On the Distributions of Sums of Symmetric Random Variables and Vectors

by

Thomas Sellke
Purdue University

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Abstract

Let F be a cumulative probability distribution function on $\mathbb{R}^1$. Then F can be the distribution of a sum X + Y, where X and Y are random variables symmetric about zero, if and only if F has mean zero or F has no (finite or infinite) mean. Also, any distribution in $\mathbb{R}^k$ can arise as the distribution of a sum $\underline{X} + \underline{Y} + \underline{Z}$, where $\underline{X}$, $\underline{Y}$ and $\underline{Z}$ are k-dimensional random vectors whose distributions are spherically symmetric about the origin.

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Thomas Sellke
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Let F be a cumulative distribution function on $\mathbb{R}$. Simons (1976) showed that a necessary condition for F to be the distribution of a sum X + Y, where X and Y are random variables symmetric about 0, is that F either have mean zero or have no mean, finite or infinite. This paper will show that this condition is also sufficient.

An obvious corollary of result is that a sum X + Y of symmetric (about 0) random variables can be symmetrically distributed about $C \neq 0$. An example of such behavior has previously been exhibited by Chen and Shepp (1983). In the Chen-Shepp construction, the summands and the sum all have Cauchy distributions. The Chen-Shepp argument can be generalized to show that, for every positive integer k, there exist k-dimensional Cauchy random vectors $\underline{X}$ and $\underline{Y}$, spherically symmetric about the origin, such that the sum $\underline{X} + \underline{Y}$ has a k-dimensional Cauchy distribution not centered at the origin. It follows easily that any distribution in $\mathbb{R}^k$ can arise from a sum of three random vectors whose distributions are spherically symmetric about the origin.

Theorem 1. Let F be a distribution function on $\mathbb{R}$. There exist random variables X and Y symmetric about zero with X + Y having distribution F if and only if

$$\int_0^\infty x dF(x) = \int_{-\infty}^0 (-x)\, dF(x).$$

Thus, a sum X + Y of symmetric (about zero) random variables can have distribution function F if and only if F has mean 0 or has no mean.

Lemma Any two-point, mean 0 distribution can arise from a sum

$$bU_1 + bU_2,$$

where U_1 and U_2 are both uniformly distributed on $[-1,1]$ and $b > 0$.

Proof of the Lemma.

Let U_1 be uniform on $[-1,1]$, and let $\theta \in [-1,1]$. Define U_2 by

$$U_2 = \begin{cases} 1 + \theta - U_1 & \text{if } U_1 \geq \theta \\ -1 + \theta - U_1 & \text{if } U_1 < \theta. \end{cases}$$

Then U_2 is also uniform on $[-1,1]$, and $U_1 + U_2$ takes on the values $1 + \theta$ and $-1 + \theta$ with probabilities $\frac{1}{2}(1-\theta)$ and $\frac{1}{2}(1+\theta)$, respectively. It follows easily that any two-point, mean zero distribution can be obtained from $bU_1 + bU_2$ if b and θ are chosen properly. Note that $U_1 + U_2$ is identically 0 if we take $\theta = \pm 1$.

Proof of Theorem 1.

In the construction of the Skorokhod representation of a mean 0 random walk, it is shown that any distribution F with mean 0 can be represented as a mixture of mean 0, two-point distributions and the point mass at 0. (See, for example, Freedman (1971), pp. 68-70. The idea seems to originate with Mulholland and Rogers (1958)). This argument also works when F has no (finite or infinite) mean. By the Lemma, any F satisfying the condition of Theorem 1 can arise as the distribution of a sum X + Y, where X and Y are both mixtures of uniform random variables symmetric about 0.

Suppose Z = X + Y, with X and Y symmetric about 0. Define

$$Z^+ = \frac{1}{2}(Z + |Z|)$$

and

$$Z^- = \frac{1}{2}(-Z + |Z|).$$

It will now be shown that $E(Z^+) = E(Z^-)$. Define X_T to equal X truncated at $\pm T$, so that

$$X_T = \begin{cases} -T & \text{if } X < -T \\ X & \text{if } -T \leq X \leq T \\ T & \text{if } X > T \end{cases},$$

and define Y_T similarly. As $T \to \infty$, $(X_T + Y_T)^+$ converges monotonically upward to $(X + Y)^+ = Z^+$, and $(X_T + Y_T)^-$ converges monotonically upward to $(X + Y)^- = Z^-$. By the monotone convergence theorem,

$$E(X_T + Y_T)^+ \to E(Z^+)$$

and

$$E(X_T + Y_T)^- \to E(Z^-)$$

as $T \to \infty$. But X_T and Y_T are, for each $T \in \mathbb{R}$, bounded random variables symmetric about 0. Thus,

$$E(X_T + Y_T) = E(X_T) + E(Y_T) = 0,$$

and

$$E(X_T + Y_T)^+ = E(X_T + Y_T)^-.$$

Since the left side of the last equality converges to $E(Z^+)$ and the right side converges to $E(Z^-)$ as $T \to \infty$, it follows that $E(Z^+) = E(Z^-)$.

Remark 1. The "only if" part of the proof is essentially the same as that given by Simons (1976) and is included here only for completeness.

Remark 2. The symmetric random variables X and Y obtained in the first part of the proof are unimodal and identically distributed. They do not necessarily have means, even when the F distribution has a mean.

Corollary 1. For any distribution F on $\mathbb{R}$, there exist symmetric (about 0) random variables X, Y, and Z such that X + Y + Z has distribution F.

Proof of Corollary 1.

Let W have distribution function F, and let Z be an independent Cauchy random variable symmetric about 0. By Theorem 1, there exist random variables $\tilde{X}$ and $\tilde{Y}$, symmetric about 0, such that $\tilde{X} + \tilde{Y}$ has the same distribution as W - Z. If the probability space on which W and Z are defined is large enough, we can use the conditional distribution of $(\tilde{X},\tilde{Y})$, given the sum $\tilde{X} + \tilde{Y}$, to construct X and Y on the (W,Z) space so that

$$W - Z = X + Y$$

and so that (X,Y) has the same distribution as $(\tilde{X},\tilde{Y})$. Then we have

$$W = X + Y + Z$$

Remark 3. Simons (1977) showed that a sum of three symmetric (about 0) random variables could have a finite but nonzero expectation.

Remark 4. By Remark 2, the random variables X, Y, and Z in Corollary 1 may be chosen to be unimodal. They are not identically distributed as constructed, but if (V_1,V_2,V_3) is defined to be a random permutation of (X,Y,Z), the permutation being independent of X,Y, and Z, then $V_1+V_2+V_3$ has distribution F, and the V_i are symmetric about 0, identically distributed, and unimodal.

Remark 5. Corollary 1 also follows from the Chen-Shepp example. Indeed, the Chen-Shepp example implies that there exist Cauchy random variables X_0, Y_0 and Z_0, symmetric about 0, such that

$$X_0 + Y_0 + Z_0 \equiv 1 .$$

If W is a random variable with distribution F, independent of X_0, Y_0 and Z_0, then we can define

$$X = WX_0, \qquad Y = WY_0, \qquad Z = WZ_0 .$$

The random variables X, Y, and Z are clearly symmetric about 0 and satisfy

$$W = X + Y + Z .$$

Theorem 2 is a generalization of the Chen-Shepp example to multiple dimensions. The theorem and its proof are due to Herman Rubin.

Theorem 2. For any positive integer k, there exist k-dimensional Cauchy random vectors $\underline{X}$ and $\underline{Y}$, symmetrically distributed about the origin, such that the sum $\underline{X} + \underline{Y}$ has a k-dimensional Cauchy distribution which is spherically symmetric about the point $(1,0,\ldots,0)$.

Corollary 2. Any distribution F in $\mathbb{R}^k$ can be attained by a sum of three random vectors whose distributions are spherically symmetric about the origin.

Proof of Corollary 2

By Theorem 2, there exist three k-dimensional Cauchy random vectors $\underline{X}_0$, $\underline{Y}_0$ and $\underline{Z}_0$, symmetric about $\underline{0}$, such that

$$\underline{X}_0 + \underline{Y}_0 + \underline{Z}_0 \equiv (1,0,\ldots,0).$$

Let $\underline{W}$ be a k-dimensional random row vector with distribution F, independent of $\underline{X}_0$, $\underline{Y}_0$, and $\underline{Z}_0$. Let M be a random k×k matrix, also independent of $\underline{X}_0$, $\underline{Y}_0$, and $\underline{Z}_0$, whose first row is $\underline{W}$ and for which $||\underline{W}||^{-1}$ M is an orthogonal matrix when $\underline{W} \neq \underline{0}$. When $\underline{W} = \underline{0}$, set M equal to the k×k matrix of all 0's. If $\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_k\}$ are the coordinate unit vectors in $\mathbb{R}^k$, a suitable M matrix can be constructed by taking the rows of $||\underline{W}||^{-1}$ M to be the orthonormal basis of $\mathbb{R}^k$ obtained by applying the Gram-Schmidt procedure to the spanning sequence $\underline{W}$, $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_k$. If $\underline{X}_0$, $\underline{Y}_0$, and $\underline{Z}_0$ are written as row vectors, then

$$\underline{X} = \underline{X}_0 M \quad , \quad \underline{Y} = \underline{Y}_0 M \quad , \quad \text{and } \underline{Z} = \underline{Z}_0 M$$

are random vectors spherically symmetric about $\underline{0}$, and they satisfy

$$\underline{W} = \underline{X} + \underline{Y} + \underline{Z} \ .$$

Proof of Theorem 2.

In this proof, $\underline{t}$, $\underline{s}$, and $\underline{x}$ will be elements of $\mathbb{R}^k$, and t_1, s_1, and x_1 will be their first coordinates. The scalar product of $\underline{t}$ and $\underline{x}$ will be written as $(\underline{t} \cdot \underline{x})$.

Let $A(x) = I_{\{||\underline{x}|| \leq 1\}}$, and for each $\alpha \in [-1,1]$, define $\varphi(\cdot,\alpha)$ on $\mathbb{R}^k$ by

$$\varphi(\underline{t},\alpha) = \int_{R^k} \frac{e^{i(\underline{t}\cdot\underline{x})} - 1 - iA(\underline{x})(\underline{t}\cdot\underline{x})}{||\underline{x}||^{k+1}} (1 + \alpha \,\mathrm{sgn}(x_1))d\underline{x}.$$

For each α , $\varphi(\cdot,\alpha)$ is the logarithm of the characteristic function (hereafter abreviated log ch.f.) of an infinitely divisible distribution. Indeed, for each $\underline{x}$, the integrand is the log ch.f. of a shifted Poisson random vector with "jumps" of size and direction $\underline{x}$, so that φ is the log ch.f. of a shifted compound Poisson random vector.

Define $\psi(\cdot)$ on $\mathbb{R}^k$ by

$$\psi(\underline{t}) = \int_{\mathbb{R}^k} \frac{e^{i(\underline{t}\cdot\underline{x})} - 1 - iA(\underline{x})(\underline{t}\cdot\underline{x})}{||\underline{x}||^{k+1}} \quad \mathrm{sgn}(x_1)\, d\underline{x} \; .$$

If $c \in \mathbb{R}$, straightforward calculation shows that

$$\varphi(c\underline{t},\alpha) = |c|\, \varphi(\underline{t},0) + c\alpha\psi(\underline{t}) - it_1 k_1 \alpha c \log |c|$$

and that

$$\varphi(\underline{t},0) = -k_2|\underline{t}| \;,$$

for some positive constants k_1 and k_2. The last formula implies that $\varphi(\cdot,0)$ is the log ch.f. of a k-dimensional Cauchy distribution centered at the origin.

Let $\underline{U}$ be a random vector in $\mathbb{R}^k$ with log ch.f. $\varphi(\cdot,\alpha)$. For each $\theta \in [0,2\pi)$, define the k-dimensional random vectors

$$\underline{V}_\theta = (\cos\theta)\underline{U} \quad \text{and} \quad \underline{W}_\theta = (\sin\theta)\underline{U} \; .$$

Then $(\underline{V}_\theta, \underline{W}_\theta)$ is an infinitely divisible 2k-dimensional random vector with log ch.f.

$$\tilde{\varphi}(\underline{t},\underline{s},\theta,\alpha) = \log E[\, \exp\{i(\underline{t}\cdot\underline{V}_\theta) + i(\underline{s}\cdot\underline{W}_\theta)\}] = \varphi(\underline{t}\cos\theta + \underline{s}\sin\theta,\, \alpha).$$

Let $\lambda(\cdot)$ be a measurable function from $[0,2\pi)$ to $[-1,1]$. Taking a "continuous convolution" of the infinitely divisible distributions associated with the $\varphi(\cdot,\cdot,\ ,\lambda(\theta))$'s produces the log ch.f.

$$\zeta(\underline{t},\underline{s},\lambda) = \int_0^{2\pi} \varphi(\underline{t},\underline{s},\theta,\, \lambda(\theta))\, d\theta \; .$$

Let $\underline{X}$ and $\underline{Y}$ be k-dimensional random vectors such that the 2k-dimensional random vector$(\underline{X},\underline{Y})$ has log ch. f. $\zeta(\cdot,\cdot,\lambda)$. Then $\underline{X}$ by itself has log ch. f.

$$\log E[\exp\{i(\underline{t}\cdot\underline{X})\}] = \zeta(\underline{t},\underline{0},\lambda) = \int_0^{2\pi} \varphi(\underline{t}\cos\theta, \lambda(\theta))\, d\theta$$

$$= \int_0^{2\pi} |\cos\theta|\,\varphi(\underline{t},0) + (\cos\theta)\,\lambda(\theta)\,\psi(\underline{t}) - i\, t_1 k_1\, \lambda(\theta)(\cos\theta)\log|\cos\theta|\, d\theta ,$$

and Y has log ch.f.

$$\log E[\exp\{i(\underline{s}\cdot\underline{y})\}] = \zeta(\underline{0},\underline{s},\lambda) = \int_0^{2\pi} \varphi(\underline{s}\sin\theta, \lambda(\theta))\, d\theta$$

$$= \int_0^{2\pi} |\sin\theta|\,\varphi(\underline{s},0) + (\sin\theta)\,\lambda(\theta)\,\psi(\underline{s}) - i\, s_1 k_1\, \lambda(\theta)(\sin\theta)\log|\sin\theta|\, d\theta .$$

The sum $\underline{X} + \underline{Y}$ has log ch.f.

$$\log E[\exp\{i(\underline{t}\cdot(\underline{X}+\underline{Y}))\}] = \zeta(\underline{t},\underline{t},\lambda) = \int_0^{2\pi} \varphi(\underline{t}(\cos\theta+\sin\theta), \lambda(\theta))\, d\theta$$

$$= \int_0^{2\pi} |\cos\theta+\sin\theta|\,\varphi(\underline{t},0) + (\cos\theta+\sin\theta)\,\lambda(\theta)\,\psi(\underline{t})$$

$$- i\, t_1 k_1\, \lambda(\theta)(\cos\theta+\sin\theta)\log|\cos\theta+\sin\theta|\, d\theta .$$

If $\lambda(\cdot)$ is chosen to be orthogonal in $L^2[0,2\pi)$ to $\sin\theta$, $\cos\theta$, $(\sin\theta)\log|\sin\theta|$, and $(\cos\theta)\log|\cos\theta|$, but not to $(\cos\theta+\sin\theta)\log|\cos\theta+\sin\theta|$, then

$$\zeta(\underline{t},\underline{0},\theta) = -k_3|t|, \qquad \zeta(0,\underline{s},\theta) = -k_3|\underline{s}| ,$$

and

$$\zeta(\underline{t},\underline{t},\lambda) = -k_4|\underline{t}| + i\, t_1 k_5,$$

where $k_3 > 0$, $k_4 > 0$, and $k_5 \neq 0$ are constants.

Thus, $\underline{X}$ and $\underline{Y}$ satisfy the conditions in the theorem, except that $\underline{X} + \underline{Y}$ is symmetric about $(k_5,0,\ldots,0)$. The random vectors $k_5^{-1}\,\underline{X}$ and $k_5^{-1}\,\underline{Y}$ are as desired.

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