BALANCED TREATMENT INCOMPLETE BLOCK (BTIB) DESIGNS FOR COMPARING TREATMENTS WITH A CONTROL: MINIMAL COMPLETE SETS OF GENERATOR DESIGNS FOR $k = 3$, $p=3(1)10$

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BALANCED TREATMENT INCOMPLETE BLOCK (BTIB) DESIGNS FOR COMPARING TREATMENTS WITH A CONTROL: MINIMAL COMPLETE SETS OF GENERATOR DESIGNS FOR $k = 3$, $p = 3(1)10$

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ABSTRACT

Bechhofer and Tamhane (1981) proposed a new class of incomplete block designs called <u>BTIB designs</u> for comparing $p \geq 2$ test treatments with a control treatment in blocks of equal size $k < p+1$. All BTIB designs for given (p,k) can be constructed by forming unions of replications of a set of elementary BTIB designs called <u>generator designs</u> for that (p,k). In general, there are many generator designs for given (p,k) but only a small subset (called the <u>minimal complete set</u>) of these suffices to obtain all admissible BTIB designs (except possibly any equivalent ones). Determination of the minimal complete set of generator designs for given (p,k) was stated as an open problem in Bechhofer and Tamhane (1981). In this paper we solve this problem for $k = 3$. More specifically, we give the minimal complete sets of generator designs for $k = 3$, $p = 3(1)10$; the relevant proofs are given only for the cases $p = 3(1)6$. Some additional combinatorial results concerning BTIB designs are also given.

1. INTRODUCTION

Consider $p+1$ treatments indexed by $0,1,\ldots,p$ with 0 denoting the control treatment and $1,2,\ldots,p$ denoting the $p \geq 2$ test treatments. It is desired to simultaneously compare the p test treatments with the control in b blocks each of size $k < p+1$. We then have an imcomplete blocks design situation.

The usual BIB designs are not, in general, appropriate for the above multiple comparisons with a control problem. Bechhofer and Tamhane (1981) proposed a new class of designs which they referred to a <u>BTIB designs</u> for this problem. The Bechhofer-Tamhane paper is basic to the developments in the present article; we shall hereinafter refer to it as BT. (We note that Pearce (1960) had earlier considered a related class of designs which he referred to as designs with <u>supplemented balance</u>. Pearce did not restrict consideration to incomplete blocks situation. Another related early paper is that by Nair and Rao (1942)).

For given (p,k) a BTIB design for any b can be built out of a set of elementary BTIB designs called <u>generator designs</u>. For given (p,k) there exists a finite number of generator designs (see Theorem 2.2 of the present paper) but this number can be very large. However, it turns out that only a small subset of these generator designs is sufficient in that essentially all <u>admissible designs</u> (see Section 2 for a definition) can be built from this set, i.e., any admissible design that cannot be constructed from this set is equivalent to some design that can be constructed from the set. This set is referred to as the <u>minimal complete set</u>. Admissible designs are important because only they can be candidates for an "optimal" design.

The primary objective of the present article is to show the minimal complete nature of the sets of generator designs that we have constructed for $k=3$ and $p=3(1)10$; the case $k=2$, $p \geq 2$ is quite trivial and is mentioned only in passing Remark 2.3. The significance of these results is that for each of the aforementioned cases we can now assert that the optimal designs can be built only out of the generator designs in the minimal complete set for that case and addition of any other generator designs to the set will not alter the "optimal" designs in an essential way. Based on the minimal complete sets presented here, Bechhofer and Tamhane (1982) have prepared catalogs of "optimal" BTIB designs for $k = 2,3$, $p = k(1)6$.

While this paper was under review, Ture (1982) in a Berkeley dissertation extended our methods to obtain minimal complete sets for $k = 4,5$, $p = k(1)10$ and $p = 6$, $k = 6$. Ture uses more refined bounds than ours but his basic method of proof is the same as ours.

2. NOTATION, DEFINITIONS AND SOME BASIC RESULTS

2.1 BTIB Designs

We assume the usual additive linear model for the observation y_{ijh} taken on the ith treatment in the hth plot of the jth block:

$$y_{ijh} = \mu + \alpha_i + \beta_j + \epsilon_{ijh} \tag{2.1}$$

where the ϵ_{ijh} are uncorrelated random errors with $E(\epsilon_{ijh}) = 0$, $var(\epsilon_{ijh}) = \sigma^2$ $(0 \le i \le p,\ 1 \le j \le b,\ 1 \le h \le k)$. We consider only connected BTIB designs for which the contrasts $\alpha_0 - \alpha_i$ $(1 \le i \le p)$ are estimable. Let $\hat{\alpha}_0 - \hat{\alpha}_i$ be the BLUE of $\alpha_0 - \alpha_i$ $(0 \le i \le p)$. Then we have the following definition.

Definition 2.1: For given (p,k,b) a BTIB design is any design for which

$$var(\hat{\alpha}_0 - \hat{\alpha}_i) = \text{const.} = \tau^2\sigma^2 \text{ (say)} \quad (1 \le i \le p), \tag{2.2}$$

and

$$corr(\hat{\alpha}_0 - \hat{\alpha}_i, \hat{\alpha}_0 - \hat{\alpha}_{i'}) = \text{const.} = \rho \text{ (say)} \quad (i \ne i', 1 \le i,i' \le p) \tag{2.3}$$

where the parameters τ^2 and ρ depend, of course, on the design.

Let r_{ij} $(0 \le r_{ij} \le k-1)$ be the number of times the ith treatment appears in the jth block $(0 \le i \le p,\ 1 \le j \le b)$ and let $\lambda_{ii'} = \Sigma_{j=1}^{b} r_{ij} r_{i'j}$ $(i \ne i',\ 0 \le i,\ i' \le p)$. The following is Theorem 3.1 of BT.

Theorem 2.1: For given (p,k,b) a design is BTIB iff

$$\lambda_{01} = \lambda_{02} = \ldots = \lambda_{0p} = \lambda_0 \text{ (say)}, \tag{2.4}$$

and

$$\lambda_{12} = \lambda_{13} = \ldots = \lambda_{p-1,p} = \lambda_1 \text{ (say)}. \tag{2.5}$$

Furthermore, τ^2 of (2.2) is given by

$$\tau^2 = \frac{k(\lambda_0 + \lambda_1)}{\lambda_0(\lambda_0 + p\lambda_1)}, \tag{2.6}$$

and ρ of (2.3) is given by

$$\rho = \frac{\lambda_1}{\lambda_0 + \lambda_1}. \tag{2.7}$$

□

Clearly, for a BTIB design to be implementable we must have $\lambda_0 > 0$. Also note that if each $r_{ij} = 0$ or 1 and if $\lambda_0 = \lambda_1 = \lambda$ (say) then we get a BIB design; thus BIB designs form a subclass of the class of BTIB designs.

2.2 Generator Designs

We start with the definition of a generator design.

Definition 2.2: For given (p,k) a generator design is a BTIB design (not necessarily connected or implementable) no proper subset of whose blocks forms a BTIB design, and no block of which contains only one of the $p+1$ treatments.

Remark 2.1: In BT it was noted that for each $p \ge 2$, $k = 2$ there are exactly two generator designs:

$$D_1 = \begin{Bmatrix} 0\,0 & \cdots & 0 \\ 1\,2 & \cdots & p \end{Bmatrix},\quad D_2 = \begin{Bmatrix} 1\,1 & \cdots & p-1 \\ 2\,3 & \cdots & p \end{Bmatrix}. \tag{2.8}$$

Although, for $p \geq 2$, $k = 2$ there are exactly two generator designs, it is not clear whether for any (p,k) there are only finitely many generator designs. This question is answered in the affirmative in Theorem 2.2. First we state two lemmas which are self-evident.

Lemma 2.1: A design with a frequency vector $\underset{\sim}{n}$ is BTIB iff

$$\underset{\sim}{M}\,\underset{\sim}{n} = \underset{\sim}{\lambda} = \begin{bmatrix} \lambda_0 \\ \cdot \\ \cdot \\ \lambda_0 \\ \lambda_1 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_1 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} p \\ \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} \frac{p(p-1)}{2} \end{matrix} \tag{2.9}$$

for some integers $\lambda_0 \geq 0$, $\lambda_1 \geq 0$ but not both zero. □

Lemma 2.2: If $\underset{\sim}{n}^{(1)}$ and $\underset{\sim}{n}^{(2)}$ are two BTIB designs with $n_h^{(2)} \geq n_h^{(1)}$ for $h = 1,2,\ldots,s$ with a strict inequality for at least some h (denoted by $\underset{\sim}{n}^{(2)} > \underset{\sim}{n}^{(1)}$) then $\underset{\sim}{n} = \underset{\sim}{n}^{(2)} - \underset{\sim}{n}^{(1)}$ is also a BTIB design. □

Theorem 2.2: For given (p,k) there exist only finitely many generator designs.

Proof: For given (p,k) let $\mathscr{S}$ be the set of all distinct blocks which can be used in a BTIB design and let us index these blocks in some manner $1,2,\ldots,s$; $\mathscr{S}$ consists of all samples of size k with replacement from integers $0,1,\ldots,p$ except those $p+1$ samples of the type $(i,i,\ldots,i)$ for $0 \leq i \leq p$. Next index the pairs $(0,1)$, $(0,2)$,...$(0,p)$, $(1,2)$, $(1,3)$, ..., $(p-1,p)$ by $1,2,\ldots$, $t = p(p+1)/2$. Let $\underset{\sim}{M} = \{m_{gh}\}$ be a $t \times s$ matrix where $m_{gh} = r_{ih} r_{i'h}$ $(0 \leq i < i' \leq p)$, g is the index of the pair (i,i') $(1 \leq g \leq t)$, and h is the index of the block $(1 \leq h \leq s)$. Then any design can be represented by a s-vector $\underset{\sim}{n} = (n_1, n_2, \ldots, n_s)$ where $n_h \geq 0$ is the frequency of the hth block in the design $(1 \leq h \leq s)$ and $b = \sum_{h=1}^{s} n_h$.

Now suppose that the theorem is not true. Then there exists an infinite sequence of generator designs $\underset{\sim}{n}^{(1)}, \underset{\sim}{n}^{(2)}, \ldots$. Choose a subsequence $\{\underset{\sim}{n}^{(i_j)}\}$ from this sequence with the property that $\underset{\sim}{n}^{(i_{j+1})} > \underset{\sim}{n}^{(i_j)}$; such a subsequence can always be chosen. Then from Lemma 2.2 it follows that $\underset{\sim}{n}^{(i_{j+1})} - \underset{\sim}{n}^{(i_j)}$ is a BTIB design. Therefore $\underset{\sim}{n}^{(i_{j+1})}$ is not a generator design. Thus we have reached a contradiction which proves the theorem. □

Remark 2.2: The representation (2.9) can be used to construct BTIB designs and in particular generator designs for at least small values of p and k. In fact, many of the generator designs given in the present paper were constructed by using (2.9); the rest were constructed by using the methods given in Section 3.2 of BT. To employ (2.9) it is first necessary to know the feasible values for the pair (λ_0,λ_1); these feasible values are obtained from Lemmas 3.2 and 3.3. Next for given (λ_0,λ_1) a lower bound on b is obtained from Lemma 3.1. For $\lambda_0 = 0$, $\lambda_1 > 0$ or for $\lambda_0 = \lambda_1 > 0$ the desired BTIB is usually a BIB design (if it exists) and known results from the theory of BIB designs can be used. For other combinations of λ_0 and λ_1, (2.9) is applied and solution is obtained by trial and error.

2.3 Admissible Designs and Minimal Complete Set of Generator Designs

Now we define the concepts of inadmissible and admissible designs. These concepts are motivated by the problem of joint confidence interval estimation of the $\alpha_0 - \alpha_i$; see Section 5 of BT for further details.

Definition 2.3: Suppose that for given (p,k) we have two BTIB designs D_1 and D_2 with parameters $(b_1,\lambda_0^{(1)},\lambda_1^{(1)})$ and $(b_2,\lambda_0^{(2)},\lambda_1^{(2)})$ and their τ^2 and ρ values are given by (2.6) and (2.7) respectively. D_2 is inadmissible with respect to (wrt) D_1 iff $b_1 \le b_2$, $\tau_1^2 \le \tau_2^2$ and $\rho_1 \ge \rho_2$ with at least one inequality strict. If a design is not inadmissible then it is said to be admissible. If $b_1 = b_2$, $\tau_1^2 = \tau_2^2$ and $\rho_1 = \rho_2$ (or equivalently $b_1 = b_2$, $\lambda_0^{(1)} = \lambda_0^{(2)}$, $\lambda_1^{(1)} = \lambda_1^{(2)}$) then D_1 and D_2 are equivalent.

For given (p,k) the candidates for an "optimal" design will be all admissible designs that can be constructed from a given set of generator designs. Now we give the definition of the minimal complete set of generator designs:

Definition 2.4: For given (p,k) the smallest set of generator designs $\{D_i \quad (1 \le i \le n)\}$ from which all admissible designs for that (p,k) (except possibly any equivalent ones) can be constructed is called the minimal complete set of generator designs.

We note that for given (p,k), the minimal complete set is unique up to substitution of any generator design in the set by an equivalent one. This fact follows from the definition of the minimal complete set.

To obtain the minimal complete set from a given set of generator designs we proceed in two steps. In the first step we delete any equivalent generator designs (except, of course, one representative of each set of equivalent generator designs). Furthermore, if the union of two or more generator designs yields an equivalent generator design, then we delete the latter design.

In the second step we delete the so-called strongly (S-) inadmissible generator designs from the set of nonequivalent generator designs obtained in the first step. The concept of S-inadmissibility is defined as follows:

Definition 2.5: If for given (p,k) we have two BTIB designs D_1 and D_2 (not necessarily generator designs), we say that D_2 is S-inadmissible wrt D_1 if D_2 is inadmissible wrt D_1, and if for any arbitrary BTIB design D_3 we have that $D_2 \cup D_3$ is inadmissible wrt $D_1 \cup D_3$.

An easily verifiable sufficient condition for S-inadmissibility of D_2 wrt D_1 is that

$$b_1 \le b_2, \ \lambda_0^{(1)} = \lambda_0^{(2)}, \ \lambda_1^{(1)} \ge \lambda_1^{(2)} \tag{2.9}$$

with at least one inequality being strict. We use a special case of (2.9) namely

$$b_1 \le b_2,\ \lambda_0^{(1)} = \lambda_0^{(2)},\ \lambda_1^{(1)} = \lambda_1^{(2)} \tag{2.10}$$

repeatedly in the sequel to decide whether a given design D_2 is S-inadmissible or equivalent wrt another design D_1.

Remark 2.3: It is easy to see that the two generator designs in (2.8) constitute the minimal complete set for $k = 2$ for each $p \ge 2$. This is because these are the only two distinct generator designs possible and neither of them is S-inadmissible wrt the other one.

Remark 2.4: For some values of (p,k) it is possible to further cut down a list of generator designs by deleting the so-called combination (C-) inadmissible designs and thus obtain the minimal complete set; for a definition of C-inadmissible designs, see Definition 5.5 of BT. For (p,k) values considered in the present paper the use of this concept is not needed. This is because it can be shown that for these values of (p,k), if any generator design is not in the minimal complete set then it is either S-inadmissible or equivalent wrt a union of the generator designs in the minimal complete set. Furthermore, it is checked that every implementable generator design in the minimal complete set is admissible at least by itself, and every generator design not containing the control is part of at least one admissible design. Therefore none of these generator designs are C-inadmissible.

3. MINIMAL COMPLETE SETS OF GENERATOR DESIGNS FOR $k = 3$, $p = 3(1)10$

3.1 Preliminary Lemmas

We need three preliminary lemmas giving relations between the parameters of a BTIB design.

Lemma 3.1: For any (p,k) consider a BTIB design D with parameters (b,λ_0,λ_1). We have the following inequalities on b:

$$\frac{2p\lambda_0 + p(p-1)\lambda_1}{k(k-1)} \le b \le \frac{2p\lambda_0 + p(p-1)\lambda_1}{2(k-1)} . \tag{3.1}$$

Furthermore, the lower inequality is an equality iff the design is binary i.e., if each $r_{ij} = 0$ or 1 $(0 \le i \le p,\ 1 \le j \le b)$.

Proof: Let r_i denote the number of replications on the ith treatment, $r_i = \Sigma_{j=1}^{b} r_{ij}$ $(0 \le i \le p)$. From (A.1) and (A.5) of BT we have

$$kr_0 = p\lambda_0 + \sum_{j=1}^{b} r_{0j}^2 \tag{3.2}$$

and

$$kr_i = \lambda_0 + (p-1)\lambda_1 + \sum_{j=1}^{b} r_{ij}^2 \qquad (1 \le i \le p). \tag{3.3}$$

Adding (3.2) and (3.3) we obtain

$$\sum_{i=0}^{p} r_i = kb = (2p\lambda_0 + p(p-1)\lambda_1 + \sum_{i=0}^{p} \sum_{j=1}^{b} r_{ij}^2)/k . \tag{3.4}$$

Now subject to the restriction that the r_{ij} are nonnegative integers satisfying $\Sigma_{i=0}^{p} r_{ij} = k$ for $1 \le j \le b$, it is easily verified that $\Sigma_{i=0}^{p} \Sigma_{j=1}^{b} r_{ij}^2$ is minimized when each $r_{ij} = 0$ or 1 and it is maximized when for each j there is a pair of treatments i_1, i_2 $(i_1 \ne i_2, 0 \le i_1, i_2 \le p)$ such that $r_{i_1 j} = k-1$, $r_{i_2 j} = 1$ and $r_{ij} = 0$ for $i \ne i_1, i_2 (0 \le i \le p, 1 \le j \le b)$. Furthermore, the minimum value of $\Sigma\,\Sigma\, r_{ij}^2$ is kb which when substituted in (3.4) yields the lower bound on b in (3.1). The maximum value of $\Sigma\,\Sigma\, r_{ij}^2$ is $b(k^2 - 2k + 2)$ which when substituted in (3.4) yields the upper bound on b in (3.1). □

Lemma 3.2: When k is odd, the quantities $p\lambda_0$ and $\lambda_0 + (p-1)\lambda_1$ must be even.

Proof: Let $b_{i\ell}$ denote the number of blocks in which the ith treatment is replicated ℓ times $(0 \le \ell \le k-1,\ 0 \le i \le p)$. Note that

$$r_i = \sum_{\ell=1}^{k-1} \ell b_{i\ell},\ \sum_{j=1}^{b} r_{ij}^2 = \sum_{\ell=1}^{k-1} \ell^2 b_{i\ell}. \tag{3.5}$$

Substituting (3.5) in (3.2) and (3.3) we get

$$p\lambda_0 = \sum_{\ell=1}^{k-1} \ell(k-\ell) b_{0\ell} \tag{3.6}$$

and

$$\lambda_0 + (p-1)\lambda_1 = \sum_{\ell=1}^{k-1} \ell(k-\ell) b_{i\ell} \quad (1 \le i \le p). \tag{3.7}$$

By noting that when k is odd, the coefficients $\ell(k-\ell)$ are even for $1 \le \ell \le k-1$ the lemma follows. □

Lemma 3.3: For any BTIB design for $p \ge 3$, $k = 3$ we have

$$(p-1)\lambda_1 \ge \lambda_0 \tag{3.8}$$

if $\lambda_1 > 0$ and if the design does not contain the generator design $\left\{\begin{matrix} 0 & 0 & & 0 \\ 1 & 2 & \ldots & p \\ 1 & 2 & & p \end{matrix}\right\}$.

Proof: The lemma follows trivially for $\lambda_0 = 0$. Thus assume that $\lambda_0 > 0$. We may change any blocks of the type $\begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}$ for $i \ge 1$ to $\begin{pmatrix} 0 \\ i \\ i \end{pmatrix}$ without affecting λ_0 or λ_1. We may also assume that for some $i \ge 1$ there are no blocks of the type $\begin{pmatrix} 0 \\ i \\ i \end{pmatrix}$ because otherwise the BTIB design would contain the generator design $\left\{\begin{matrix} 0 & 0 & & 0 \\ 1 & 2 & \ldots & p \\ 1 & 2 & & p \end{matrix}\right\}$. For that particular i we can write

$$(p-1)\lambda_1 - \lambda_0 = \sum_{\substack{i'=1 \\ i' \ne i}}^{p} \sum_{j=1}^{b} r_{ij} r_{i'j} - \sum_{j=1}^{b} r_{ij} r_{0j} \tag{3.9}$$

$$= \sum_{j=1}^{b} r_{ij} \left(\sum_{\substack{i'=1 \\ i' \neq i}}^{p} r_{i'j} - r_{0j} \right)$$

$$\geq 0.$$

The last step of (3.9) follows because the summand is negative iff $r_{ij} = 2$, $r_{0j} = 1$ and $r_{i'j} = 0$ for $i' \neq i$, a possibility that is ruled out. □

3.2 Proofs of Minimal Complete Sets for $k = 3$, $p = 3(1)6$

In this section we give minimal complete sets of generator designs for $k = 3$, $p = 3(1)6$ and prove the minimal complete nature of the set in each case. The method of proof in each case is the same and we outline the general method here thus avoiding the repetitive details in each proof. In addition, in Section 3.3 we give minimal complete sets of generator designs for $k = 3$, $p = 7(1)10$ without the accompanying proofs which the reader can easily construct once he understands the general method illustrated in this section for $p = 3(1)6$.

To show that for given (p,k), a set of generator designs $\{D_1, D_2, \ldots, D_n\}$ is minimal complete, we consider an arbitrary BTIB design D for that (p,k) having parameters $(b, \lambda_0, \lambda_1)$. Then for that D we show that there exists a BTIB design $D^* = \bigcup_{i=1}^{n} f_i D_i$ (i.e., constructed out of the set $\{D_1, \ldots D_n\}$) such that $\lambda_0^* = \lambda_0$, $\lambda_1^* = \lambda_1$ and $b^* \leq b$. Thus D is either equivalent to or S-inadmissible wrt D^* (cf. (2.10)). The proof is completed by finally noting that the set $\{D_1, \ldots D_n\}$ consists of nonequivalent generator designs, none of which is S-inadmissible wrt to any other ones or unions of any other ones and therefore that set is minimal complete.

In the proofs below for given (p,k) typically we must consider several cases depending on the values of (λ_0, λ_1); in each case we simply construct the desired D^* such that $\lambda_0^* = \lambda_0$, $\lambda_1^* = \lambda_1$ and explain why D^* requires the smallest possible number of blocks (by using Lemma 3.1) which implies that $b^* \leq b$.

Theorem 3.1: For $p = 3$, $k = 3$ the minimal complete set of generator designs is as given in Table 3.1.

Table 3.1

Minimal Complete Set of Generator Designs for $p = 3$, $k = 3$

D_i	Design	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	$\begin{Bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 3 \end{Bmatrix}$	3	2	1
D_2	$\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$	1	0	1

Proof: Parameters λ_0, λ_1 of an arbitrary BTIB design D must satisfy the following: (i) $\lambda_1 - \lambda_0 / 2 \geq 0$ by Lemma 3.3, and (ii) $\lambda_1 - \lambda_0 / 2$ must be an integer because λ_0 is even by Lemma 3.2. Let $f_1 = \lambda_0 / 2$, $f_2 = \lambda_1 - \lambda_0 / 2$ and $D^* = f_1 D_1 \cup f_2 D_2$. Note that D^* requires the smallest possible number of blocks because D_1 and D_2 do, both being binary designs. $\square$

Summaries of the proofs of the next three theorems will be given in condensed form. The summaries will list cases that need to be considered, values of λ_0 and λ_1 for these cases, the values of the f_i yielding D*, and brief explanatory remarks. From these summaries the reader should be able to construct a detailed proof that the proposed set of generator designs is indeed minimal complete.

Theorem 3.2. For $p = 4$, $k = 3$ the minimal complete set of generator designs is as given in Table 3.2.

Table 3.2

Minimal Complete Set of Generator Designs for $p=4$, $k=3$

D_i	Design	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	$\begin{Bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&2&3&4 \end{Bmatrix}$	4	2	0
D_2	$\begin{Bmatrix} 0&0&0&0&0&0 \\ 1&1&1&2&2&3 \\ 2&3&4&3&4&4 \end{Bmatrix}$	6	3	1
D_3	$\begin{Bmatrix} 0&0&0&0&1&1&2 \\ 0&1&1&2&2&3&3 \\ 3&2&4&4&3&4&4 \end{Bmatrix}$	7	2	2
D_4	$\begin{Bmatrix} 0&0&1&1&1&1&2&2 \\ 1&3&2&2&3&3&3&4 \\ 2&4&3&4&4&4&3&4 \end{Bmatrix}$	8	1	3
D_5	$\begin{Bmatrix} 0&0&0&0&0&0&0&0&1&2 \\ 0&0&1&1&1&1&2&2&3&3 \\ 3&4&2&2&3&4&3&4&4&4 \end{Bmatrix}$	10	4	2
D_6	$\begin{Bmatrix} 1&1&1&2 \\ 2&2&3&3 \\ 3&4&4&4 \end{Bmatrix}$	4	0	2

Proof. Consider an arbitrary BTIB design D with parameters (b,λ_0,λ_1) for $p = 4$, $k = 3$.

Case 1. $\lambda_1 = 0$.

$f_1 = \lambda_0/2$, $f_2 = f_3 = f_4 = f_5 = f_6 = 0$.

Remarks: D* must be equivalent to D. By Lemma 3.2 λ_0 must be even.

Case 2. $\lambda_0 \equiv 0 \pmod 3$, $\lambda_1 > 0$.

$f_1 = 0$, $f_2 = \lambda_0/3$, $f_3 = f_4 = f_5 = 0$, $f_6 = (\lambda_1-\lambda_0/3)/2$.

Remarks: $\lambda_1 - \lambda_0/3$ is ≥ 0 and even by lemmas 3.2 and 3.3. D* achieves the lower bound on b of Lemma 3.1.

Case 3. $\lambda_0 = 1$, $\lambda_1 > 0$.

$f_1 = f_2 = f_3 = 0$, $f_4 = 1$, $f_5 = 0$, $f_6 = (\lambda_1-3)/2$.

Remarks: λ_1 is odd by lemma 3.2. $\lambda_1 \neq 1$ since no BIBD exists with $v = p+1 = 5$, $k = 3$, and $\lambda = 1$ (see Raghavarao, 1971, p. 86). D* achieves the smallest integer value of $b \geq$ lower limit of lemma 3.1.

Case 4. $\lambda_0 \equiv 1 \pmod 3$, $\lambda_0 \geq 4$, $\lambda_1 > 0$.

$f_1 = 0$, $f_2 = (\lambda_0-4)/3$, $f_3 = f_4 = 0$, $f_5 = 1$, $f_6 = \{\lambda_1-2-(\lambda_0-4)/3\}/2$.

Remarks: f_6 is ≥ 0 and even by lemmas 3.2 and 3.3. $\lambda_0 \equiv 1 \pmod 3$ and lemma 3.1 suffice to show D* has the minimum number of blocks possible.

Case 5. $\lambda_0 \equiv 2 \pmod 3$, $\lambda_1 > 0$.

$f_1 = 0$, $f_2 = (\lambda_0-2)/2$, $f_3 = 1$, $f_4 = f_5 = 0$, $f_6 = \{\lambda_1-2-(\lambda_0-2)/3\}/2$.

Remarks: f_6 is ≥ 0 and even by lemmas 3.2 and 3.3. $\lambda_0 \equiv 2 \pmod 3$ and lemma 3.1 suffice to show that D* has the minimum number of blocks possible.

This covers all possible cases and hence completes the proof of the theorem. □

Theorem 3.3. For $p = 5$, $k = 3$ the minimal complete set of generator designs is as given in Table 3.3.

Table 3.3

Minimal Complete Set of Generator Designs for $p = 5$, $k = 3$

D_i	Design	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	$\begin{Bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&2&3&4&5 \end{Bmatrix}$	5	2	0
D_2	$\begin{Bmatrix} 0&0&0&0&0&1&2 \\ 0&1&1&3&4&2&3 \\ 2&3&5&4&5&4&5 \end{Bmatrix}$	7	2	1
D_3	$\begin{Bmatrix} 0&0&0&0&0&1&1&1&2&2 \\ 1&1&2&3&4&2&3&4&3&3 \\ 2&3&4&5&5&5&4&5&4&5 \end{Bmatrix}$	10	2	2
D_4	$\begin{Bmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 1&1&1&1&2&2&2&3&3&4 \\ 2&3&4&5&3&4&5&4&5&5 \end{Bmatrix}$	10	4	1
D_5	$\begin{Bmatrix} 1&1&1&2&2&3&1 \\ 2&2&3&3&4&4&4 \\ 3&5&5&4&5&5&4 \end{Bmatrix}$	7	0	2
D_6	$\begin{Bmatrix} 1&1&1&1&1&1&2&2&2&3 \\ 2&2&2&3&3&4&3&3&4&4 \\ 3&4&5&4&5&5&4&5&5&5 \end{Bmatrix}$	10	0	3

Proof. Consider an arbitrary BTIB design D with parameters (b,λ_0,λ_1) for $p = 5$, $k = 3$.

To start with λ_0 must be even by lemma 3.2. The cases to be considered are thus.

Case 1. $\lambda_1 = 0$.

$f_1 = \lambda_0/2$, $f_2 = f_3 = f_4 = f_5 = f_6 = 0$.

Remarks: D must be equivalent to D*.

Case 2a. $\lambda_0 = 0$, $\lambda_1 \equiv 0 \pmod 3$, $\lambda_1 > 0$.

$f_1 = f_2 = f_3 = f_4 = f_5 = f_6 = 0$.

Remarks: D* achieves the lower bound on b of lemma 3.1.

Case 2b. $\lambda_0 = 0$, $\lambda_1 \equiv 1 \pmod 3$, $\lambda_1 > 0$.

$f_1 = f_2 = f_3 = f_4 = 0$, $f_5 = 2$, $f_6 = (\lambda_1-4)/3$.

Remarks: $\lambda_1 > 1$ since no BIBD exists with $v = p+1 = 5$, $k = 3$, and $\lambda = 1$ (see Raghavarao, 1971, p. 86). D* achieves the smallest integer value of $b \geq$ lower bound of lemma 3.1.

Case 2c. $\lambda_0 = 0$, $\lambda_1 \equiv 2 \pmod 3$, $\lambda_1 > 0$.

$f_1 = f_2 = f_3 = f_4 = 0$, $f_5 = 1$, $f_6 = (\lambda_1-2)/3$.

Remarks: D* achieves the smallest integer value of $b \geq$ lower bound of lemma 3.1.

Case 3a. $\lambda_0 \equiv 0 \pmod 4$, $\lambda_0 \geq 4$, $\lambda_1 - \lambda_0/4 \equiv 0 \pmod 3$, $\lambda_1 > 0$ $f_1 = f_2 = f_3 = 0$, $f_4 = \lambda_0/4$, $f_5 = 0$, $f_6 = (\lambda_1-\lambda_0/4)/3$.

Remarks: D* achieves lower bound on b of lemma 3.1.

Case 3b. $\lambda_0 \equiv 0 \pmod 4$, $\lambda_0 \geq 4$, $\lambda_1 - \lambda_0/4 \equiv 1 \pmod 3$, $\lambda_1 > 0$.

$f_1 = 0$, $f_2 = 2$, $f_3 = 0$, $f_4 = \lambda_0/4-1$, $f_5 = 0$, $f_6 = (\lambda_1-1-\lambda_0/4)/3$.

Remarks: D* achieves the minimum possible number of blocks $\geq$ lower bound of lemma 3.1.

Case 3c. $\lambda_0 \equiv 0 \pmod 4$, $\lambda_0 \geq 4$, $\lambda_1 - \lambda_0/4 \equiv 2 \pmod 3$, $\lambda_1 > 0$.

$f_1 = f_2 = f_3 = 0$, $f_4 = \lambda_0/4$, $f_5 = 1$, $f_6 = (\lambda_1-2-\lambda_0/4)/3$.

Remarks: D* achieves the minimum possible integer $b \geq$ lower bound of lemma 3.1.

Case 4a. $\lambda_0 \equiv 2 \pmod 4$, $\lambda_0 \geq 4$, $\lambda_1 - (\lambda_0+2)/4 \equiv 0 \pmod 3$, $\lambda_1 > 0$.

$f_1 = 0$, $f_2 = 1$, $f_3 = 0$, $f_4 = (\lambda_0-2)/4$, $f_5 = 0$, $f_6 = \{\lambda_1-(\lambda_0+2)/4\}/3$.

Remarks: D* achieves the minimum possible integer $b \geq$ lower bound of lemma 3.1.

Case 4b. $\lambda_0 \equiv 2 \pmod 4$, $\lambda_0 \geq 4$, $\lambda_1 - (\lambda_0+2)/4 \equiv 1 \pmod 3$, $\lambda_1 > 0$.

$f_1 = f_2 = 0$, $f_3 = 1$, $f_4 = (\lambda_0-2)/4$, $f_5 = 0$, $f_6 = \{\lambda_1-1-(\lambda_0+2)/4\}/3$.

Remarks: D* achieves the lower bound on b of lemma 3.1.

Case 4c. $\lambda_0 \equiv 2 \pmod 4$, $\lambda_0 \geq 4$, $\lambda_1-(\lambda_0+2)/4 \equiv 2 \pmod 3$, $\lambda_1 > 0$.

$f_1 = 0$, $f_2 = 1$, $f_3 = 0$, $f_4 = (\lambda_0-2)/4$, $f_5 = 1$, $f_6 = \{\lambda_1-2-(\lambda_0+2)/4\}/3$.

Remarks: D* achieves the minimum possible integer $b \geq$ lower bound of lemma 3.1.

This covers all possible cases.

Theorem 3.4. For p = 6, k = 3 the minimal complete set of generator designs is as given in Table 3.4.

Table 3.4
Minimal Complete Set of Generator Designs for $p = 6$, $k = 3$

D_i	Design	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}\right\}$	6	2	0
D_2	$\left\{\begin{matrix} 0 & 0 & 0 & 1 & 1 & 2 & 3 \\ 1 & 2 & 4 & 2 & 5 & 3 & 4 \\ 3 & 6 & 5 & 4 & 6 & 5 & 6 \end{matrix}\right\}$	7	1	1
D_3	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 3 & 5 \\ 2 & 5 & 6 & 3 & 4 & 5 & 6 & 5 & 6 & 5 & 6 \end{matrix}\right\}$	11	3	1
D_4	$\left\{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 4 & 5 & 6 & 5 & 6 & 6 \end{matrix}\right\}$	15	5	1
D_5	$\left\{\begin{matrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 4 \\ 2 & 2 & 3 & 3 & 4 & 3 & 3 & 4 & 5 & 5 \\ 5 & 6 & 4 & 6 & 5 & 4 & 5 & 6 & 6 & 6 \end{matrix}\right\}$	10	0	2

Proof. Consider an arbitrary BTIB design D with parameters (b,λ_0,λ_1) for $p = 6$, $k = 3$.

Case 1. $\lambda_1 = 0$.

$f_1 = \lambda_0/2$, $f_2 = f_3 = f_4 = f_5 = f_6 = 0$.

Remarks: By lemma 3.2 λ_0 must be even. D* must be equivalent to D.

Case 2. $\lambda_0 \equiv 0 \pmod 5$, $\lambda_1 > 0$.

$f_1 = f_2 = f_3 = 0$, $f_4 = \lambda_0/5$, $f_5 = (\lambda_1 - \lambda_0/5)/2$, $f_6 = 0$.

Remarks: $\lambda_1 - \lambda_0/5$ is even and non-negative by lemmas 3.2 and 3.3. D* achieves the lower bound on b given in lemma 3.1.

Case 3. $\lambda_0 \equiv 1 \pmod 5$, $\lambda_1 > 0$.

$f_1 = 0$, $f_2 = 1$, $f_3 = 0$, $f_4 = (\lambda_0-1)/5$, $f_5 = \{\lambda_1-1-(\lambda_0-1)/5\}/2$, $f_6 = 0$.

Remarks: $\lambda_1-1-(\lambda_0-1)/5$ is even and non-negative by lemmas 3.2 and 3.3. D* achieves the lower bound on b given in lemma 3.1.

Case 4. $\lambda_0 \equiv 2 \pmod 5$, $\lambda_1 > 0$.

$f_1 = 0$, $f_2 = 2$, $f_3 = 0$, $f_4 = (\lambda_0-2)/5$, $f_5 = \{\lambda_1-2-(\lambda_0-2)/5\}/2$, $f_6 = 0$.

Remarks: $\lambda_1-2-(\lambda_0-2)/5$ can be shown to be non-negative and even using lemmas 3.2 and 3.3. D* achieves the lower bound on b given in lemma 3.1.

Case 5. $\lambda_0 \equiv 3 \pmod 5$, $\lambda_1 > 0$

$f_1 = f_2 = 0$, $f_3 = 1$, $f_4 = (\lambda_0-3)/5$, $f_5 = \{\lambda_1-1-(\lambda_0-3)/5\}/2$, $f_6 = 0$.

Remarks: $\lambda_1-1-(\lambda_0-3)/5$ can be shown to be non-negative and even using lemmas 3.2 and 3.3. D* achieves the lower bound on b given in lemma 3.1.

Case 6. $\lambda_0 \equiv 4 \pmod 5$, $\lambda_1 > 0$.

$f_1 = 0$, $f_2 = f_3 = 1$, $f_4 = (\lambda_0-4)/5$, $f_5 = \{\lambda_1-2-(\lambda_0-4)/5\}/2$, $f_6 = 0$.

Remarks: $\lambda_1-2-(\lambda_0-4)/5$ can be shown to be non-negative and even using lemmas 3.2 and 3.3. D* achieves the lower bound on b given in lemma 3.1.

This covers all possible cases and hence completes the proof of the theorem.

3.3 Minimal Complete Sets for k = 3, p = 7(1)10

In this section we give the minimal complete sets of generator designs for k = 3, p = 7(1)10 without the accompanying proofs. The interested reader can construct the proofs for himself along the lines of the proofs in the previous section or he can obtain the proofs by writing to one of the authors.

Table 3.5

Minimal Complete Set of Generator Designs for p = 7, k = 3

D_i	Design	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7	7	2	0
D_2	0 0 0 0 0 0 0 1 1 2 2 3 1 1 2 3 4 5 7 4 6 3 5 4 2 3 4 5 6 6 7 5 7 6 7 7	12	2	1
D_3	0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 4 1 1 1 1 2 2 2 3 3 3 4 5 6 7 6 3 5 2 3 4 5 4 5 7 4 5 7 6 6 6 7 7 6 7	17	4	1
D_4	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 2 2 2 2 2 3 3 3 3 4 4 4 5 5 6 2 3 4 5 6 7 3 4 5 6 7 4 5 6 7 5 6 7 6 7 7	21	6	1
D_5	1 1 1 2 2 3 3 2 4 6 4 5 4 5 3 5 7 6 7 7 6	7	0	1

Table 3.6

Minimal Complete Set of Generator Designs for p = 8, K = 3

D_i	Design	b_i'	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8	8	2	0
D_2	0 0 0 0 1 1 1 2 2 2 3 4 1 3 5 7 3 4 6 3 4 5 6 5 2 4 6 8 5 8 7 7 6 8 8 7	12	1	1
D_3	0 0 0 0 0 0 0 0 0 0 0 0 1 1 2 3 4 5 1 1 1 2 2 2 3 3 4 6 7 8 2 6 3 4 5 6 3 4 5 4 5 7 5 7 6 6 8 8 8 7 6 8 7 8	18	3	1
D_4	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 5 1 1 1 1 2 2 2 2 2 3 3 3 4 4 4 4 5 6 6 7 8 2 6 3 4 6 7 3 4 5 6 7 5 6 7 5 6 7 8 8 7 8 8 8 8 7	23	5	1
D_5	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 2 2 2 2 2 2 3 3 3 3 3 4 4 4 4 5 5 5 6 6 7 2 3 4 5 6 7 8 3 4 5 6 7 8 4 5 6 7 8 5 6 7 8 6 7 8 7 8 8	28	7	1
D_6	1 1 1 1 1 1 1 2 2 2 2 2 3 3 3 3 4 5 7 2 2 3 4 4 6 6 3 4 4 5 5 4 4 5 5 6 6 7 3 8 8 5 5 7 7 6 5 8 7 7 7 7 6 8 8 8 8	19	0	2
D_7	All 56 combinations of three digits of 1,2,...,8.	56	0	6

Table 3.7

Minimal Complete Set of Generator Designs for p = 9, k = 3

D_i	Design	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8 9	9	2	0
D_2	0 0 0 0 0 0 0 0 0 1 1 1 2 2 3 3 4 5 1 1 2 2 3 4 6 7 8 2 4 6 3 5 4 7 5 6 3 5 4 7 5 6 9 8 9 8 9 7 6 9 8 9 7 8	18	2	1
D_3	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 2 3 4 5 1 1 1 1 2 2 2 2 3 3 3 4 4 5 6 6 7 7 2 6 3 4 5 8 3 4 5 8 4 5 7 8 5 7 9 6 9 6 8 9 8 9 9 7 6 8 7 9	24	4	1
D_4	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 3 5 1 1 1 1 1 1 2 2 2 2 2 2 3 3 3 3 4 4 4 4 5 5 6 6 7 7 8 2 4 6 3 4 5 6 7 8 3 4 5 6 7 8 5 6 7 9 5 6 7 9 8 9 8 9 8 9 9 9 8 7	30	6	1
D_5	{All 36 combinations of two digits of 1,2,...,9 bordered by a single row of zeros}	36	8	1
D_6	1 1 1 1 2 2 2 3 3 3 4 5 2 4 6 8 4 5 7 4 5 6 7 6 3 5 7 9 6 9 8 8 7 9 9 8	12	0	1

Table 3.8

Minimal Complete Set of Generator Deseigns for p = 10, k = 3

D_i	Design[1]	b_i	$\lambda_0^{(i)}$	$\lambda_1^{(i)}$
D_1	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 2 3 4 5 6 7 8 9 X	10	2	0
D_2	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 2 2 3 3 4 5 8 1 1 1 2 2 2 3 3 4 4 5 6 6 7 7 2 4 6 3 5 4 7 5 6 9 3 5 8 4 7 8 5 9 6 X X 9 X 8 9 X 9 7 6 9 8 X 7 8 X	25	3	1
D_3	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 2 3 4 5 7 0 1 1 1 1 1 2 2 2 2 2 3 3 3 3 4 4 4 5 5 6 6 6 7 8 2 6 3 4 5 8 9 X 3 4 5 8 9 4 5 7 8 9 5 7 9 X 6 7 9 6 7 8 9 X 8 X X 7 6 8 X 9 X	32	5	1
D_4	0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 2 2 2 2 2 2 3 3 3 3 3 4 4 4 5 5 6 7 8 0 1 1 2 2 3 4 6 7 8 2 2 3 4 4 5 6 6 3 3 4 5 5 7 4 4 5 7 7 5 5 6 6 6 9 8 9 X 3 5 4 7 5 6 9 8 9 8 8 X 9 9 X 7 7 6 6 X 9 9 X 8 8 X 9 9 7 7 X 8 8 X X X	37	2	2
D_5	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 3 5 7 0 0 1 1 1 1 1 1 1 2 2 2 2 2 2 2 3 3 3 3 3 4 4 4 4 4 5 5 5 6 6 6 7 8 8 2 4 6 9 X X 3 4 5 6 7 8 9 3 4 6 7 8 9 X 5 6 7 9 X 5 5 6 7 X 7 8 9 7 8 9 8 9 X X 8 X X	39	7	1
D_6	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 2 2 2 2 2 3 3 3 3 4 4 4 5 6 7 8 0 0 1 1 1 1 2 2 2 2 3 3 3 4 4 5 6 6 7 7 8 2 2 3 4 5 6 6 3 3 4 5 7 4 4 5 7 5 5 6 6 9 8 9 X X 3 4 5 8 4 5 7 8 5 7 9 6 9 6 8 9 8 9 9 8 9 X 9 X 7 7 6 6 X 9 X 8 8 X 9 7 7 X 8 X X X	44	4	2
D_7	{All 45 combinations of two digits of 1,2,...,10 bordered by a single row of zeros}	45	9	1
D_8	1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 3 3 3 3 3 3 4 4 4 5 5 5 6 7 2 2 2 3 4 5 6 7 9 3 4 4 5 5 7 8 4 4 5 6 6 7 6 7 9 6 6 8 8 8 3 6 7 8 5 8 9 X X 9 6 X 7 9 8 X 5 8 X 7 9 X 8 9 X 7 X 9 X 9	30	0	2

[1] X denotes treatment number 10.

REFERENCES

Bechhofer, R. E. and Tamhane, A. C. (1981). Incomplete block designs for comparing treatments with a control: General theory. _Technometrics_ _23_, 45-57.

Bechhofer, R. E. and Tamhane, A. C. (1982). Tables of admissible and optimal balanced treatment incomplete block (BTIB) designs for comparing treatments with a control. Submitted for publication to _Selected Tables in Mathematical Statistics_.

Nair, K. R. and Rao, C. R. (1942). Incomplete block designs involving several groups of varieties. _Science and Culture_ _7_, 625.

Pearce, S. C. (1960). Supplemented balance. _Biometrika_ _47_, 263-271.

Raghavarao, D. (1971). _Construction and Combinatorial Problems in Design of Experiments_. John Wiley & Sons, Inc., New York.

Ture, T. E. (1982). On the construction and optimality of balanced treatment incomplete block designs. Ph.D. Dissertation, Univ. of California, Berkeley, California.

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