TRANSVERSING THE SMALLEST POSSIBLE CORRIDOR UNDER ENERGY AND VELOCITY CONSTRAINTS

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ABSTRACT

For $\eta > 0$ a fixed constant minimize the functional

$$\tilde{\rho}_\eta(f) = \sup_{a \le x \le b} |f(x)|^2 + \eta \int_a^c (Lf(x))^2 dx$$

over f in the Sobolev space $W_m^2[a,c]$ which satisfy $f(c) = 1$, where $a < b < c$ and $L = D^m + a_{m-1}D^{m-1} + \ldots + a_0 I$ are fixed. There is always a solution to this problem and if the m-dimensional null space of L is spanned by a Chebyshev system on [a,c] then the solution is unique and equioscillates on [a,b]. An explicit solution is given for $L = D^2$ and is shown to yield the solution to a control problem described below. A solution to the $\tilde{\rho}_\eta$ problem is conjectured.

Running Title: CORRIDOR PROBLEM

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TRAVERSING THE SMALLEST POSSIBLE CORRIDOR

UNDER ENERGY AND VELOCITY CONSTRAINTS

by

M. C. Spruill*

1. Introduction. A particle of unit mass travels in the x - y plane. Its coordinates at time t are $\underset{\sim}{x}(t) = (x(t),y(t))$ where $x'(t) \equiv -1$ and $\underset{\sim}{x}(0) = (c,1)$. The initial velocity $y'(0)$ in the y direction and the acceleration y'' are subject to control. For those controls which have the property that the particle hits the line $x = a$, $a < c$, the energy expended is $E = \int_0^{c-a} (y''(t))^2 dt$. Denote the corridor $\{(x,y): x \in [a,b], |y| \leq \gamma\}$ by $C(\gamma)$ where $b \in (a,c)$ is fixed. Given that $E \leq E_0$ is to be spent, what is the minimum $\gamma \geq 0$ for which the particle may be made to pass through the corridor $C(\gamma)$? The explicit solutions to the corridor problems C_{E_0}, $E_0 \in [0,\infty)$, are given in Theorem 4.1.

The solutions are found by solving another class of problems P_η for $\eta > 0$, which may be stated as follows. Given $\eta > 0$ minimize the functional

$$\rho_\eta(f) = \sup_{a \leq x \leq b} |f(x)|^2 + \eta \int_a^c (f''(x))^2 dx$$

over $f \in W_2^2[a,c]$ subject to $f(c) = 1$. The explicit solutions to these problems may be found in Theorem 3.2. They equioscillate on [a,b] and are splines on the entire interval [a,c] possessing two knots in [a,b]. The locations of the knots coincide with those at which the solution achieves its maximum

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absolute value on [a,b]. The solutions to this class of problems may also be used in optimal robust regression. One assumes that observations on the random variables $y(x) = f(x) + \varepsilon$ may be taken in the interval $x \in [a,b]$. The value $f(c)$ is to be estimated under the assumption that f is almost a linear function in the sense that $||f''||_2^2 \leq \frac{1}{\eta}$. The estimator used was developed by Speckman in [2] and employs the solution to P_η. Some optimal experimental designs employing Speckman's estimator are studied in [4].

The optimal design problem motivates the study of the minimization of a more general functional than ρ_η. For the fixed linear differential operator $L = D^m + a_{m-1}D^{m-1} + \ldots + a_0 I$, where $a_j \in C^j[a,c]$, $j=0,\ldots,m-1$, defined on $W_m^2[a,c]$ define

$$\tilde{\rho}_\eta(f) = \sup_{a \leq x \leq b} |f(x)|^2 + \eta \int_a^c [Lf(x)]^2 dx.$$

The function minimizing $\tilde{\rho}_\eta$ subject to $f \in W_m^2[a,c]$ and $f(c) = 1$ will be termed a solution to $\tilde{P}_\eta$. In the statistical context it could be used in extrapolation of a function which is almost a solution to $Lf = 0$ in the sense that $\int_a^c (Lf(s))^2 ds \leq \frac{1}{\eta}$. It is shown below that $\tilde{P}_\eta$ always has a solution and that if the m-dimensional null space of L is spanned by a Chebyshev system on [a,c] then the solution is unique. In particular, the solutions gotten in Theorem 3.2 must be unique since $L = D^2$.

We offer the following.

<u>Conjecture</u>: If L has a null space spanned by a Chebyshev system then there exist m points $\{x_1,\ldots,x_m\}$ contained in [a,b] such that the unique solution to $\tilde{P}_\eta$ is given by

$$(1.1) \qquad \delta_0(x) = [s^2(\eta)+\eta z^2(\eta)]^{-1}[\eta z(\eta) \sum_{i=1}^m (-1)^{i+q} \phi_{x_i}(x) + \int_a^c h_x(s) h_c(s) ds],$$

where $\{\phi_{x_i}\}_{i=1}^{m}$ are the Lagrange interpolation polynomials for the null space of L on [a,c] to the points x_i, i.e.; $\phi_{x_i}(x_j) = \delta_{ij}$, q is either zero or one,

$$h_x(s) = G(x,s) - \sum_{i=1}^{m} \phi_{x_i}(x)G(x_i,s),$$

G(x,s) solves $LG(\cdot,s)=0$ on [s,c] subject to $\frac{d^i}{dx^i} G(x,s)\Big|_{x=s} = \delta_{i,m-1}, i=0,\ldots,m-1,$

$$s^2(\eta) = ||h_c||_2^2 \text{ (on [a,c]), and } z(\eta) = \sum_{i=1}^{m} |\phi_{x_i}(c)| .$$

The functional ρ_η is not differentiable as the following example shows. Let $a = 0$, $b = 1$, $c = 2$, $f_0(t) = -1 + 2t$, and $h_n(t) = t/n$ for $n \geq 1$ and $t \in [0,2]$. Then $\rho_\eta(f_0+h_n) - \rho_\eta(f_0) = 2/n + 1/n^2$, $\rho_\eta(f_0-h_n) - \rho_\eta(f_0) \equiv 0$, $||h_n||^2 = 14/3n^2$, and if a bounded linear functional Λ were to exist satisfying $|\rho_\eta(f_0+h)-\rho_\eta(f_0)-\Lambda h| = o(||h||)$ it would entail both $\Lambda nh_n \to 2$ by consideration of h_n and $\Lambda nh_n \to 0$ by consideration of $-h_n$. The methods employed in the solution to P_η are adapted from the literature on the theory of optimal design of experiments as found for example in [3].

2. Solving $\tilde{P}_\eta$. Suppose that $f_1,\ldots,f_m$ is a T-system on [a,c] and $b \in (a,c)$ as above. In general the subspace $\{\sum_{j=1}^{m} \alpha_j f_j : \alpha'd=0\}$, where d is a fixed arbitrary vector, does not have the Chebyshev property.

Lemma 2.1. The collection of functions $\mathfrak{F} = \{\sum_{j=1}^{m} \beta_j f_j : \beta' f(c)=0\}$ has the Chebyshev property on [a,b]. In particular at least one of $f_j(c)$, $j \in \{1,\ldots,m\}$, say $f_1(c)$, must be non-zero and the system $\{g_i\}_{i=2}^{m}$ is a T-system for $\mathfrak{F}$, where

$$g_i(x) = f_i(x) - \frac{f_i(c)}{f_1(c)} f_1(x). \tag{2.1}$$

Proof: It is easily checked that the span of $\{g_i\}_{i=2}^{m}$ is $\mathfrak{F}$ if $f_1(c) \neq 0$. Let $a \leq \tau_1 < \tau_2 < \ldots < \tau_{m-1} \leq b$ and form the determinants

$$D = \begin{vmatrix} g_2(\tau_1)\ldots g_2(\tau_{m-1}) \\ \vdots \\ g_m(\tau_1)\ldots g_m(\tau_{m-1}) \end{vmatrix} \quad \text{and} \quad A = \begin{vmatrix} f_1(\tau_1)\ldots f_1(c) \\ \vdots \\ f_m(\tau_1)\ldots f_m(c) \end{vmatrix} .$$

Since $A = (-1)^m f_1(c)D$ and the determinants A do not change sign the determinants D must not change sign. □

Theorem 2.1. The problem $\tilde{P}_\eta$ has a solution $f_0 \in W_m^2[a,c]$. If the null space of L has the Chebyshev property then there exist points $a \leq x_1 < x_2 < \ldots < x_m \leq b$ and $q = 0$ or 1 such that

$$f_0(x_i) = (-1)^{i+q} \sup_{a\leq x\leq b} |f_0(x)|$$

and the solution is unique if $\eta > 0$.

Proof: If $\eta = 0$ clearly $\tilde{\rho}_\eta(f_0) = 0$ and there are many solutions. If $\eta > 0$ let $f_n \in W_m^2[a,c]$ be such that $\tilde{\rho}_\eta(f_n) \downarrow \inf \tilde{\rho}_\eta(f)$. The sequence Lf_n satisfies $||Lf_n||_2^2 \leq \tilde{\rho}_\eta(f_1)$. Since $||h||_2 \leq k$ is weakly sequentially compact in $L_2[a,c]$ there is an element $u \in L_2[a,c]$ and a subsequence n' such that $Lf_{n'} \xrightarrow{w} u$. We may write, for $a \leq \tau_1 < \tau_2 < \ldots < \tau_m \leq b$,

$$f_{n'}(x) = \sum_{i=1}^{m} f_{n'}(\tau_i)\phi_{\tau_i}(x) + \int_a^c h_x(s)Lf_{n'}(s)ds.$$

The integrals $\int_a^c h_x(s)Lf_{n'}(s)ds$ converge to $\int_a^c h_x(s)u(s)ds$ for all $x \in [a,c]$. Since $\sup_{a\leq x\leq b} |f_{n'}(x)| \leq \tilde{\rho}_\eta(f_1)$ all of the m sequences $\{f_{n'}(\tau_i)\}_{i=1}^{m}$ are bounded. Appealing again to sequential compactness of $\mathbb{R}^1$ we may assume that the sequence n' has been chosen to satisfy $f_{n'}(\tau_i) \to \alpha_i$ also. Define the function $f_0 \in W_m^2[a,c]$ by

$$f_0(x) = \sum_{i=1}^{m} \alpha_i \phi_{\tau_i}(x) + \int_a^c h_x(s)u(s)ds.$$

Since $||u||_2 \leq \underline{\lim}||Lf_{n'}||_2$ it is clear that $\tilde{\rho}_\eta(f_0) \leq \inf \tilde{\rho}_\eta(f)$. Since $f_n(x) \to f_0(x)$ for all x, $f_0(c) = 1$. The first assertion of the theorem has been proven.

For the following arguments we shall employ the notation of [1] in counting the zeros of a continuous function on [a,b]. If $x_0 \in (a,b)$ is an isolated zero of f and f does not change sign at x_0 then x_0 is termed a non-nodal zero. All other zeros, including zeros at the endpoints are nodal zeros. For any such function $\tilde{Z}(f)$ is the number of zeros in [a,b] counting one for each nodal zero and two for each non-nodal zero. Suppose that g is a continuous function on [a,b] and there are points $a \leq x_1 < x_2 < \dots < x_m \leq b$ and $q \in \{0,1\}$ such that $g(x_i) = (-1)^{i+q}||g||_\infty$, i=1,...,m. If there exists a point $x_0 \in \{x_1,\dots,x_m\}$ and a continuous function h such that $h(x_0) = g(x_0)$ then $||h||_\infty \leq ||g||_\infty$ entails $\tilde{Z}(g-h) \geq m - 1$. Consider the collection of functions on [a,c] $\{\sum_{j=1}^{m} \alpha_j f_j : \alpha' f(c)=0\}$. Their restriction to [a,b] is a Chebyshev system spanned by the Chebyshev system $g_2,\dots,g_m$ which may be defined, if $f_1(c) \neq 0$, except for the sign of one of them, from (2.1). By Bernstein's theorem (see [1]) there are constants $\beta_1,\dots,\beta_m$ such that $\beta' f(c) = 0$, and the function $\sum_{j=1}^{m} \beta_j f_j = g_0$ is the minimax approximant to f_0 on [a,b]. Therefore there exist m points $a \leq x_1 < \dots < x_m \leq b$ and a $q \in \{0,1\}$ such that

$$(g_0(x_i)-f_0(x_i))(-1)^{i+q} = ||g_0-f_0||_\infty, \tag{2.2}$$

where $||g_0-f_0||_\infty = \sup_{a \leq x \leq b} |g_0(x)-f_0(x)|$. Also

$$||g_0-f_0||_\infty = \inf\{||\Sigma\beta_j f_j - f_0||_\infty : \beta' f(c)=0\}.$$

Since $Lg_0 = 0$ on $[a,c]$ and $\beta' f(c) = 0$ entails $(f_0-g_0)(c) = 1$ we must have $||f_0||_\infty = ||f_0-g_0||_\infty$. Furthermore there must be a point $x \in [a,b]$ at which $f_0(x) = f_0(x) - g_0(x) = \pm||f_0||_\infty$. We conclude $\tilde{Z}(f_0-(f_0-g_0)) \geq m - 1$ on $[a,b]$. However if $f_0 - (f_0-g_0) = g_0$ is a non-trivial polynomial in the system $g_2,\ldots,g_m$ we must have by Theorem 4.2 of [1] $\tilde{Z}(g_0) \leq m - 2$. We conclude that g_0, the best approximant from $\mathfrak{F}$ on $[a,b]$ is zero. Consequently f_0 itself equioscillates in the sense of (2.2).

We now verify the uniqueness of the solution when the null space of L is spanned by a T-system $\{f_1,\ldots,f_m\}$. Because the norm on the Hilbert space $L_2[a,c]$ is strictly convex, if h_0 and h_1 are two solutions to the $\tilde{P}_\eta$ problem then $\tilde{\rho}_\eta(\alpha h_0+(1-\alpha)h_1) < \tilde{\rho}_\eta(h_0)$ unless $kLh_1 = Lh_0$, a.e. on $[a,c]$ for some constant k. Since $||\alpha h_0+(1-\alpha)h_1||_\infty^2 \leq \alpha||h_0||_\infty^2 + (1-\alpha)||h_1||_\infty^2$, where $||h||_\infty = \sup_{a\leq x\leq b} |h(x)|$, we have $\tilde{\rho}_\eta(\alpha h_0+(1-\alpha)h_1) \leq \psi(\alpha)$ where the function ψ is defined by

$$\psi(\alpha) = \alpha||h_0||_\infty^2 + (1-\alpha)||h_1||_\infty^2 + \eta(k+\alpha(1-k))^2||Lh_0||_2^2 .$$

Note that $\psi(0) = \psi(1) = \tilde{\rho}_\eta(h_0)$ and $\psi''(\alpha) = 2\eta(1-k)^2||Lh_0||_2^2$. Thus, whether $||Lh_0|| = 0$ or $||Lh_0|| > 0$ we must have $||h_0||_\infty = ||h_1||_\infty$. Again it must be the case that h_0 and h_1 share a common extreme value at one of their points of equioscillation. Therefore $\tilde{Z}(h_0-h_1) \geq m - 1$. If $Lh_0 = 0$ then $h_0 - h_1 \in \mathfrak{F}$ and of necessity $\tilde{Z} \leq m - 2$ unless $h_0 = h_1$. If $||Lh_0|| > 0$ then $k = 1$ so that again, $h_0 - h_1$ is in $\mathfrak{F} = \{\Sigma\beta_j f_j : \beta' f(c)=0\}$ and $\tilde{Z} \leq m - 2$. We conclude that in any case $h_0 = h_1$ proving that the solution is unique. □

3. <u>Exact solution to P_η</u>. A requirement of our method of proof is the following theorem. Let $\mathcal{X}$ be an index set, $\mathcal{H}_0$, $\mathcal{H}_1$, and $\mathcal{H}_2$ be three Hilbert spaces,

$\{m_x\}_{x\in\mathcal{X}}$ be a collection of bounded linear mappings from $\mathcal{H}_0$ into $\mathcal{H}_1$, and T be a bounded linear mapping from $\mathcal{H}_0$ into $\mathcal{H}_2$. Let h* be a given non-zero bounded linear functional on $\mathcal{H}_0$ and set $U = \{h\in\mathcal{H}_0 : h^*(h)=1\}$. Equip $\mathcal{X}$ with a topology for which every point of $\mathcal{X}$ is a Borel measurable set and let Ξ denote the collection of all Borel probability measures ξ on $\mathcal{X}$ whose supports $S(\xi)$ are finite sets.

Theorem 3.1. Suppose there is a point $h_0 \in U$, a measure $\xi_0 \in \Xi$, and $\alpha > 0$ satisfying

i) $S(\xi_0) \subset \{x: ||m_x(h_0)||_{\mathcal{H}_1} = \sup_{x\in\mathcal{X}} ||m_x(h_0)||_{\mathcal{H}_1}\}$,

ii) $\int (m_x^* m_x h_0 + T^*Th_0)d\xi_0(x) = \alpha h^*$, and

iii) $\int (||m_x h||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2})d\xi_0(x) = 0$ implies $h = 0$.

Then among all $h\in U$ h_0 minimizes $\sup_{\mathcal{X}} ||m_x(h)||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2}$.

Proof: Let $N(\xi) = \{h\in\mathcal{H}_0 : \int(||m_x(h)||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2})d\xi(x) > 0\}$ and

$$d(h^*,\xi) = \sup_{h\in N(\xi)} \frac{[h^*(h)]^2}{\int(||m_x(h)||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2})d\xi(x)} .$$

Clearly

$$(3.1) \qquad d(h^*,\xi) \geq [\inf_{h\in N(\xi)\cap U} \int(||m_x(h)||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2})d\xi(x)]^{-1}.$$

Since

$$\inf_{N\cap U} \int(||m_x h||^2 + ||Th||^2)d\xi \leq \inf_{N\cap U} \sup_{\mathcal{X}} (||m_x h||^2 + ||Th||^2)$$

$$\leq \sup_{\mathcal{X}} ||m_x h_0||^2 + ||Th_0||^2 = S$$

we have $d(h^*,\xi) \geq \frac{1}{S}$ for all $\xi\in\Xi$.

Using ii) we have for ξ_0

$$d(h^*,\xi_0) = \sup_{N(\xi_0)} \frac{\alpha^{-2}(\int[(m_x h, m_x h_0)_{\mathcal{H}_1} + (Th, Th_0)_{\mathcal{H}_2}]d\xi_0(x))^2}{\int(||m_x h||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2})d\xi_0(x)}$$

$$\leq \sup_{N(\xi_0)} \frac{S\alpha^{-2}[\int(\sqrt{||m_x(h)||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2}})\, d\xi_0(x)]^2}{\int(||m_x h||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2})d\xi_0(x)}$$

$$\leq S\alpha^{-2} = S(S^{-2}) = \frac{1}{S}.$$

Since by (3.1) $d(h^*,\xi_0) = \frac{1}{S} \geq [\inf_{U\cap N(\xi_0)} \sup_{\mathcal{X}} (||m_x h||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2})]^{-1}$ we have $\inf_{U\cap N(\xi_0)} \sup_{\mathcal{X}} (||m_x h||^2_{\mathcal{H}_1} + ||Th||^2_{\mathcal{H}_2}) \geq \sup_{\mathcal{X}} (||m_x(h_0)||^2_{\mathcal{H}_1} + ||Th_0||^2_{\mathcal{H}_2})$. By iii) $N(\xi_0) = \mathcal{H}_0 - \{0\}$ and we have proven the theorem. □

We shall prove below that there are points $x_1 < x_2$ in $[a,b]$ such that the representation (1.1) holds when $L = D^2$. In the general case their locations in $[a,b]$ will depend upon η. For this particular case we show that $x_2(\eta) \equiv b$ and the location of $x_1(\eta)$ is determined as follows. Let

$$\eta_0 = \frac{(b-a)^2}{24} \left[\frac{1}{b-a} + \frac{1}{2(c-b)} \right]^{-1}.$$

For $\eta \geq \eta_0$, $x_1(\eta) = a$. For $0 < \eta < \eta_0$, $x_1(\eta)$ is the unique real solution to

$$\frac{(b-x)^2}{24\,\eta} - \frac{1}{2(c-b)} = \frac{1}{b-x} \tag{3.2}$$

in $[a,b)$. By $x_1(\eta)$ below we shall mean the function which has just been defined. In the case of $L = D^2$, $\phi_{x_1}(x) = \frac{x-x_2}{x_1-x_2}$, and $\phi_{x_2}(x) = \frac{x-x_1}{x_2-x_1}$.

<u>Theorem 3.2</u>. With $x_1(\eta)$ as above $x_2(\eta) \equiv b$ and $q = 0$ the conjectured solution (1.1), with $L = D^2$, solves P_η.

Proof: Note that $f_0(c) = 1$. Fix η and let ξ_0 be the probability measure on $[a,b]$ satisfying $\xi_0(x_1) = \frac{|\phi_{x_1}(c)|}{|\phi_{x_1}(c)|+|\phi_{x_2}(c)|}$ and $\xi_0(x_2) = 1 - \xi_0(x_1)$. Every function f in $W_2^2[a,c]$ may be written

$$f(x) = \sum_{i=1}^{2} f(x_i)\phi_{x_i}(x) + \int_a^c h_x(s)Lf(s)ds.$$

We have, setting $\mathcal{H}_0 = W_2^2[a,c]$, $\mathcal{H}_1 = \mathbb{R}$, $\mathcal{H}_2 = L_2[a,c]$, h^* the evaluation functional in $W_2^2[a,c]$ at c, the mappings $m_x(h) = h(x)$, $Tf = \sqrt{\eta}\, f''$,

$$\int(m_x^* m_x f_0 + T^*Tf_0)d\xi_0(x)(f)$$

$$= \sum_{i=1}^{2} \xi_0(x_i)f(x_i)f_0(x_i) + \eta \int_a^c f_0''(s)f''(s)ds$$

$$= [s^2(\eta)+\eta z^2(\eta)]^{-1} \left\{ \frac{\eta z(\eta)}{z(\eta)} \sum_{i=1}^{2} |\phi_{x_i}(c)|(-1)^i f(x_i) + \eta \int_a^c h_c(s)f''(s)ds \right\}$$

$$= \eta[s^2(\eta)+\eta z^2(\eta)]^{-1}[\sum_{i=1}^{2} \phi_{x_i}(c)f(x_i) + \int_a^c h_c(s)f''(s)ds]$$

$$= \eta[s^2(\eta)+\eta z^2(\eta)]^{-1}f(c).$$

Therefore ii) of Theorem 3.1 holds with $\alpha = \eta[s^2(\eta)+\eta z^2(\eta)]^{-1}$ and $h_0 = f_0$. The condition iii) is also satisfied by ξ_0.

In order to verify i) notice that

$$-f_0(x_1) = f_0(x_2) = \eta z(\eta)[s^2(\eta)+\eta z^2(\eta)]^{-1}$$

and by straightforward but tedious computation, for $x \in [a,b]$

$$f_0'(x) = \frac{4\eta(c-(\frac{x_1-b}{2}))}{(b-x_1)^2} - \frac{(b-x_1)(c-b)}{6} + \int_a^x h_c(s)ds,$$

where

$$\int_a^x h_c(s)ds = \begin{cases} 0 & \leq x \leq x_1 \\ \tfrac{1}{2}(x-x_1)^2 \dfrac{(c-b)}{b-x_1} & x_1 \leq x \leq b\ . \end{cases}$$

With $x_1(\eta)$ as defined above one can check that for $0 < \eta < \eta_0$, $f_0'(x) = 0$ for $x \in [a,x_1]$ and $f_0'(x) > 0$ for $x \in (x_1,b)$. For $\eta \geq \eta_0$, $f_0'(x) \geq 0$ for $x \in [a,b]$. Therefore i) holds and the theorem has been proven. □

4. <u>Solution to the control problems C_E</u>. The corridor problem C_E may be solved by minimizing $||f||_\infty = \sup_{a\leq x\leq b} |f(x)|$ over $f \in W_2^2$ subject to $f(c) = 1$ and $\int_a^c (f''(t))^2dt = ||f''||_2^2 \leq E$. If f_0 does this the trajectory of the solution $\underset{\sim}{x}(t)$ to C_E satisfies $\underset{\sim}{x}(t) = (c-t, f_0(c-t))$, for $t \in [0,c-a]$.

<u>Lemma 4.1</u>. If $E \geq 3(c-b)^{-3}$ then there is a solution f_0 satisfying $||f_0||_\infty = 0$ and $||f_0''||_2^2 = 3(c-b)^{-3}$. If $E < 3(c-b)^{-3}$ the problem C_E has a unique solution f_0. The solution f_0 must satisfy $||f_0''||_2^2 = E$.

<u>Proof</u>: Every function f in $W_2^2[a,c]$ may be written

$$f(x) = f(b) + f'(b)(x-b) + \int_b^x \int_b^s f''(t)dt\, ds.$$

If we take $f(b) = f'(b) \equiv f''(t) \equiv 0$ on $[a,b]$ then

$$1 = f(c) = \int_b^c \int_b^s f''(t)dtds = \int_b^c (c-t)f''(t)dt$$

and by Schwarz's inequality the latter integral is no larger than $||f''||_2\sqrt{\frac{(c-b)^3}{3}}$. Therefore if f is to satisfy $||f||_\infty = 0$ in addition to the other conditions, then $||f''||_2^2 \geq 3(c-b)^{-3}$ with equality if and only if $f''(t) = k(c-t)$ on $[b,c]$. The first assertion has been verified.

The verification that a solution exists is similar to the proof of the existence of a solution to $\tilde{P}_\eta$ and will not be repeated here.

We next verify that if $E < 3(c-b)^{-3}$ then every solution must satisfy $||f_0''||_2^2 = E$. Suppose, to the contrary, that $||f_0''||_2^2 = \alpha < E$. Consider for $\varepsilon \in (0,1)$ the functions g_ε satisfying $g_\varepsilon(s) = (1-\varepsilon)f_0(s)$ for $s \in [a,b]$ and $g_\varepsilon(s) = k(c-s)$ for $s \in [b,c]$ where k satisfies

$$1 = (1-\varepsilon)[f_0(b)+f_0'(b)(c-b)] + k\,\frac{(c-b)^3}{3} .$$

Using the fact that $f_0(s) = k_2(c-s)$, $s \in [b,c]$ and

$$1 = f_0(b) + f_0'(b)(c-b) + k_2\,\frac{(c-b)^3}{3}$$

it can be verified that

$$||g''||_2^2 \leq ||f_0''||_2^2 + K\varepsilon = \alpha + K\varepsilon.$$

For ε_0 sufficiently small we have $||g''_{\varepsilon_0}||_2^2 < E$ and

$$||g_{\varepsilon_0}||_\infty = (1-\varepsilon_0)||f_0||_\infty < ||f_0||_\infty$$

contradicting the assumption that f_0 solves C_E.

An argument similar to that used in Theorem 2.1 may now be used to prove that the solutions equioscillate and are unique. □

Let g_η solve P_η and define the function $\varphi(\eta) = ||g_\eta''||_2^2$. Straightforward but tedious calculation reveals that

$$\varphi(\eta) = \left[\left(\frac{(c-b)^2(c-a)}{3}\right)\left(1+3\eta\,\frac{(c-a)}{(c-b)(b-a)^2}\right)^2\right]^{-1}, \quad \eta \geq \eta_0$$

and

$$\varphi(\eta) = \left[\left(\frac{(c-b)^2(c-x_1)}{3}\right)\left(1+\left(\frac{b-x_1}{4}\right)\left(\frac{1}{c-x_1} + \frac{1}{c-b}\right)\right)^2\right]^{-1}, \quad 0 < \eta < \eta_0.$$

Clearly $\lim_{\eta\to\infty} \varphi(\eta) = 0$ and from (3.2) $\lim_{\eta\to 0} \varphi(\eta) = \frac{3}{(c-b)^3}$. Also $\varphi'(\eta) < 0$.

Theorem 4.1. If $E > 0$ is a given positive number then the function f_E minimizing $\sup_{a\le x\le b} |f(x)|$ among all functions $f \in W_2^2[a,c]$ for which $f(c) = 1$ and $\int_a^c (f''(x))^2 dt \le E$ yielding the solution $\underset{\sim}{x}_E(t) = (c-t, f_E(c-t))$, $t \in [0,c-a]$, to the corridor problem C_E may be found as follows.

i) If $E \ge 3(c-b)^{-3}$ then there are many solutions. The one using the least energy satisfies $||f||_\infty = 0$ and $f''(x) = \frac{3(c-x)}{(c-b)^3}$.

ii) If $E < 3(c-b)^{-3}$ there is a solution to $\varphi(\eta) = E$ and the unique solution to C_E is the solution to $P_{\varphi^{-1}(E)}$.

Proof: The assertion i) has already been proven in Lemma 4.1. The range of φ is $(0, 3(c-b)^{-3})$ and φ^{-1} exists by the remarks preceding the theorem. Let $\eta = \varphi^{-1}(E)$. By lemma 4.1 C_E has a unique solution, call it g_0, and furthermore $||g_0''||_2^2 = E$. By theorem 3.2 P_η has the unique solution f_0. By definition of the function φ, $||f_0''||_2^2 = E$. Our claim is that $f_0 = g_0$. Since f_0 solves P_η we have $\rho_\eta(g_0) \ge \rho_\eta(f_0)$. This implies that $||g_0||_\infty \ge ||f_0||_\infty$ and therefore that f_0 also solves C_E. The unicity of the solution shows that $f_0 = g_0$ as asserted. □

If $E = 0$ f_0 has a graph which is a straight line passing through $(\frac{a+b}{2}, 0)$ and $(c,1)$.

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