A WEAK SYSTEM OF AXIOMS FOR "RATIONAL" BEHAVIOR AND THE NON-SEPARABILITY OF UTILITY FROM PRIOR

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ABSTRACT

We give a weak system of consistency axioms for "rational" behavior. The axioms do not even assume the existence of an ordering for axioms. The conclusions are still that utility functions exist, both unconditionally and conditionally given the state of nature; the unconditional utility is a weighted linear combination of the conditional utilities; and the separation of the weights from the conditional scales is not necessary and even the possibility is questioned.

0. Introduction. There have been many axiomatic approaches to "rationality", all of which obtain as a conclusion a Bayesian approach. Some of these, e.g. Ramsey (1926), de Finetti (1937), only consider the "truth" of statements about the state of nature, or about uncertain events. Others, e.g. Rubin (1949a,b), Chernoff (1954), were not influenced by these sources but took an approach based on the simple construction of a cardinal utility function by von Neumann and Morgenstern. Some, such as Savage (1954), were influenced by both sources.

To be as convincing as possible to a non-Bayesian, the axiomatic assumptions should be intuitively reasonable, and as weak as possible. The approach we take is based solely on the notion of choice of action to be taken; while we certainly allow the consideration of hypothetical actions, we do not consider the abstract question of "truth".

Two observations led to the weakening of the axioms and ultimately to this paper. The first, an oral communication from Herman Chernoff, was that there is no way to prevent the decision maker from randomizing, and so the question of whether one action is preferred to another may even be meaningless. We show that this is not so, but we do not assume it away. The second was made by the author many years ago; formally the problem of passing from one state of nature to an unknown state of nature is the same as that of passing from one individual to society. The implications are clear; unfortunately, while the conditions in Arrow (1951) are not satisfied because of the treatment of randomization, the conclusions even get worse; how can one compare the utility scales of two individuals? From that, I came to question the comparison of the utility scales for different states of nature. That this is a problem was already noted by Ramsey; in this article the attempt by de Finetti, and also by Savage, to get around this problem is questioned, and I reject their solution.

The only effect of the rejection of the necessary comparability of utility scales for different states of nature is that one can no longer separate utility from probability, the impact on "rational behavior" is essentially unchanged. I am not at all disturbed by this inability to make this separation; the use of "weights" rather than "personal probabilities" has no observable consequences for the rationalist, and may be more palatable to the user.

Another important observation is that the "prior" is essentially a mathematical consequence of the existence of utility, and does not come from a separate argument.

This does not mean that one is required not to use a separate utility function and prior, or even to add axioms requiring this, but that it is not necessary for coherent behavior to require this, and coherent behavior still remains Bayesian.

1. Utility. We give an axiom system for utility which we feel is substantially weaker than most which have been proposed. The basic concept is that of a choice set. We permit randomization from the beginning; if S is any set, H(S) is the set of all random combinations of elements of the set. We identify an element x of S with the probability distribution assigning probability 1 to x. Implicit in our notation is the assumption that a lottery of lotteries is equivalent to the obvious single lottery. This enables us to use ordinary algebraic notation, and equate finite probability mixtures with finite convex combinations, as is usually done. Formally, we assume

Axiom 1. There is a choice-set function C defined on all subsets of the action space $\mathcal{A}$. For all E, $C(E) \subseteq H(E)$, and if E has 1, 2, or 3 elements, $C(E) \neq \emptyset$.

Axiom 2. If $T \subseteq H(S)$ and $H(T) \cap C(S) \neq \emptyset$, $C(T) = H(T) \cap C(S)$.

Restricting the choice to a smaller set has the obvious consequences.

Axiom 3. If $C(S) \neq \emptyset$ and $0 < a < 1$, $C(aS+(1-a)\{x\}) = aC(S) + (1-a)\{x\}$.

This is the statement that if a choice is given with probability a, the choice made given this information is the same as the free choice.

The remaining two axioms are more technical. The first seems eminently reasonable, and is absolutely necessary for the existence of choice sets if there are an infinite number of distinct choices, such as choosing an integer. The second is essentially an Archimedean axiom, and we shall discuss the consequences of removing it in the appendix.

Axiom 4. *If* $x \in H(S)$ *and for all* $V \subseteq H(S)$, $x \in V$ *and* $C(V) \neq \emptyset$ *implies* $x \in C(V)$, *then* $x \in C(S)$.

In other words, if the possible choice x is not in the choice set of S, it is not in the choice set of some subset of H(S) which has a non-empty choice set.

To state the next axiom, let $C(\{x,y\}) = \{x\}$ and $C(\{y,z\}) = \{y\}$. For $0 < a < 1$, let $u_a = ax + (1-a)z$. Then we need one of the axioms.

Axiom 5k. *If* x,y,z, *and* a *are as above, there is an* $a \in (0.1)$ *such that*

(a) $C(\{y,u_a\}) = \{y\}$

(b) $C(\{y,u_a\})) = u_a$

(c) $y \in C(\{y,u_a\})$

(d) $u_a \in C(\{y,u_a\})$

(e) $\{y,u_a\} \subseteq C(\{y,u_a\})$

It is only necessary to assume one of these forms for all x,y,z as above. The actual use made will be to show that for some a, $C(\{y,u_a\}) = [y,u_a]$, or y is equivalent to u_a. We shall now proceed to prove

Theorem 1. *Given axioms* 1-5, *there is a unique (except for positive linear transformation) "utility function"* U *such that for all sets* E, $C(E) = \{x: x \in H(E)$ *and for all* $y \in E$, $U(x) \geq U(y)\}$.

We prove this in several stages.

Lemma 1. *Using* Axiom 1 *for two element sets*, Axiom 2, *and* Axiom 3, $x \neq y$, $C(\{x,y\})$ *is one of* $\{x\}$, $\{y\}$, $[x,y]$.

First, let $u \neq x$, $u \in C(\{x,y\})$. Then $u \in C(\{x,u\})$. For any $v \in (u,y]$, $u = ax + (1-a)v$, $0 < a < 1$, and hence $v \in C(\{x,v\})$. But also $u \in C(\{x,v\})$, so $v \in C(\{x,y\})$.

Thus if any interior point of $[x,y]$ belongs to $C(\{x,y\})$, all of $[x,y]$ does. If both x and y belong to $C(\{x,y\})$ and $0 < a < 1$, $u = ax + (1-a)y \in C(\{x,u\})$. Since also $x \in C(\{x,u\})$, $u \in (\{x,y\})$.

We define the preference relation $\succ$ by

Definition 1. $x \succsim y$ if $x \in C(\{x,y\})$, $x \approx y$ if $\{x,y\} \subseteq C(\{y,x\})$, $x \prec y$ if $x \notin C\{x,y\}$.

Most treatments of utility start with the assumption that the relation $\succ$ is a complete transitive reflexive relation and define $x \in C(E)$ only for $x \in E$ by $x \in C(E)$ if for all $y \in E$, $x \succsim y$. This assumes that only nonrandomized strategies are to be considered in establishing the choice set and that there is a "preference relation" among all randomized strategies. They then adjoin randomized strategies by closing $C(E)$ under mixtures, whereas we approach things in the other direction.

Lemma 2. The relation $\succsim$ is transitive, i.e., $x \in C(\{x,y\})$ and $y \in C(\{y,z\})$ imply $x \in C(\{x,z\})$.

We need only show $x \in C(\{x,y,z\})$. By the same argument as in Lemma 1, at least one of x,y, and z is an element of $C(\{x,y,z\})$. If $z \in C(\{x,y,z\})$ then since $y \in C(\{y,z\})$, $y \in C(\{x,y,z,\})$. If $y \in C(\{x,y,z,\})$ then since $x \in C(\{x,y\})$, $x \in C(\{x,y,z,\})$, q.e.d.

Lemma 3. For all E, $x \in C(E)$ if and only if $x \in H(E)$ and for all $y \in E$, $x \succsim y$.

First, if $x \in C(E)$ then $x \in H(E)$. If $y \in E$, then $\{x,y\} \cap C(E) \neq \emptyset$, and hence $x \in C(\{x,y\})$, so $x \succsim y$.

Now suppose $x \in H(E)$ and $x \succsim y$ for all $y \in E$. It is easily seen that $x \succsim y$ for all $y \in H(E)$. Let V be any subset of H(E) such that

$x \in V$ and $C(V) \neq \emptyset$. Let $y \in C(V)$. Then $x \in C(\{x,y\})$. Then by Axiom 2, $x \in C(V)$, q.e.d.

Lemma 4. Let $x \succcurlyeq y$ and $y \succcurlyeq z$, and not $z \succcurlyeq x$. Then there exists a unique a, $0 \leq a \leq 1$, such that $y \approx ax + (1-a)z$.

Consider the relation $y \succ bx + (1-b)z$, $0 \leq b \leq 1$. This holds if b=0, and it is easily seen that if it holds for any b it holds for all smaller b. Thus there is a unique a, $0 \leq a \leq 1$ such that $y \succcurlyeq bx + (1-b)z$ holds if $b < a$, and fails if $b > a$. We wish to show that $y \approx ax + (1-a)z$. This requires the use of Axiom 5. We require that a fixed form holds for all triples.

Before doing this, let us observe some geometrical consequences of the axioms, in particular of Axiom 3 and the fact that partial ordering is a connected transitive relation. In particular, if we fix x then the transformation

$$y' = \lambda y + (1-\lambda)x$$

preserves $\succcurlyeq$ if λ is positive and interchanges $\succcurlyeq$ and $\preccurlyeq$ if λ is negative, provided the transformed points correspond to actions. This enables us to assume that $a \neq 0$ and $a \neq 1$, and that if one undesirable inequality confronts us, we can reverse that inequality.

Let us now return to the proof. Form (5e) of the axiom gives us the result immediately. Otherwise, assume $y \not\approx v = ax + (1-a)$.

If $y \succ v$, then $x \succ y$ and $y \succ v$. (5a) or (5c) immediately make $y \succ u$ for some $u \in (v,x)$; however this contradicts the definition of a. If $y \nsucc v$, then $v \succ y$ and $y \succ z$. However, we can find an affine transformation yielding y', v, and z' such that $z' \succ y'$ and $y' \succ v'$. Then (5a) or (5c) immediately make $y' \succ u'$ for some $u' \varepsilon (z',v')$. Then the preimage $u \varepsilon (z,v)$ and $y \prec u$; however this also contradicts the definition of a. A similar argument holds for (5b) or (5d), reversing the order.

We can now define a utility function. Let x and y be given, $x \succ y$. Let $\xi > \eta$ be real numbers. By Lemma 4, for any z there is a unique a_z such that $z \approx a_z x + (1-a_z)y$, where this is interpreted by the usual rules of algebra if $a_z < 0$ or $a_z > 1$. It can be shown that $a_{\lambda z+(1-\lambda)w} = \lambda a_z + (1-\lambda)a_w$. Define $U(z|x,y,\xi,\eta) = a_z\xi + (1-a_z)\eta$. Then it is easily seen that

Lemma 6: For any $x \succ y$, $\xi > \eta$ the function $U(z|x,y,\xi,\eta)$ is a convex-linear mapping of A into the reals, and $z \succcurlyeq w$ if and only if $U(z|(x,y,\xi,\eta) \geq U(w|x,y,\xi,\eta)$. Furthermore, if $U(u|x,y,\xi,\eta) = \mu$, $U(v|x,y,\xi,\eta) = \nu$, $\mu > \nu$, then $U(z|x,y,\xi,\eta) = U(z|u,v,\mu,\nu)$.

Lemma 7: If $x \succ y$, $\xi > \eta$, $\xi' > \eta'$, then there exist σ, τ such that $U(z|x,y,\xi',y') = \sigma\, U(z|x,y,\xi,\eta) + \tau$, $\sigma > 0$.

Let us now prove the theorem. By Lemma 6 and the properties of $\succcurlyeq$, if we define $Z = \{x\colon\ x \varepsilon H(E) \text{ and for all } y \varepsilon E,\ U(x) \geq U(y)\}$, Z must be a subset of C(E). If $C(E) \neq \emptyset$, Axiom 2 then tells us $Z = C(E)$. Now let $V \subseteq E$, $Z \cap H(V) \neq \emptyset$, and $C(V) \neq \emptyset$. Then the same argument shows that $C(V) = Z \cap H(V)$. Hence the theorem follows from Axiom 4.

2. Choice with many "states". We assume that there is a set Ω such that the choice problem is envisioned for each $\omega \varepsilon \Omega$ and also an overall choice problem is considered. Two immediate examples come to mind: Ω may be the class of states of nature, or Ω may be the set of all individuals in a population. Suppose we assume that the choice process given ω is "reasonable"

for each $\omega \varepsilon \Omega$, and the overall process is also reasonable. We will need an additional "obvious" condition.

We assume that there are choice sets C, and for each $\omega \varepsilon \Omega, D_\omega$, such that C and all D_ω satisfy the assumptions of the preceding section. We also assume that if one action is at least as good as another for all ω, it is at least as good, i.e.,

Axiom 0: If $x \varepsilon D_\omega \{x,y\}$ for all $\omega \varepsilon \Omega$, $x \varepsilon C\{x,y\}$.

By Theorem 1, there exist utility functions U and V_ω, $\omega \varepsilon \Omega$, corresponding to the choice sets C, D_ω. Axiom 0 then yields

Lemma 8: If $V_\omega(x) \geq V_\omega(y)$ for all ω, $U(x) \geq U(y)$,

Corollary 9: If $V_\omega(x) = V_\omega(y)$ for all ω, $U(x) = U(y)$.

Let us define $v_x(\omega) = V_\omega(x)$. Then Corollary 9 states that U(x) is determined by the function v_x; i.e., there is a functional Φ such that for all actions x, $U(x) = \Phi(v_x)$. Now lemma 8 states that if $f = \omega_x$, $g = v_y$, $f \geq g$ implies $\Phi(f) \geq \Phi(y)$. Also from the convex-linear properties of utility, $\Phi(\alpha f + (1-\alpha)g) = \alpha\Phi(f) + (1-\alpha)\Phi(g)$. This implies that

Lemma 10: If $\{f_1,f_2,g_1,g_2\} \subseteq \mathcal{D}(\Phi)$ and $f_1-f_2 = g_1-g_2$, then $U(f_1)-U(f_2) = U(g_1) - U(g_2)$.

This holds because $\frac{1}{2} f_1 + \frac{1}{2} g_2 = \frac{1}{2} f_2 + \frac{1}{2} g_1$ and we can then use the linear-convexity of Φ.

Let us now define, for $f,g \varepsilon \mathcal{D}(\Phi)$, $\Psi(f-g) = \Phi(f)-\Phi(g)$. Then

Lemma 11: Ψ is a well-defined function on $\{f-g: f,g \varepsilon \mathcal{D}(\Phi)\}$; also Ψ is linear-convex and for all α with $|\alpha| \leq 1$, $\Psi(\alpha h) = \alpha\Psi(h)$, and if $h \geq 0$, $\Psi(h) \geq 0$.

Lemma 10 gives the uniqueness, and Lemma 8 the positivity. The rest follows from the linear-convexity of Φ.

We may also extend Ψ to all multiples of elements in its domain. We also call this extension Ψ. Then

Lemma 12: Ψ is a positive linear functional on a linear space.

Since "positive linear functional" is another term for "finitely additive integral", we have established the following.

Theorem 2: Under the assumptions of this section, "rational" choice is equivalent to maximizing a utility function, and the difference of the utilities of two actions is a finitely additive integral of the difference of their utility functions as functions on Ω.

3. Scaling and the existence of a "prior". We have already observed that utility U can be replaced by $U' = \sigma U + \tau$. Similarly V_ω can be replaced by $\rho_\omega V_\omega + \lambda_\omega$. Then $\Psi(f) = \frac{1}{\sigma}\Psi'(\rho\cdot f)$. In the finitely additive case, an integral need not correspond to a measure. For example, suppose Ψ is the mean with respect to the "diffuse" prior on the integers, and let $\rho_\omega = \omega$. Then $\Psi'(h)$ is the mean of $\frac{h(\omega)}{\omega}$. If this is not 0, the function h is unbounded; yet the finitely additive integral can be well-defined for all those h for which $\frac{h(\omega)}{\omega}$ is bounded. In the countably additive case, the integral still may be the integral with respect to a measure, but this measure need not be finite. Thus the problem of the meaning of "prior" is raised.

In the second situation in section 2, we are considering the problem of a so-called "social welfare function". It is extremely unclear how one could possibly compare the utility scales of two individuals. Our axioms are not quite compatible with Arrow's [1],but our conclusions are even stronger -- a social welfare function must be dictatorial if it corresponds to "rational" behavior and can be computed from the individuals' orderings only.

In the case of unknown state of nature, there is a <u>conceivable</u> way of establishing a separation between the utility scale and the measure scale. To show why it is necessary to be so finicky, let us consider a typical argument about establishing a posterior. The usual method is to say that a set $\Theta \subseteq \Omega$ has prior probability p iff for ε small (positive or negative) the amount to pay to receive ε if $\omega \in \Theta$ is $\varepsilon p_\varepsilon$, and p_ε approaches p as $\varepsilon \to 0$. To see why this is not so, suppose that for $\omega \in \Theta$ there is high inflation and for $\omega \notin \Theta$ there is no inflation. Clearly, the equating of equal amounts of money with equal changes in utility does not seem reasonable.

In an earlier unpublished report of the author (with stronger axioms), it was assumed that there are two non-equivalent actions which lead to the <u>same history of the universe</u> independent of ω. Since the state ω is usually highly associated with the history, and may in fact even have some components determined by it, this gives some difficulty. This is not an absolute difficulty, as the action space may contain purely hypothetical elements, but the assumption no longer convinces this author. In this case we can use these two actions to normalize the utility function and obtain a prior. If we further assume that only the history of the universe affects the utility, then the integral is the integral with respect to that prior.

<u>4. Appendix</u>. (1) If we assume that the action space is closed under countable combinations, then we can show that the corresponding utility is bounded.

(2) We can ascertain what happens if Axiom 5 is violated.

<u>Theorem 3</u>: <u>From Axioms 1-4 there exists an ordered set</u> Q <u>of utility functions, such that for any actions</u> x <u>and</u> y, $x \approx y$ <u>if all</u> $q \in Q$ <u>have</u> $q(x) = q(y)$, <u>and if</u> $x \not\approx y$, <u>the first</u> $q \in Q$ <u>with</u> $q(x) \neq q(y)$ <u>determines whether</u> $x \succ y$ <u>or</u> $y \succ x$.

Proof: Consider pairs of pairs of actions $x_1 \prec x_2$, $y_1 \prec y_2$. Define $z_{ij} = \frac{1}{2} x_i + \frac{1}{2} y_j$. Then $z_{11} \prec z_{12}, z_{21} \prec z_{22}$. Then if we consider mixtures of z_{11} and z_{22}, there is a cut α such that if $0 \leq \beta < \alpha$, $(1-\beta)z_{11}+\beta z_{22} \prec z_{12}$, and if $\alpha < \beta \leq 1$, the reverse ordering holds. If $0 < \alpha < 1$, the differences $x_2 - x_1$ and $y_2 - y_1$ are <u>comparable</u>; if $\alpha = 0$ $x_2 - x_1$ <u>is infinitely greater than</u> $y_2 - y_1$. Let $x_2 \succ x_1$; we can set up a scale of all differences comparable to, or infinitesimal compared to, $x_2 - x_1$. This gives a utility difference scale to those differences. Let D be the collection of all comparability classes; we then have a set of functions Δ such that for each $d \in D$ there is a $\delta \in \Delta$ with δ giving a utility difference scale on all intervals comparable to or infinitestmal compared to an element of d. It can be shown by use of the axiom of choice that there can be constructed a set of linear-convex functions Q corresponding to elements of D such that if $x \succ y$ and $x-y \in d$ then (a) if $d' > d$, $q_{d'}(x) = q_{d'}(y)$ and (b) if δ corresponds to d, $\delta(x-y) = q_d(x) - q_d(y)$. This establishes the theorem.

If the action space is also closed under countable combinations, Q is well-ordered.

It is not clear how to extend this to the multiple state case, as a given overall utility may involve different levels of individual utility functions, and we may even be involved with infinitestimal sets of states with one level of utility comparable to large sets of states at a lower level.

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