A CLASS OF MINIMAX ESTIMATORS OF A NORMAL QUANTILE

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I. Introduction

Let $X_1, \ldots, X_n$, $n \geq 2$, be a random sample from a normal population with unknown mean ξ and unknown variance σ^2. We shall be concerned with the statistical estimation of the parametric function $\theta = \xi + b\sigma$ where b is a given constant. Clearly θ is a p-quantile of the normal distribution if $p = \Phi(b)$. We assume that $(\delta-\theta)^2 \sigma^{-2}$ is taken to be the loss function, so that the loss is invariant under location and scale transformations.

The problem of estimating θ is invariant under the affine group and there exists the best equivariant estimator δ_0 which is minimax. If $b = 0$, (i.e. median ξ is to be estimated) then δ_0 clearly coincides with the sample mean and is admissible. (The latter fact follows from the admissibility of δ_0 for each fixed value of σ and the independence of δ_0 of this value). Moreover the admissibility of the sample mean in this problem is a characteristic property of the normal law (cf. Kagan and Zinger, 1973). The situation is different when $b \neq 0$. Zidek (1971) had established the inadmissibility of δ_0 for $b \neq 0$ by exhibiting an estimator which improves upon δ_0. However Zidek's procedure coincides with δ_0 outside of a compact subset of the sample space and its relative improvement over δ_0 is small.

The inadmissibility of the "standard" estimator δ_0 of some quantiles of exponential distribution has been demonstrated by Rukhin and Strawderman (1982). Their minimax procedures also coincide with δ_0 with positive probability, and the general conditions on minimax estimators which would guarantee that they are generalized Bayes rules or are admissible remain

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Quantile Estimation

ABSTRACT

The estimation problem of the quantiles of a normal distribution with both parameters unknown, is considered. We construct a class of minimax procedures each of which improves upon the traditional (best equivariant) estimator of a quantile different from the median. For this purpose a differential inequality is introduced and a family of its solutions is found.

Key word: normal quantiles, estimation, quadratic loss, minimaxness, equivariant procedures, admissibility, differential inequality.

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unknown in quantile estimation problem.

In this paper for $b \neq 0$ we construct a class of minimax procedures each of which improves upon the best equivariant estimator δ_0 of the normal quantile $\xi + b\sigma$. These estimators are different from δ_0 with probability one, and some of them possibly are generalized Bayes rules and could be admissible.

2. Minimax Estimators of a Quantile

Let $X = \sum_1^n X_j/n$ and $Y^2 = \sum_1^n (X_j - X)^2$ be sufficient statistics for ξ and σ. If $\delta(X,Y)$ is an estimator which depends only on X and Y and is equivariant under the affine group then $\delta(pX + r, pY) = p\delta(X,Y) + r$ for all positive p and all real r. Therefore δ must be of the form, $\delta(X,Y) = X + \gamma Y$ for some real constant γ. The examination of the risk of δ (which is independent of ξ and σ) reveals that the best choice of γ is $\gamma = c$ where

$$c = b\,\Gamma(n/2)/(2^{1/2}\Gamma((n+1)/2)) = ba.$$

Thus the best equivariant estimator has the form $\delta_0(X,Y)=X+abY$ with

$$a = \Gamma(n/2)/(2^{1/2}\,\Gamma((n+1)/2)). \qquad (1)$$

It is easy to see that if $\delta(X,Y)$ is a minimax procedure, then $-\delta(-X,Y)$ is a minimax estimator of $\xi - b\sigma$. Therefore it suffices to obtain minimax estimators of guantiles $\xi + b\sigma$ of order larger than 1/2, i.e. for positive b. Thus throughout this paper without loss of generality we assume that b is a positive constant.

Following Zidek (1971) we consider "shrinkage" estimators of the form

$$\delta(X,Y) = \delta_0(X,Y) - 2cYh(n^{1/2}X/Y). \qquad (2)$$

Here h is a bounded continuous nonnegative function which is differentiable almost everywhere.

Theorem 1. Estimator δ of the form (2) is minimax if there exists a differentiable bounded function g such that for all z under condition $1 - g(z)/2 > h(z) > 0$ the following differential inequality holds:

$$(2-g(z)-2h(z))\ [ng(z)+\{h(z)(2/(cn^{1/2}) - zg(z))\}'/h(z)] \geq a^{-2}. \quad (3)$$

Here a is defined by (1) and ' denotes the differentiation with respect to z.

Proof. Because of the scale equivariance of procedures (2) it suffices to compare their risk function with that of δ_0 for $\sigma = 1$ only. One has

$$\Delta = E_\xi\ (X+cY-\xi-b)^2 - E_\xi\ (X+cY-2cYh-\xi-b)^2$$
$$= 4cE_\xi\ y(X+cY-cYh-\xi-b)h,$$

so that δ is minimax if $\Delta \geq 0$ for all ξ. To evaluate $E_\xi Y(X-\xi)h$ we integrate by parts

$$\int(x-\xi)h(n^{1/2}x/y)\ \exp\{-\ n(x-\xi)^2/2\}\,dx$$
$$= \int h'(n^{1/2}x/y)\ \exp\{-\ n(x-\xi)^2/2\}\,dx/(n^{1/2}y).$$

Thus

$$\Delta = 4cE_\xi\ [Y(cY-cYh-b)h + h'n^{-1/2}]\ .$$

A sufficient condition for the non-negativity of Δ is

$$\int_0^\infty y^{n-2}\ \exp\{-y^2/2\}\ [cy^2h(u/y)(1-h(u/y)) - byh(u/y)$$
$$+ h'(u/y)n^{-1/2}]\ dy \geq 0 \quad (4)$$

for all real u.

Since for any bounded differentiable function k

$$\int_0^\infty k(u/y)y^n\ \exp\{-y^2/2\}\,dy$$
$$=\int_0^\infty [(n-1)k(u/y) - (u/y)k'(u/y)]y^{n-2}\ \exp\{-y^2/2\}\,dy$$

one can rewrite (4) in the following form

$$\int_0^\infty y^{n-2}\ \exp\{-y^2/2\}\ [(c(1-h(u/y))h(u/y) - k(u/y))y^2$$
$$- byh(u/y) + h'(u/y)n^{-1/2} + (n-1)k(u/y) - (u/y)k'(u/y)]dy \geq 0.$$

The latter integral is nonnegative for all u if the integrand quadratic form in y is nonnegative, i.e. if for all real z

$$c(1-h(z))h(z) - k(z) \geq 0$$

and

$$4(c(1-h(z))h(z) - k(z))(h'(z)n^{-1/2}+(n-1)k(z) - zk'(z)) \geq b^2h^2(z). \quad (5)$$

Now let $k(z) = ch(z)g(z)/2$ for some bounded differentiable function g. Then (5) takes the form with a defined by (1)

$$(2-2h-g)\ [ng + (h(2/(cn^{1/2}) - zg))'/h] \geq a^{-2}$$

for all values of the argument z such that $1-g(z)/2 > h(z) > 0$, and Theorem 1 is proved.

Corollary. Let I denote the interval,

$$I = \{z: |z - 2n^{1/2}a/b\ | < 2(na^2-1)^{1/2} / b\}.$$

(Because of the properties of the gamma-function $a^2 > n^{-1}$, so that I is non-empty). Assume that h vanishes outside of I and within I

$$h(z) \leq 1 - (cn^{1/2}z)^{-1} - cz/(4a^2n^{1/2}).$$

Then the corresponding estimator (2) is minimax. To prove this Corollary just put in (3) $g(z) = 2/(cn^{1/2}z)$.

Notice that the minimaxity of Zidek's estimator which corresponds to

$$h(z) = (1 - (cn^{1/2}z)^{-1} - cz/(4a^2n^{1/2}))/2,\ z \in I\ ,$$ follows from our Corollary.

Theorem 1 shows that any positive solution h of the differential inequality (3) provides a minimax procedure δ. The use of similar differential inequalities in multiparameter estimation problems started by Stein (1973) has been developed by many authors (see Brown, 1979 and Berger, 1980).

As was mentioned, strictly positive solutions h are of interest. To find such solutions of (3) it is convenient to view g as independent variable

$g_1 < g < g_2$, and let z be a one-to-one function of g. In this case (3) can be rewritten in the following form

$$(2-g-2h)\ [ng + \{h(2/(cn^{1/2}) - zg)\}'/(z'h)] \geq a^{-2}. \qquad (6)$$

Here h = h(g) and the differentiation is understood with respect to g.

To obtain strictly positive solutions h of (6) we need the following notation.

Let

$$g_1 = 1 - (1-1/(na^2))^{1/2},\ g_2 = 2 - g_1,$$

so that

$$-g^2 + 2g - 1/(na^2) = (g - g_1)(g_2 - g).$$

Also denote by g_0 the largest root of the equation

$$(1 - g)(-g^2 + 2g - 1/(na^2))$$
$$= (n-1)\ [g^2 - 2g + 1/((n-1)a^2)]\ (1/(na^2 g^2) - 1),$$

which does not exceed $(na^2)^{-1/2}$. Thus for $g_0 < g < (na^2)^{-1/2}$

$$(1- g)(-g^2 + 2g - 1/na^2))$$
$$> (n-1)\ [g^2 - 2g + 1/((n-1)a^2)]\ (1/(na^2g^2) - 1). \qquad (7)$$

Theorem 2. Let $\hat{g}$ be any number under condition $g_0 < \hat{g} < (na^2)^{-1/2}$. Put

$$A = (1-\hat{g})\hat{g}[n(1/(na^2\hat{g}^2) - 1)(\hat{g}-g_1)(g_2-\hat{g})]^{-1},$$

and let $g = g(z)$, $g_1 < g < g_2$, be defined as the solution of the equation

$$z = (2/(cn^{1/2}))(1/g - (g-\hat{g})\ [A(g-g_1)(g_2-g)]^{-1}).$$

Any estimator (2) with h of the form $h = h(g) = (g-g_1)(g_2-g)f/(2g)$, where $f = f(g)$ is a positive unimodal differentiable function with the maximum at $\hat{g}$,

$$f(\hat{g}) \leq 1 - (na^2)^{-1}/[(\hat{g}-g_1)(g_2-\hat{g})(n-1 + (1-\hat{g})(1/na^2\hat{g}^2) -1)^{-1}],$$

is minimax.

Proof. To prove Theorem 2 we verify inequality (6). One has

$$h(2/(cn^{1/2}) - zg) = (g-\hat{g})f/(cn^{1/2}A).$$

and

$$z' = dz/dg$$

$$= -(2/(cn^{1/2}))\ [1/g^2 + (g^2-2\hat{g}g + 2\hat{g} - 1/(na^2))A^{-1}(g-g_1)^{-2}(g_2-g)^{-2}].$$

Inequality (6) can be rewritten in the following form

$$(2g - g^2 - (g-g_1)(g_2-g)f)$$

$$\times [1 + 2((g-\hat{g})f)'/(cn^{1/2}A(g-g_1)(g_2-g)Z'f)] \geq (na^2)^{-1}. \qquad (8)$$

Now we replace z' by its expression above and obtain

$$[(g-g_1)(g_2-g) + 1/(na^2) - (g-g_1)(g_2-g)f]$$

$$\times \{1 - ((g-\hat{g})f)'(g-g_1)(g_2-g)/nf$$

$$\times [A(g-g_1)^2(g_2-g^2g^{-2} + g^2 - 2\hat{g}g + 2\hat{g} - 1/(na^2)]^{-1}\} \geq 1/(na^2).$$

This inequality can be reduced to the following form after cancelling the (positive) factor $(g-g_1)(g_2-g)$

$$(1-f)n[A(g-g_1)^2(g_2-g)^2g^{-2} + g^2 - 2\hat{g}g + 2\hat{g} - 1/(na^2)]$$

$$\geq [(g-g_1)(g_2-g)(1-f) + 1/(na^2)]\ \overline{((g-\hat{g})f)'}/f.$$

Here $((g-\hat{g})f)' = d((g-\hat{g})f)/dg \leq f$ because of the assumed unimodality of the function f. Therefore it suffices to solve the inequality

$$(1-f)n[A(g-g_1)^2(g_2-g)^2g^{-2} + g^2 - 2\hat{g}g + 2\hat{g} - 1/(na^2)]$$

$$\geq (g-g_1)(g_2-g)(1-f) + 1/(na^2)$$

or

$$1 - f \geq (na)^{-2}$$

$$\times [A(g-g_1)^2(g_2-g)^2g^{-2} + g^2 - 2\hat{g}g + 2\hat{g} - 1/(na^2) - (g-g_1)(g_2-g)n^{-1}] \qquad (9)$$

Notice that because of the choice of the constant A the function in the right-hand side of (9) attains its maximum at $g = \hat{g}$, which is also the point of minimum of the function $1 - f$. Therefore (9) holds for all $g, g_1 < g < g_2$, if

$$f(\hat{g}) \leq 1 - (na)^{-2}$$
$$\times [A(g-g_1)^2(g_2-\hat{g})^2\hat{g}^{-2} - \hat{g}^2 + 2\hat{g} - 1/(na^2) - (\hat{g}-g_1)(g_2-\hat{g})n^{-1}]$$
$$= 1 - (na^2)^{-1}$$
$$\times [(\hat{g}-g_1)(g_2-\hat{g})(n-1 + (1-\hat{g})(1/(na^2\hat{g}^2) - 1)^{-1})]^{-1}.$$

Because of (7) f is a positive solution of (8) and Theorem 2 is proved.

3. Sufficient Condition for Inadmissibility

A slight modification of Theorem 1 gives a sufficient condition for the inadmissibility of estimators (2). Indeed let now

$$\delta_0(X,Y) = X + cY - cYh_0(n^{1/2}X/Y) \tag{10}$$

with some continuous function h_0. If $\delta(X,Y)$ is defined by (2) then the comparison of the risk functions of δ_0 and δ shows that the following is true.

Theorem 3. Estimator (10) is inadmissible if there exist a nonnegative continuous bounded function h which is differentiable almost everywhere and a bounded differentiable function g such that for all z under the condition $1-h_0(z) - g(z)/2 > h(z) > 0$ the following differential inequality holds:

$$(2(1-h_0-h)-g)[ng + \{h(2/(cn^{1/2}) - zg)\}'/h] \geq a^{-2}.$$

Notice that in this Theorem as well as in Theorem 1 it suffices to define function g only on the set $H = \{z: h(z) > 0\}$.

Corollary 1. Estimator (9) is inadmissible if for any positive z

$$h_0(z) < 1 - 1/(cn^{1/2}z) - cz/(4n^{1/2}a^2).$$

In particular procedure (9) is inadmissible if

$$h_0(2/b) < 1 - (a^2 n)^{-1/2} .$$

To prove this Corollary put

$$h(z) = \max[0, 1-h_0(z) - 1/(cn^{1/2}z) - cz/(4n^{1/2}a^2)], \ z > 0$$

$$h(z) = 0, \ z < 0$$

and let $g(z) = 2/(cn^{1/2}z)$, $z \in H$. Then h is a nonnegative continuous function which is differentiable almost everywhere and which does not vanish on the set H of positive Lebesque measure. Thus the resulting estimator (10) is strictly better that δ_0.

Corollary 2. Estimator (9) is inadmissible if for all z

$$h_0(z) \leq 0.$$

This Corollary follows from Theorem 3 with functions h and g defined as in Theorem 2.

Corollary 1 of Theorem 3 can be used to establish the inadmissibility of some generalized Bayes procedures which correspond to (improper) prior densities

$$\lambda(\xi,\sigma) = \exp\{-\xi^2/(2\tau^2\sigma^2)\}\sigma^{-\alpha} .$$

These procedures are of the form

$$\delta_0(X,Y) = X\tau^2/(1+\tau^2) + dbY[1+nX^2Y^{-2}/(1+\tau^2)]^{1/2},$$

where

$$d = \Gamma((n+\alpha)/2)/(2^{1/2}\Gamma((n+\alpha+1)/2)).$$

Thus

$$h_0(z) = [1-\rho(1+z^2/(1+\tau^2))^{1/2} + z/(n^{1/2}(1+\tau^2))]/2$$

with $\rho = d/a$.

According to Corollary 1, $\delta_0(X,Y)$ is inadmissible if

$$1 - \rho[1+4/(b^2(1+\tau^2))]^{1/2} + 2/(bn^{1/2}(1+\tau^2))$$
$$\leq 1 - (na^2)^{-1/2}.$$

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