|B(v)| \leq c(v+T-S)^{\frac{1}{2}}(\omega) + c(T-S)^{\frac{1}{2}}(\omega) \quad \forall v \leq \Delta-(T-S)(\omega))$$

(4.4) $$= P(|B(v)| \leq c(v+1)^{\frac{1}{2}}+c \quad \forall v \leq \Delta(T-S)(\omega)^{-1}-1)$$

$$\leq K(T(\omega)-S(\omega))^{\lambda_0(c)} \qquad \text{(Theorem 3.5)},$$

where K depends on (Δ,c).

Choose stopping times $S_i^n \leq T_i^n$ as in the statement of the theorem. Then

$$E\left(\sum_{i=1}^{\infty}(T_i^n-S_i^n)^{d-\lambda_0(c)} I([S_i^n,T_i^n] \cap S_\Delta^+(c)(\omega) \neq \emptyset)\right)$$

$$\leq K\, E\left(\sum_{i=1}^{\infty}(T_i^n-S_i^n)^d\right) \qquad \text{(by (4.4))}$$

$$\to 0 \qquad \text{as } n \to \infty.$$

Therefore $\emptyset_{d-\lambda_0(c)}\text{-m}(A(\omega) \cap S_\Delta^+(c)(\omega)) = 0$ a.s. and the proof is completed by letting $\Delta \downarrow 0$.

(b) The proof of (4.3) is obvious from the previous argument. If $d = \lambda_0(c)$, (4.3) means that $A(\omega) \cap S^+_{n^{-1}}(c)(\omega)$ is a.s. finite, implying $A(\omega) \cap S^+(c)(\omega)$ is a.s. countable. □

<u>Corollary 4.2</u>. Let $c \in (0,\infty)$. If $A \subset [0,\infty)$ satisfies $\emptyset_d\text{-m}(A) = 0$ then w.p.1 $\emptyset_{d-\lambda_0(c)}(A \cap S^+(c)(\omega)) = 0$. In particular if $d = \lambda_0(c)$, then $A \cap S^+(c)(\omega) = \emptyset$ a.s. If $\emptyset_{\lambda_0(c)}\text{-m}(A) < \infty$ then $A \cap S^+(c)(\omega)$ is a.s. countable.

<u>Proof</u>. Immediate from the above. □

The proof of the following refinement of Theorem B is essentially the same as the proof of Theorem 15 in [9]; the only difference is that Corollary 4.2 above is used in place of Theorem 13 (a) in [9].

<u>Theorem 4.3</u>. Assume $\emptyset \in \mathcal{H}$ and $\alpha \geq \lambda_0(c)$ satisfy

(4.5) $$\int_0^\varepsilon \emptyset(t)t^{-\alpha-1}dt < \infty \quad \text{for some } \varepsilon > 0.$$

Let $\{A_n\}$ be a sequence of optional subsets of $[0,\infty)\times\Omega$ such that $A_n(\omega)$ is a.s. closed and set $A = \bigcup_{n=1}^{\infty} A_n$. Then

$$\{\omega | \phi\text{-}m(A(\omega)) > 0\} \subset \{\omega | \phi_{\alpha-\lambda_0(c)}\text{-}m(A(\omega) \cap S^+(c)(\omega)) > 0\} \quad \text{a.s.}$$

In particular if (4.5) holds with $\alpha = \lambda_0(c)$ then

$$\{\omega | \phi\text{-}m(A(\omega)) > 0\} \subset \{\omega | A(\omega) \cap S^+(c)(\omega) \neq \emptyset\}. \quad \square$$

A similar result, but with both the hypotheses and the conclusion weakened will be given in the next section (Theorem 5.8).

Corollary 4.4. Let $c \in (0,\infty)$ and A be an analytic subset of $[0,\infty)$. If ϕ is as in Theorem 4.3 and $\phi\text{-}m(A) > 0$, then $\phi_{\alpha-\lambda_0(c)}\text{-}m(A \cap S^+(c)(\omega)) > 0$ a.s. In particular if (4.5) holds with $\alpha = \lambda_0(c)$, and $\phi\text{-}m(A)>0$, then $A \cap S^+(c)(\omega)\neq\emptyset$ a.s.

Proof. There is a compact set $K \subset A$ such that $\phi\text{-}m(K) > 0$ (see Rogers [10, p. 122]). The result is now immediate from Theorem 4.3. $\square$

Theorems A and B are immediate from Theorems 4.1 and 4.3, respectively. For example, we can find a single null set outside of which the conclusion of Theorem 4.3 holds simultaneously for all rational (α,c) satisfying $\alpha \geq \lambda_0(c)$ and for all the functions $\phi^\alpha(t) = t^\alpha(\log 1/t)^{-2}$. This gives us Theorem B. Nonetheless, there is still a "gap" between Theorems 4.1 and 4.3. Neither result settles the still unresolved question: Is $S^+(1) = \emptyset$ a.s.? However, an immediate consequence of Corollary 4.2 is (take $A = [0,n)$ in Corollary 4.2, and let $n \to \infty$)

Corollary 4.5. $S^+(1)$ is at most countable a.s. $\square$

Since $\phi_{\frac{1}{2}}\text{-}m(Z(\omega)) = 0$ a.s. (recall that Z is the zero set of B), Theorem 4.1 (a) suggests that $Z(\omega) \cap S^+(\lambda_0^{-1}(1/2)) = \emptyset$ a.s. The hypotheses of this result, however, require that we cover Z by stopping times,

$\bigcup_{i=1}^{\infty} [[S_i^n, T_i^n]]$ so that $\lim_{n\to\infty} E(\sum_{i=1}^{\infty} (T_i^n - S_i^n)^{\frac{1}{2}}) = 0$. Recall that $E((T_i^n - S_i^n)^{\frac{1}{2}}) \geq c\, E(L^0_{T_i^n} - L^0_{S_i^n})$ for some universal constant c, where L_t^0 is the local time of B. (In fact much more is true -- see Barlow and Yor [1]). It is now easy to show that there are no stopping times satisfying the above conditions. Nonetheless we now give a direct proof that $Z \cap S^+(\lambda_0^{-1}(1/2)) = \emptyset$ a.s., similar to the earlier argument showing that $I^+(1) = \emptyset$ a.s.

The proof of the following lemma is elementary.

Notation. $T_B(x) = \inf\{t | B_t = x\}$.

Lemma 4.6. If $\varepsilon > 0$, there are positive constants K_1^ε, K_2^ε such that

$$K_1^\varepsilon |x| t^{\frac{1}{2}} \leq E^x(T_B(0) \wedge t) \leq K_2^\varepsilon |x| t^{\frac{1}{2}}$$

whenever $t/x^2 \geq \varepsilon$. □

Theorem 4.7. $Z(\omega) \cap S^+(\lambda_0^{-1}(1/2))(\omega) = \emptyset$ a.s.

Proof. Let $c = \lambda_0^{-1}(1/2) (\approx 1.3069\ldots)$. For each $n \in \mathbb{N}$, inductively define stopping times $\{S_i^n, T_i^n | i \in \mathbb{N}\}$ as follows:

$$S_0^n = 0,\ T_i^n = \inf\{t \geq S_i^n |\ |B_t - B_{S_i^n}| > c(t - S_i^n + n^{-1})^{\frac{1}{2}}\} \wedge (S_i^n + 1), \text{ and}$$

$$S_{i+1}^n = \inf\{t > T_i^n | B_t - B_{S_i^n} = 0\} \wedge (T_i^n + 1).$$

Let

$$T = \inf\{t |\ |B_t| > c(t+1)^{\frac{1}{2}}\} \wedge n$$

$$S = \inf\{t > T | B_t = 0\} \wedge (T+n).$$

A scaling argument shows that the i.i.d. random variables $\{S^n_{i+1}-S^n_i \mid i \in \mathbb{N}_0\}$ are equal in law to S/n and the i.i.d. random variables $\{T^n_i-S^n_i \mid i \in \mathbb{N}_0\}$ are equal in law to T/n. Therefore

$$\begin{aligned} E(S^n_{i+1}-S^n_i) &\geq n^{-1}\int E^{c(t+1)^{\frac{1}{2}}}(T_B(0)\wedge(t+n))I(t<n)P(T\in dt) \\ &\geq K\, n^{-1}\int I(t<n)(t+1)^{\frac{1}{2}}(t+n)^{\frac{1}{2}}\; P(T\in dt) \qquad \text{(Lemma 4.6)} \\ &\geq K\, n^{-\frac{1}{2}}\int_0^n s^{-\frac{1}{2}}\; P\;(T\geq s)ds \\ &\geq K\, n^{-\frac{1}{2}}\int_1^n s^{-1}ds \qquad \text{[9, Lemma 10 (a)]} \\ &= K\, n^{-\frac{1}{2}}\log n. \end{aligned}$$

If $\Delta \in (0,1)$, then

$$P(T^n_i-S^n_i \geq \Delta \mid \mathcal{F}_{S^n_1}) = P(T\geq n\Delta) \equiv P(n,\Delta) \leq K_\Delta\, n^{-\frac{1}{2}}.$$

Therefore $\lim\limits_{n\to\infty} \dfrac{E(S^n_{i+1}-S^n_i)}{P(n,\Delta)} \geq \lim\limits_{n\to\infty} K'_\Delta \log n = \infty.$

Lemma 2.1 shows that if $N_n = \min\{i \mid S^n_i > 1\}$, then

(4.6) $$\max_{i\leq N_n} T^n_i - S^n_i \overset{P}{\to} 0.$$

Fix $\Delta \in (0,1)$ and let

$$A = \{\omega \mid [t,t+1] \cap Z(\omega) \neq \emptyset \quad \forall t \leq 1\}.$$

Suppose $\omega \in A$ and $t_0 \in Z(\omega) \cap S^+_\Delta(c)(\omega) \cap [0,1]$. The definition of A implies that for all $i < N_n$, $S^n_i = \inf\{t > T^n_{i-1} \mid B_t = 0\}$ and therefore $t_0 \in [S^n_{i_0}, T^n_{i_0}]$ for some $i_0 < N_n$. If $u \in [t_0, t_0+\Delta]$, then

$$|B_u - B_{S^n_{i_0}}| = |B_u| = |B_u - B_{t_0}| \le c(u-t_0)^{\frac{1}{2}} \le c(u - S^n_{i_0} + n^{-1})^{\frac{1}{2}},$$

and therefore $T^n_{i_0} - S^n_{i_0} \ge \Delta$. It follows that

$$A \cap \{\omega | Z(\omega) \cap S^+_\Delta(c)(\omega) \cap [0,1] \neq \emptyset\} \subset \{\max_{i \le N_n} T^n_i - S^n_i \ge \Delta \text{ for all } n\}.$$

As the set on the right is null by (4.6), we obtain

$$P(Z \cap S^+(c) \neq \emptyset) \le P(A^c) < 1.$$

A scaling argument and the Blumenthal 0-1 law now shows that $P(Z \cap S^+(c) \neq \emptyset) = 0$. □

In [7] and [9] sets of asymmetric slow points were also considered, i.e., sets of the form

$$S^+(c_1,c_2) = \{(t,\omega) | \exists \Delta > 0 \ni B(t+h)-B(t) \in [c_1 h^{\frac{1}{2}}, c_2 h^{\frac{1}{2}}] \quad \forall 0 \le h \le \Delta\} \quad (-\infty \le c_1 < c_2 \le \infty).$$

Actually theorems A and B were proven for sets of this form where $-\lambda_0(c)$ is replaced by $\lambda_0(c_1,c_2)$, the largest eigenvalue of

$$\frac{1}{2}\psi''(x) - \frac{1}{2}x\psi'(x) = -\lambda\psi(x), \ \psi \in L^2([c_1,c_2], e^{-x^2/2}dx)$$

$$\psi(c_i) = 0 \quad \text{if } |c_i| < \infty$$

(then $\lambda_0(c) = \lambda_0(-c,c)$).

In [7, Theorem 19] it was shown that

$$S^+(c_1,c_2)(\omega) \cap Z(\omega) \begin{cases} \neq \emptyset \text{ a.s. if } \lambda_0(c_1,c_2) < 1/2 & \forall -\infty \le c_1 < c_2 \le \infty. \\ = \emptyset \text{ a.s. if } \lambda_0(c_1,c_2) > 1/2 \end{cases}$$

The proof of Theorem 4.7 in fact goes through in the asymmetric case and shows

$$S^+(c_1,c_2)(\omega) \cap Z(\omega) = \emptyset \text{ a.s. if } \lambda_0(c_1,c_2) = 1/2 \text{ and } |c_1| < \infty,\ |c_2| < \infty.$$

We mention this because it is interesting to note that

$$S^+(0,\infty)(\omega) \cap Z(\omega) \neq \emptyset \quad \text{a.s.} \quad \text{and } \lambda_0(0,\infty) = 1/2.$$

The first statement is clear as $S^+(0,\infty) \cap Z$ is the countable set of the starts of positive excursions and for the second see Proposition 1 in [9].

5. On the size of $S^+(c^+)$

In this section we obtain more precise results for $S^+(c^+)$, similar in spirit to those obtained in the previous section for $S^+(c)$. Recall that

$$S^+(c^+) = \{(t,\omega) | \overline{\lim_{h\to 0^+}} |B_{t+h}-B_t| h^{-\frac{1}{2}} \leq c\},$$

and

$$\mathcal{H}^\lambda = \{\psi \in \mathcal{H} | \overline{\lim_{n\to\infty}} \psi(2^{-n-1})\psi(2^{-n})^{-1} \leq 2^{-\lambda}\}.$$

We first give sufficient conditions (Theorem 5.2) and necessary conditions (Theorem 5.3) for a random set $A \subset [0,\infty)\times\Omega$ to intersect $S^+(c^+)$. These conditions extend to points which are simultaneously slow for 2 independent Brownian motions and this independent Brownian motion allows one to estimate the size of $A \cap S^+(c^+)(\omega)$ itself (Theorem 5.5). Theorem 1.1 and the other results concerning $S^+(c^+)$ mentioned in the introduction follow as corollaries.

In order to accommodate an independent Brownian motion B' we enlarge our probability space. Start with a Brownian motion B and a probability space for which $(\Omega_1,\mathcal{F}_1,\mathcal{F}_t^1,B_t,\theta_t,P_1^x)$ is a Hunt process and let

$(\Omega_2,\mathcal{F}_2,\mathcal{F}_t^2,B_t',\theta_t^2,P_2^y)$ denote another such space. We will work on

$$(\Omega,\mathcal{F},P^{(x,y)}) = (\Omega_1\times\Omega_2,\ \overline{\mathcal{F}_1\times\mathcal{F}_2},\ \overline{P^x\times P^y}),$$

where $\overline{\mathcal{F}_1\times\mathcal{F}_2}$ denotes the usual completion of $\mathcal{F}_1\times\mathcal{F}_2$ with respect to the measures $\{P^x\times P^y\}$ and $\overline{P^x\times P^y}$ is the extension of $P^x\times P^y$. Let $\{\mathcal{F}_t\}$, respectively $\{\mathcal{G}_t\}$, denote the filtration obtained by completing $\{\mathcal{F}_t^1\times\{\emptyset,\Omega_2\}\}$, respectively $\{\mathcal{F}_t^1\times\mathcal{F}_t^2\}$, in the usual way. Let $\{\mathcal{F}_t'\}$ denote the larger filtration obtained by completing $\{\mathcal{F}_1\times\mathcal{F}_t^2\}$. Our primary concern is with the original Brownian motion, B, and hence with the filtration $\{\mathcal{F}_t\}$. We abuse the notation slightly by considering B as a Brownian motion on $(\Omega,\mathcal{F},\mathcal{F}_t,P^x)$. P^x will denote $P^{(x,0)}$ and the underlying measure is understood to be $P = P^0$ unless otherwise indicated.

Define $S^{+'}(c)$ and $S^{-'}(c)$ as before but with B' in place of B. We often write $S^{\pm}(c)(\omega_1)$ for $S^{\pm}(c)(\omega)$ and $S^{\pm'}(c)(\omega_2)$ for $S^{\pm'}(c)(\omega)$, where $\omega = (\omega_1,\omega_2)$ for some $\omega_i \in \Omega_i$.

Lemma 5.1. If $\psi \in \mathcal{H}^\lambda$ (for some $\lambda \geq 0$) then there is a $\emptyset$ in $\mathcal{H}^\lambda$ such that $\lim_{t\to 0^+} \psi(t)\emptyset(t)^{-1} = 0$.

Proof. If $\lambda = 0$, $\emptyset(t) = \psi(t)^{\frac{1}{2}}$ does the job. If $\lambda > 0$, then $(\log (n+1))\psi(2^{-n-1}) \leq (\log n)\psi(2^{-n})$ for large enough n, say $n \geq N$. Define $\emptyset$ on $[0,2^{-N-1}]$ by setting $\emptyset(2^{-n-1}) = (\log n)\psi(2^{-n})$, $\emptyset(0) = 0$, and then extending $\emptyset$ by linear interpolation. □

Theorem 5.2. Let $\{A_n\}$ be a sequence of $(\mathcal{G}_t)$-optional sets such that $A_n(\omega)$ is a.s. closed, and let $A = \bigcup_{n=1}^{\infty} A_n$.

(a) If $\psi \in \mathcal{H}^{\lambda_0(c)}$ $(0 < c < \infty)$, then

$$\{\omega|\psi\text{-}m(A(\omega)) > 0\} \subset \{\omega|A(\omega) \cap S^+(c^+)(\omega) \neq \emptyset\} \quad \text{a.s.}$$

(b) If $\psi \in \mathcal{H}^{\lambda_0(c)+\lambda_0(c')}$, then

$$\{\omega|\psi\text{-}m(A(\omega)) > 0\} \subset \{\omega|A(\omega) \cap S^+(c)(\omega) \cap S^{+'}(c'^+)(\omega) \neq \emptyset$$
$$\text{and } A(\omega) \cap S^+(c^+)(\omega) \cap S^{+'}(c')(\omega) \neq \emptyset\} \quad \text{a.s.}$$

Proof. We only prove (b) as the proof of (a) will then be clear. It suffices to show that

$$\{\omega|\psi\text{-}m(A(\omega)) > 0\} \subset \{\omega|A \cap S^+(c) \cap S^{+'}(c'^+)(\omega) \neq \emptyset\},$$

as the other inclusion follows by symmetry. We may and do assume $\lambda_0(c) + \lambda_0(c') \leq 1$.

We assume first that for each ω, $A(\omega)$ is a closed subset of $[0,1]$ and will show

$$\{\omega|\psi\text{-}m(A(\omega)) = \infty\} \subset \{\omega|A \cap S^+(c) \cap S^{+'}(c'^+)(\omega) \neq \emptyset\}. \tag{5.1}$$

We may, and shall, assume ψ is defined on $[0,1]$. Let $\{c_j|j = 0,1,2,\ldots\}$ decrease strictly to c' (the exact values of $\{c_j\}$ will be specified later) and define g_n on $(0,2^n)$ by

$$g_n(u) = c_j(u+1)^{\frac{1}{2}} \text{ if } u \in [2^{n-j-1}, 2^{n-j}) \quad j = 0,1,2,\ldots$$

Let $U_n = \inf\{t|\ |B'(t)| > g_n(t)\} \wedge 2^n$. Then for $k \in \{0,\ldots,n-1\}$,

$$P(U_n \geq 2^{k+1}|U_n \geq 2^k) = \int_0^\infty P^y(|B_{t-2^k}| \leq c_{n-k-1}(t+1)^{\frac{1}{2}}\ \forall t \in [2^k,2^{k+1}])\, d\mu^{n,k}(y), \tag{5.2}$$

where

$$\mu^{n,k}(dy) = P(|B'(2^k)| \in dy | U_n \geq 2^k).$$

Proposition 3.1 and Lemma 3.3 imply that

$$\text{(5.3)} \qquad \mu^{n,k}([0,y]) \geq P^{{}^n c_{n-k-1}}(|B_{2^k}| \leq y | T(c_{n-k-1}) > 2^k).$$

As the integrand on the right side of (5.2) is non-increasing (use Lemma 3.4), we can use (5.3) and integration by parts to obtain

$$\begin{aligned} P(U_n \geq 2^{k+1} | U_n \geq 2^k) &\geq P^{{}^n c_{n-k-1}}(T(c_{n-k-1}) > 2^{k+1} | T(c_{n-k-1}) > 2^k) \\ &\geq 2^{-\lambda_0(c_{n-k-1})} \quad \text{(by (3.5))}. \end{aligned}$$

A similar, but simpler, argument shows that

$$P(T(c) \geq 2^{k+1} | T(c) \geq 2^k) \geq 2^{-\lambda_0(c)}.$$

Combine these results to see that

$$P(T(c) \wedge U_n \geq 2^{k+1} | T(c) \wedge U_n \geq 2^k) \geq 2^{-\lambda_0(c)-\lambda_0(c_{n-k-1})}, \quad k = 0,\dots,n-1.$$

Equality here would have simplified things. We use random stopping to achieve the same effect. Let $V_n \in \{2^k | k = 0,1,\dots,n\} \cup \{\infty\}$ denote a random variable whose distribution satisfies

$$P(V_n \geq 2^{k+1} | V_n \geq 2^k) = \frac{2^{-\lambda_0(c)-\lambda_0(c_{n-k-1})}}{P(T(c)\wedge U_n \geq 2^{k+1} | T(c)\wedge U_n \geq 2^k)}, \quad k = 0,\dots,n-1$$

$$P(V_n = \infty) = 1 - \sum_{k=0}^{n} P(V_n = 2^k).$$

By enlarging our original space $(\Omega_1, \mathcal{F}_1, \mathcal{F}_t^1, P_1^x)$, if necessary, we may assume there is a sequence of independent $\mathcal{F}_0$-measurable random variables

$\{V_i^n | n \in \mathbb{N}, i \in \mathbb{N}\}$, independent of B and B', such that V_i^n is equal in law to V_n. Let $A' = A \cup [[2,\infty))$, and define $\tilde{g}_n$ on (0,1) by

$$\tilde{g}_n(t) = c_j(t+2^{-n})^{\frac{1}{2}} \text{ if } t \in [2^{-j-1}, 2^{-j}), \quad j = 0,1,2,\ldots$$

If $D_E = \inf\{t \geq 0 \mid (t,\omega) \in E\}$ denotes the début of a set $E \subset [0,\infty)\times\Omega$ define sequences of stopping times as follows:

$$S_1^n = D_{A'}$$

$$T_i^n = \inf\{t > S_i^n \mid |B'_t - B'_{S_i^n}| > \tilde{g}_n(t-S_i^n) \text{ or } |B_t - B_{S_i^n}| > c(t-S_i^n+2^{-n})^{\frac{1}{2}}\}$$

$$\wedge (S_i^n + (V_i^n \wedge 2^n)2^{-n})$$

$$S_{i+1}^n = D_{A' \cap [[T_i^n,\infty))}.$$

A scaling argument shows that the i.i.d. random variables $\{T_i^n - S_i^n | i \in \mathbb{N}\}$ are equal in law to $2^{-n}(U_n \wedge T(c) \wedge V_n)$, where V_n is independent of U_n and T(c). If $W_n = U_n \wedge T(c) \wedge V_n$ then the definition of the law of V_n guarantees that

$$P(W_n \geq 2^{k+1} | W_n \geq 2^k) = 2^{-\lambda_0(c)-\lambda_0(c_{n-k-1})} \text{ for } k \in \{0,\ldots,n-1\}, \text{ giving}$$

$$(5.4) \qquad P(W_n \geq 2^k) = 2^{-k\lambda_0(c) - \sum_{i=n-k}^{n-1} \lambda_0(c_i)} P(U_n \wedge T(c) \geq 1), \quad k = 0,1,\ldots,n.$$

Let $\emptyset(t) = t^{-(\lambda_0(c)+\lambda_0(c'))}\psi(t)$, $d_i = \lambda_0(c') - \lambda_0(c_i) \downarrow 0$, and

$$N_n = \max\{i | S_i^n \leq 1\} \leq N'_n \equiv \min\{m | \sum_{j=1}^{m} (T_j^n - S_j^n) > 1\}.$$

Then

$$E\left(\sum_{i=1}^{N_n}\psi(T_i^n-S_i^n)\right) \le E(N_n')E\left((T_1^n-S_1^n)^{\lambda_0(c)+\lambda_0(c')}\phi(T_1^n-S_1^n)\right)$$

$$\le 2^{1+n}E(W_n)^{-1}E\left(W_n^{\lambda_0(c)+\lambda_0(c')}\phi((W_n2^{-n})\wedge 1)\right)2^{-n(\lambda_0(c)+\lambda_0(c'))}$$

$$\le 2^{n(1-\lambda_0(c)-\lambda_0(c'))+1}\Big[\sum_{k=0}^{n-1}\int I(2^k \le W_n < 2^{k+1})2^{(k+1)(\lambda_0(c)+\lambda_0(c'))}\phi(2^{k+1-n})dP$$

$$+ 2^{n(\lambda_0(c)+\lambda_0(c'))}\phi(1)P(W_n \ge 2^n) + \phi(2^{-n})\Big]$$

$$\div \Big[\sum_{k=0}^{n} P(W_n \ge 2^k)2^{k-1}\Big]$$

$$\le K2^{n(1-\lambda_0(c)-\lambda_0(c'))}\Big[\sum_{k=0}^{n-1} 2^{(k+1)\lambda_0(c')-\sum_{i=n-k-1}^{n-1}\lambda_0(c_i)}\phi(2^{k+1-n})+\phi(2^{-n})\Big]$$

$$\div \Big[\sum_{k=0}^{n} 2^{-k\lambda_0(c)-\sum_{i=n-k}^{n-1}\lambda_0(c_i)+k}\Big] \quad \text{(by (5.4))}$$

$$\le K\Big[\sum_{k=0}^{n} 2^{\sum_{i=n-k}^{n-1} d_i}\phi(2^{-(n-k)})\Big] \div \Big[\sum_{k=0}^{n} 2^{(n-k)(\lambda_0(c)+\lambda_0(c')-1)+\sum_{i=n-k}^{n-1} d_i}\Big]$$

$$\le K\Big[\sum_{\ell=0}^{n} 2^{-\sum_{i=0}^{\ell-1} d_i}\phi(2^{-\ell})\Big] \div \Big[\sum_{\ell=0}^{n} 2^{-\ell(1-\lambda_0(c)-\lambda_0(c'))-\sum_{i=0}^{\ell-1} d_i}\Big]$$

$$\le K\Big[\sum_{\ell=0}^{n} 2^{-\sum_{i=0}^{\ell-1} d_i}\phi(2^{-\ell})\Big].$$

Now choose $c_j \downarrow c'$ so slowly that

$$d_j \ge [2(j+1)^{-1} + \log(\phi(2^{-j-1})\phi(2^{-j})^{-1})](\log 2)^{-1}.$$

This is possible since $\overline{\lim}_{j\to\infty} \emptyset(2^{-j-1})\emptyset(2^{-j})^{-1} \leq 1$. Then

$$2^{-\sum_{i=0}^{\ell-1} d_i} \leq \ell^{-2}\emptyset(1)\emptyset(2^{-\ell})^{-1}$$

and therefore

$$E\left(\sum_{i=1}^{N_n} \psi(T_i^n - S_i^n)\right) \leq K\emptyset(1) \sum_{\ell=0}^{\infty} \ell^{-2} = K_1.$$

By Fatou's lemma we have

$$(5.5) \qquad \underline{\lim}_{n\to\infty} \sum_{i=1}^{N_n} \psi(T_i^n - S_i^n) < \infty \quad \text{a.s.}$$

Fix ω so that (5.5) holds and $\psi\text{-}m(A(\omega)) = \infty$. Choose $n_k \uparrow \infty$ so that $\sum_{i=1}^{N_{n_k}} \psi(T_i^{n_k} - S_i^{n_k})$ remains uniformly bounded in k. As $A(\omega) \subset \bigcup_{i=1}^{N_{n_k}} [S_i^{n_k}, T_i^{n_k}]$, one must have a $\Delta = \Delta(\omega) > 0$ such that

$$\max_{i \leq N_{n_k}} T_i^{n_k} - S_i^{n_k} \geq \Delta \quad \forall k.$$

If $i_k \leq N_{n_k}$ satisfies $T_{i_k}^{n_k} - S_{i_k}^{n_k} \geq \Delta$, then by taking a further subsequence we may assume $S_{i_k}^{n_k} \to t_0$. It is easy to check that

$t_0 \in A(\omega) \cap S^+(c)(\omega) \cap S^{+'}(c'^+)(\omega)$ and the proof of (5.1) is complete for $A(\omega)$ a closed subset of $[0,1]$.

In general we may assume that $A_n \subset [[0,n]]$. By Lemma 5.1 there is a $\tilde{\psi} \in \mathcal{H}^{\lambda_0(c)+\lambda_0(c')}$ satisfying $\lim_{t\to 0^+} \psi(t)\tilde{\psi}(t)^{-1} = 0$. By the above we may fix ω outside a null set so that (5.1) holds for each A_n with $\tilde{\psi}$ in place of ψ. If $\psi\text{-m}(A(\omega)) > 0$ then $\psi\text{-m}(A_n(\omega)) > 0$ for some n and hence $\tilde{\psi}\text{-m}(A_n(\omega)) = \infty$. (5.1) implies that $A_n(\omega) \cap S^+(c)(\omega) \cap S^{+'}(c'^+)(\omega) \neq \emptyset$ and the proof is complete. □

The next result gives sufficient conditions for $A \cap S^+(c^+)(\omega)$ to be a.s. empty.

Theorem 5.3. Let $A \subset [0,\infty)\times\Omega$ be $\mathcal{B}([0,\infty))\times\mathcal{F}$-analytic, and let $c, c' \in (0,\infty)$.

(a) Assume for each $\psi \in \mathcal{H}^{\lambda_0(c)}$ there are $(\mathcal{G}_t)$-stopping times $S_i^n \leq T_i^n$ $(i,n \in \mathbb{N})$ such that $\psi(T_i^n - S_i^n)$ is defined and

$$A \subset \bigcup_{i=1}^{\infty} [[S_i^n, T_i^n]] \text{ for each } n, \text{ and } \lim_{n\to\infty} E\Big(\sum_{i=1}^{\infty} \psi(T_i^n - S_i^n)\Big) = 0. \tag{5.6}$$

Then $A \cap S^+(c^+)(\omega) = \emptyset$ a.s.

(b) Assume for each $\psi \in \mathcal{H}^{\lambda_0(c)+\lambda_0(c')}$ there are $(\mathcal{G}_t)$-stopping times satisfying (5.6).

Then $A \cap S^+(c^+) \cap S^{+'}(c'^+)(\omega) = \emptyset$ a.s.

Proof. It suffices to prove (b) as (a) will follow by simply assuming $c' = \infty$ throughout the argument. If $\{c_j\}_{j\geq 0}$ is a sequence decreasing to c, let

$$S^+(\{c_j\}) = \{(t,\omega) \mid \sup_{0<h\leq 2^{-j}} |B_{t+h} - B_t| h^{-\frac{1}{2}} \leq c_j \quad \text{for } j = 0,1,\ldots\}.$$

We claim it suffices to fix such a sequence $\{c_j\}$ and a corresponding sequence $\{c_j'\}$ decreasing to c' and then show $A \cap S^+(\{c_j\}) \cap S^{+'}(\{c_j'\})(\omega) = \emptyset$ a.s. ($S^{+'}(\{c_j'\})$ is defined using B'). Indeed if $A \cap S^+(c^+) \cap S^{+'}(c'^+)(\omega) \neq \emptyset$ with positive probability, then by the Section Theorem (see [6, p. 64]) there is a random variable $R \in [0,\infty]$ such that $[[R]] \subset A \cap S^+(c^+) \cap S^{+'}(c'^+)$ and

$P(R < \infty) > \varepsilon > 0$. Define random variables X_j and Y_j by

$$X_j = \begin{cases} \sup_{0<h\leq 2^{-j}} (|B_{R+h}-B_R|h^{-\frac{1}{2}}) \vee c, & \text{if } R < \infty \\ c+2^{-j} & \text{if } R = \infty \end{cases}$$

$$Y_j = \begin{cases} \sup_{0<h\leq 2^{-j}} (|B'_{R+h}-B'_R|h^{-\frac{1}{2}}) \vee c' & \text{if } R < \infty \\ c'+2^{-j} & \text{if } R = \infty. \end{cases}$$

Then $\{X_j\}$ and $\{Y_j\}$ decrease a.s. to c and c', respectively. An elementary argument shows there are sequences $\{c_j\}$ and $\{c'_j\}$ decreasing to c and c', respectively such that

$$P(X_j \leq c_j \text{ and } Y_j \leq c'_j \quad \forall j \in \mathbb{N}_0) \geq 1-\varepsilon/2.$$

Therefore

$$P(S^+(\{c_j\}) \cap S^{+'}(\{c'_j\}) \cap A(\omega) \neq \emptyset) \geq P(R < \infty, X_j \leq c_j \text{ and } Y_j \leq c'_j \quad \forall j \in \mathbb{N}_0) \geq \varepsilon/2.$$

This proves the claim.

Now fix sequences $\{c_j\}$ and $\{c'_j\}$ as above and define functions g_n and $\tilde{g}_n$ on $[0,2^n-1)$ by

$$\left.\begin{aligned} g_n(u) &= c_n+c_j(u+1)^{\frac{1}{2}} \\ \tilde{g}_n(u) &= c'_n+c'_j(u+1)^{\frac{1}{2}} \end{aligned}\right\} \text{if } \ 2^{n-j-1}-1 \leq u < 2^{n-j}-1, \quad j = 0,1,\ldots,n-1.$$

Let

$$U_n = \inf\{t \mid |B_t| > g_n(t) \text{ or } |B'_t| > \tilde{g}_n(t)\} \wedge (2^n-1),$$

and write $c_{\sqrt{n}}$ for $c_{[\sqrt{n}]}$. Then

$$
\begin{aligned}
P(U_n = 2^n-1) &\le P(|B_u| \le c_n + c_{\sqrt{n}}(u+1)^{\frac{1}{2}} \quad \forall u < 2^{n-\sqrt{n}}-1) \\
&\quad \times P(|B'_u| \le c'_n + c'_{\sqrt{n}}(u+1)^{\frac{1}{2}} \quad \forall u < 2^{n-\sqrt{n}}-1) \qquad (5.7)\\
&\le K_0(2^{n-\sqrt{n}}-1)^{-\lambda_0(c_{\sqrt{n}})-\lambda_0(c'_{\sqrt{n}})} \le K\, 2^{-(n-\sqrt{n})(\lambda_0(c_{\sqrt{n}})+\lambda_0(c'_{\sqrt{n}}))}.
\end{aligned}
$$

where K is independent of n (by Theorem 3.5). Now define $\tilde{\psi}: [0,1] \to [0,\infty)$ by letting

$$
\begin{aligned}
\tilde{\psi}(2^{-n}) &= 2^{-(n-\sqrt{n})(\lambda_0(c_{\sqrt{n}})+\lambda_0(c'_{\sqrt{n}}))} \qquad n = 1,2,3\ldots \\
\tilde{\psi}(0) &= 0
\end{aligned}
$$

and extending $\tilde{\psi}$ by linear interpolation. As $\tilde{\psi}(2^{-n})$ decreases to 0 as $n \to \infty$, it is clear that $\tilde{\psi} \in \mathcal{H}$. In fact $\tilde{\psi} \in \mathcal{H}^{\lambda_0(c)+\lambda_0(c')}$ because

$$\overline{\lim_{n\to\infty}}\, \tilde{\psi}(2^{-n-1})\tilde{\psi}(2^{-n})^{-1} \le 2^{-\lambda_0(c)-\lambda_0(c')},$$

which is easily verified because $\lambda_0(c_i)$ and $\lambda_0(c'_i)$ increase to $\lambda_0(c)$ and $\lambda_0(c')$ respectively. Let $\psi(t) = \tilde{\psi}(2t) \in \mathcal{H}^{\lambda_0(c)+\lambda_0(c')}$.

Let $\{S_i^n, T_i^n \mid i,n \in \mathbb{N}\}$ be stopping times satisfying (5.6), and let

$$g(t) = c_j t^{\frac{1}{2}} \text{ if } 2^{-j-1} < t \le 2^{-j} \text{ and } \tilde{g}(t) = c'_j t^{\frac{1}{2}} \text{ if } 2^{-j-1} < t \le 2^{-j}.$$

If $t \in [S_i^n(\omega), T_i^n(\omega)] \cap S^+(\{c_j\})(\omega)$, then for each $h \in [0,1-(T_i^n-S_i^n)]$,

$$|B(T_i^n+h)-B(T_i^n)| \le |B(T_i^n+h)-B(t)| + |B(T_i^n)-B(t)|$$

$$\le g(T_i^n+h-t)+g(T_i^n-t)$$

$$\le g(h+T_i^n-S_i^n)+g(T_i^n-S_i^n).$$

Similarly if $t \in [S_i^n, T_i^n] \cap S^{+'}(\{c_j'\})$ then

$$|B'(T_i^n+h)-B'(T_i^n)| \le \tilde{g}(h+T_i^n-S_i^n)+\tilde{g}(T_i^n-S_i^n).$$

Therefore on $\{\omega | T_i^n-S_i^n(\omega) \in (2^{-m-1}, 2^{-m}]\}$ $(m \in \mathbb{N})$ we have

$$P([S_i^n,T_i^n] \cap S^+(\{c_j\}) \cap S^{+'}(\{c_j'\}) \neq \emptyset | \mathcal{F}_{T_i^n})$$

$$\le P(|B(h)| \le g(h+2^{-m})+g(2^{-m}),\ |B'(h)| \le \tilde{g}\,(h+2^{-m})+\tilde{g}(2^{-m}) \quad \forall h \in [0,1-2^{-m}])$$

$$= P(|B(h)| \le g_m(h) \text{ and } |B'(h)| \le \tilde{g}_m(h) \quad \forall h \in [0,2^m-1]) \qquad \text{(scaling)}$$

$$= P(U_m = 2^m-1)$$

$$\le K\,\tilde{\psi}(2^{-m}) \qquad \text{(by (5.7) and the definition of } \tilde{\psi})$$

$$= K\,\psi(2^{-m-1}) \le K\,\psi(T_i^n-S_i^n).$$

It follows that there is a constant K such that

$$P([S_i^n,T_i^n] \cap S^+(\{c_j\}) \cap S^{+'}(\{c_j'\}) \neq \emptyset\ |\mathcal{F}_{T_i^n}) \le K\,\psi(T_i^n-S_i^n),$$

as this is trivial if $T_i^n-S_i^n > 1$. Therefore

$$P(A \cap S^+(\{c_j\}) \cap S^{+'}(\{c_j'\})(\omega) \neq \emptyset) \le E(\sum_{i=1}^{\infty} P([S_i^n,T_i^n] \cap S^+(\{c_j\}) \cap S^{+'}(\{c_j'\}) \neq \emptyset | \mathcal{F}_{T_i^n}))$$

$$\le K\,E(\sum_{i=1}^{\infty} \psi(T_i^n-S_i^n))$$

$$\to 0 \quad \text{as} \quad n \to \infty. \quad \square$$

Theorem 1.1 is now an easy consequence of Theorems 5.2 and 5.3.

Proof of Theorem 1.1. Let A be an analytic subset of $[0,\infty)$. Assume that $\psi\text{-m}(A) > 0$ for some $\psi \in \mathcal{H}^{\lambda_0(c)}$. By Rogers [10, p. 122] there is a compact subset K of A such that $\psi\text{-m}(K) > 0$. Theorem 5.2 now shows that $K \cap S^+(c^+)(\omega) \neq \emptyset$ a.s.

Assume now that $\psi\text{-m}(A) = 0$ for each $\psi \in \mathcal{H}^{\lambda_0(c)}$. The stopping times in (5.6) may now be taken to be constant times. Therefore Theorem 5.3 implies that $A \cap S^+(c^+)(\omega) = \emptyset$ a.s. □

Recall that Z denotes the zero set of B.

Corollary 5.4. (a). $S^+(1^+)(\omega)$ is an uncountable dense subset of $[0,\infty)$ a.s.
(b) $S^+(\lambda_0^{-1}(\frac{1}{2})^+) \cap Z(\omega)$ is dense in $Z(\omega)$ a.s.

Proof. (a) Suppose $S^+(1^+)(\omega)$ is a.s. countable. Then there are random variables $\{R_j | j \in \mathbb{N}\}$ $(0 \leq R_j \leq \infty)$ such that $S^+(1^+)(\omega)$ and $\bigcup_{j=1}^{\infty} [[R_j]]$ are indistinguishable subsets of $[0,\infty)\times\Omega$ (see [6, p. 167]). By Lemma 5.1 there is a $\emptyset \in \mathcal{H}^1$ such that $\lim_{t\to 0^+} t\emptyset(t)^{-1} = 0$. Then $\emptyset$-m is not a σ-finite measure on $[0,1]$ (see Rogers [10, p. 79]) and so there is an uncountable collection of disjoint compact subsets of $[0,1]$, $\{A_i | i \in I\}$, such that $\emptyset\text{-m}(A_i) = \infty$ for each i (see Rogers [10, Theorems 57 and 59]). As

$$\tilde{I} = \{i \in I | P(R_j \in A_i) > 0 \text{ for some } j \in \mathbb{N}\}$$

is a countable set we may choose $i_0 \in I-\tilde{I}$. Therefore we have $P(S^+(1^+)(\omega) \cap A_{i_0}) = 0$, a result that contradicts Theorem 1.1 since $\emptyset\text{-m}(A_{i_0}) = \infty$. It follows that $S^+(1^+) \cap [0,1]$ is uncountable with probability $p > 0$ (here we are using the non-obvious fact that

$\{\omega | S^+(1^+)(\omega) \cap [0,1]$ is uncountable$\} \in \mathcal{F}$ -- see [6, p. 163]). A scaling argument now shows that $S^+(1^+)(\omega) \cap [0,\varepsilon]$ is uncountable with the same probability, p, for each $\varepsilon > 0$. The 0-1 law shows that $p = 1$.

The fact that $S^+(1^+)(\omega)$ is a.s. dense is obvious.

(b) If $\psi(t) = (2\, t \log\log 1/t)^{\frac{1}{2}}$, then by Taylor and Wendel [11], $\psi\text{-m}(Z \cap [0,t]) = L_t^0(B)$ for all $t \geq 0$, a.s., where $L_t^0(B)$ is the local time of B. It follows that if $\emptyset(t) = (t \log 1/t)^{\frac{1}{2}} \in \mathcal{H}^{\frac{1}{2}}$ then w.p. 1 for any rationals $0 \leq r < s$,

$$Z(\omega) \cap [r,s] \neq \emptyset \Rightarrow \emptyset\text{-m}(Z \cap [r,s]) = \infty.$$

Apply Theorem 5.2(a) to the countable collection of optional sets $\{Z \cap [[r,s]] | 0 \leq r < s \text{ rationals}\}$ to conclude that w.p. 1 whenever $0 \leq r < s$ are rationals then

$$Z(\omega) \cap [r,s] \neq \emptyset \Rightarrow Z \cap S^+(\lambda_0^{-1}(1/2)^+)(\omega) \cap [r,s] \neq \emptyset. \quad \square$$

We next use the independent Brownian motion B' to estimate the size of $A(\omega) \cap S^+(c^+)(\omega)$ for certain random sets $A \in \mathcal{B}([0,\infty)) \times \mathcal{F}_1 \times \{\emptyset,\Omega_2\}$. Note this is no real restriction on A as we are interested in the Brownian motion B on the original $(\Omega_1, \mathcal{F}_1, \mathcal{F}_t^1, P_1)$.

<u>Theorem 5.5</u>. Let $c \in (0,\infty)$ and $\alpha \geq \lambda_0(c)$. Let $A \in \mathcal{B}([0,\infty)) \times \mathcal{F}_1 \times \{\emptyset,\Omega_2\}$.

(a) Assume $A = \bigcup_n A_n$ where A_n is $(\mathcal{F}_t)$-optional and $A_n(\omega)$ is a.s. closed. If $\psi \in \mathcal{H}^\alpha$, then $\{\omega | \psi\text{-m}(A(\omega)) > 0\} \subset \{\omega | \emptyset_{\alpha-\lambda_0(c)}(A \cap S^+(c^+)(\omega)) > 0\}$ a.s.

(b) Assume for each $\psi \in \mathcal{H}^\alpha$ there are $(\mathcal{G}_t)$-stopping times $S_i^n \leq T_i^n$ such that $A \subset \bigcup_{i=1}^{\infty} [[S_i^n, T_i^n]]$ for each n and $\lim_{n\to\infty} E(\sum_{i=1}^{\infty} \psi(T_i^n - S_i^n)) = 0$. Then

$$\psi\text{-m}(A(\omega) \cap S^+(c^+)(\omega)) = 0 \quad \forall\psi \in \mathcal{H}^{\alpha-\lambda_0(c)} \quad \text{a.s.}$$

Proof. (a) If $\alpha = \lambda_0(c)$, this is just Theorem 5.2 (a). Assume $\alpha > \lambda_0(c)$ and choose c' so that $\lambda_0(c)+\lambda_0(c') = \alpha$. Theorem 5.2 (b) implies that for all ω_1 outside a P_1^0-null set N_1,

$$\text{(5.8)} \qquad \psi\text{-m}(A(\omega_1,\omega_2)) > 0 \Rightarrow A\cap S^+(c^+)\cap S^{+'}(c')(\omega_1,\omega_2) \neq \emptyset \text{ for } P_2^0\text{-a.a.}\omega_2.$$

The measurability condition on A allows us to write $A(\omega_1)$ for $A(\omega_1,\omega_2)$. Fix $\omega_1 \notin N_1$. If $\psi\text{-m}(A(\omega_1)) > 0$, then (5.8) and Corollary 4.2 imply that $\emptyset_{\lambda_0(c')}\text{-m}(A(\omega_1)\cap S^+(c^+)(\omega_1)) > 0$. Note that in applying Corollary 4.2, $A(\omega_1) \cap S^+(c^+)(\omega_1)$ plays the role of the deterministic set A and we work with the Brownian motion B'. As $\lambda_0(c') = \alpha-\lambda_0(c)$ this completes the proof of (a).

(b) If $\alpha = \lambda_0(c)$, this is immediate from Theorem 5.3 (a). Assume $\lambda_0(c) < \alpha$ and choose c' so that $\lambda_0(c) + \lambda_0(c') = \alpha$. By Theorem 5.3 (b) we may fix ω_1 outside a P_1^0 null set so that

$$A(\omega_1) \cap S^+(c^+)(\omega_1) \cap S^{+'}(c'^+)(\omega_2) = \emptyset \quad \text{for } P_2^0\text{-a.a.}\omega_2.$$

Now apply Theorem 1.1, using $A(\omega_1) \cap S^+(c^+)(\omega_1)$ as our deterministic set and B' as our Brownian motion. We obtain $\psi\text{-m}(A(\omega_1) \cap S^+(c^+)(\omega_1)) = 0$ for <u>every</u> $\psi \in \mathcal{H}^{\lambda_0(c')} = \mathcal{H}^{\alpha-\lambda_0(c)}$. □

An immediate corollary to (a) is

Corollary 5.6. For all $c \geq 1$, $\emptyset_{1-\lambda_0(c)}(S^+(c^+)(\omega)) > 0$ a.s. □

Our final objective is to refine (1.6) on the existence of points which are simultaneously slow from the left and right. Although we have not been able to decide whether or not $S(\lambda_0^{-1}(1/2))(\omega)$ is empty, Theorem 5.9 below settles this question for $S(\lambda_0^{-1}(1/2)^+)(\omega)$ and in fact goes a little farther.

The following result is undoubtedly known in greater generality but as we could not find it in the literature, we include a proof of the simple case we need.

Lemma 5.7. Let $(\Omega,\mathcal{F},P)$ be a probability space and let $\mathcal{F}^*$ denote the universal completion of $\mathcal{F}$. Assume $A_n \in \mathcal{B}([0,\infty))\times\mathcal{F}$ is such that $A_n(\omega)$ is closed for each ω. If $\psi \in \mathcal{H}$ and $A = \bigcup_{n=1}^{\infty} A_n$ then $\{\omega|\psi\text{-}m(A(\omega)) = 0\} \in \mathcal{F}^*$, and if $P(\psi\text{-}m(A(\omega)) = 0) = 1$, there are $\mathcal{F}^*$-measurable random variables $\{S_i^n, T_i^n|i,n \in \mathbb{N}\}$ such that $A \subset \bigcup_{i=1}^{\infty}[[S_i^n,T_i^n]]$ and $\lim_{n\to\infty} E(\sum_{i=1}^{\infty}\psi(T_i^n-S_i^n)) = 0$.

Proof. Assume first that $A(\omega)$ is compact for each ω. Let $\{S_j|j \in \mathbb{N}\}$ be an enumeration of all the finite sets of open intervals with rational endpoints. Define a sequence of functions on Ω by

$$X_j(\omega) = \begin{cases} \sum_{I\in S_j} \psi(|I|) & \text{if } A(\omega) \subset \bigcup_{I\in S_j} I \\ \infty & \text{otherwise.} \end{cases}$$

(Here $|I|$ denotes the length of I.) Then $\{\omega|X_j(\omega) = \infty\}$ is the projection onto Ω of $A\backslash(\bigcup_{I\in S_j} I\times\Omega)$ and hence is in $\mathcal{F}^*$ (see [6, p. 43]). Let $S_j = \{(u_i^j,v_i^j)|i=1,\dots,n_j\}$ and let $J_n = \min\{j|X_j < 2^{-n}\}$ $(\min \emptyset = \infty)$. On $\{\omega|J_n(\omega) < \infty\}$ define

$$S_i^n = \begin{cases} u_i^{J_n} & \text{if } i \le n_{J_n} \\ 1 & \text{if } i > n_{J_n} \end{cases}$$

$$T_i^n = \begin{cases} v_i^{J_n} & \text{if } i \le n_{J_n} \\ 1 & \text{if } i > n_{J_n} \end{cases}$$

and on $\{\omega | J_n(\omega) = \infty\}$, let $(S_i^n, T_i^n) = (0,\infty)$. Then $\{\psi\text{-m}(A(\omega)) = 0\} = \bigcap_{n=1}^{\infty} \{J_n < \infty\} \in \mathscr{F}^*$.

Assume now that $\bigcap_n \{J_n(\omega) < \infty\}$ is a set of probability one. On this set we have for all n

$$\sum_{i=1}^{\infty} \psi(T_i^n - S_i^n) \le 2^{-n}$$

and hence $E(\sum_{i=1}^{\infty} \psi(T_i^n - S_i^n)) \to 0$ as $n \to \infty$. It is clear from their definition that $\{S_i^n, T_i^n\}$ are universally measurable and $A \subset \bigcup_{i=1}^{\infty} [[S_i^n, T_i^n]]$.

In the general case we may assume $A = \bigcup_n A_n$ where $A_n(\omega)$ is compact for each ω. It is now obvious that $\{\psi\text{-m}(A(\omega)) = 0\} \in \mathscr{F}^*$. Assume this set has probability 1. By the above there are universally measurable random variables $\{S_i^{n,m}, T_i^{n,m} | i,n,m \in \mathbb{N}\}$ such that $A_n \subset \bigcup_i [[S_i^{n,m}, T_i^{n,m}]]$ and $E(\sum_{i=1}^{\infty} \psi(|T_i^{n,m} - S_i^{n,m}|)) \le 2^{-n-m}$. One now has $A \subset \bigcup_{i,n} [[S_i^{n,m}, T_i^{n,m}]]$ and

$$\lim_{m\to\infty} E(\textstyle\sum_{i,n} \psi(|T_i^{n,m} - S_i^{n,m}|)) = 0. \quad \square$$

<u>Theorem 5.8.</u> Assume $\{A_n\}$ is a sequence of $\{\mathfrak{F}_t\}$-optional sets such that $A_n(\omega)$ is a.s. closed and let $A = \bigcup_{n=1}^{\infty} A_n$. If $\alpha > \lambda_0(c)$ and there is a $\psi \in \mathcal{H}^{\alpha}$

such that $P(\psi\text{-}m(A(\omega)) > 0) > 0$, then there is a $\emptyset \in \mathcal{H}^{\alpha-\lambda_0(c)}$ such that $P(\emptyset\text{-}m(A(\omega) \cap S^+(c)(\omega)) > 0) > 0$.

Proof. Choose c' so that $\alpha = \lambda_0(c') + \lambda_0(c)$. Theorem 5.2 (b) implies that $A \cap S^+(c) \cap S^{+'}(c'^+)(\omega) \neq \emptyset$ with positive probability. Now apply Theorem 5.3 (a) to the Brownian motion B' on $(\Omega,\mathcal{F},\mathcal{F}_t',P)$, using $A \cap S^+(c)$ as the $\mathcal{B}([0,\infty))\times\mathcal{F}$-analytic set. That result shows there is a

$\emptyset \in \mathcal{H}^{\lambda_0(c')} = \mathcal{H}^{\alpha-\lambda_0(c)}$ and an $\varepsilon > 0$ such that whenever $\{S_i,T_i | i \in \mathbb{N}\}$ are $\mathcal{F}_t'$-stopping times satisfying $A \cap S^+(c) \subset \bigcup_{i=1}^{\infty} [[S_i,T_i]]$ then $E(\sum_{i=1}^{\infty} \emptyset(T_i-S_i)) \geq \varepsilon$. In particular this is true whenever S_i and T_i are $\mathcal{F}_0'$-measurable random variables. Now apply Lemma 5.7 to the $\mathcal{B}([0,\infty))\times\mathcal{F}_0'$ measurable set $A \cap S^+(c)$. Note that the hypotheses of that result are satisfied since we can write $A \cap S^+(c) = \bigcup_{n=1}^{\infty} A_n \cap S^+_{n^{-1}}(c)$. Therefore one sees that $P(\emptyset\text{-}m(A \cap S^+(c)(\omega)) > 0) > 0$. □

Theorem 5.9. If $c,c' \in (0,\infty)$ satisfy $\lambda_0(c) + \lambda_0(c') = 1$, then $S^+(c^+) \cap S^-(c')(\omega) \neq \emptyset$ a.s., and is therefore a.s. dense. In particular with probability 1 there is a $t \geq 0$ such that

$$\overline{\lim_{h\to 0}} |B(t+h)-B(t)|h^{-\frac{1}{2}} = \lambda_0^{-1}(1/2) \quad (\approx 1.3069)$$

but there is no $t \geq 0$ for which

$$\overline{\lim_{h\to 0}} |B(t+h)-B(t)|h^{-\frac{1}{2}} < \lambda_0^{-1}(1/2).$$

Proof. The previous theorem with $A = [0,\infty)$ and $\alpha = 1$ show there is a $\psi \in \mathcal{H}^{1-\lambda_0(c')} = \mathcal{H}^{\lambda_0(c)}$ such that $\psi\text{-}m(S^-(c')(\omega)) > 0$ with positive probability.

Now apply Theorem 5.2 (a) with $A = S^-(c')$ to conclude that $S^-(c') \cap S^+(c^+)(\omega) \neq \emptyset$ with positive probability (note that $A = \bigcup_{n=1}^{\infty} S^-_{n^{-1}}(c')$ so that the hypotheses are met). A scaling argument and the Blumenthal 0-1 law shows this probability must be one. The second statement is immediate from the above and (1.6) (the latter is also an easy consequence of Theorem 4.1 (a)). □

A similar argument yields

Theorem 5.10. If $c, c' \in (0,\infty)$ satisfy $\lambda_0(c) + \lambda_0(c') = 1/2$ then $S^+(c^+) \cap S^-(c') \cap Z(\omega) \neq \emptyset$ a.s. In particular w.p. 1 there exists $t \geq 0$ such that $B(t) = 0$ and

$$\overline{\lim_{h\to 0}} |B(t+h)| h^{-\frac{1}{2}} = \lambda_0^{-1}(1/4) \quad (\approx 1.65\ldots)$$

but there are no t such that $B(t) = 0$ and

$$\overline{\lim_{h\to 0}} |B(t+h)| h^{-\frac{1}{2}} < \lambda_0^{-1}(1/4).$$

Proof.

As $A = Z$ satisfies the hypotheses of Theorem 5.8 with $\alpha = 1/2$, there is a $\psi \in \mathcal{H}^{\frac{1}{2}-\lambda_0(c')}$ such that $P(\psi\text{-}m(Z \cap S^+(c')(\omega)) > 0) > 0$. By reversing B from $\tau_N = \inf\{t | L_t^0(B) > N\}$ we can conclude that $P(\psi\text{-}m(Z \cap S^-(c')(\omega)) > 0) > 0$ (see the proof of Theorem 7 in [9]). An application of Theorem 5.2 (a) now shows that $Z \cap S^-(c') \cap S^+(c^+)(\omega) \neq \emptyset$ with positive probability and hence with probability one. See Proposition 1 of [9] for the estimated value of $\lambda_0^{-1}(1/4)$. □

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