ESTIMATING A POSSIBLY RATIONAL MEAN

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Abstract

Let X be normal with unknown mean θ and known variance σ^2. We consider estimating θ under certain priors giving positive weight to the rational numbers with squared error loss. The results indicate the nature of the effect of this on the estimate, and also that procedures of this type can have considerable robustness.

<u>0. Preliminaries</u>. Let X be an $N(\theta,\sigma^2)$ random variable with θ having the formal prior measure ξ. Then the observation has the marginal "density"

$$f(x) = \int \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} d\xi(\theta) \tag{1}$$

and the formal Bayes estimator with squared error loss is

$$\hat{\theta}(x) = \frac{\int \theta \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} d\xi(\theta)}{\int \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} d\xi(\theta)} \tag{2}$$

$$= x + \frac{(x-\theta) \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta)^2} d\xi(\theta)}{\int \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-\theta^2)} d\xi(\theta)} = x - \sigma^2 \frac{f'(x)}{f(x)} .$$

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If ξ is a proper prior, then the Bayes risk improvement is

(3) $$I(0) = \int \sigma^4 \frac{f'^2}{f} dx ,$$

and in fact the improvement of the Bayes estimator over $X - q(X)$ is

(4) $$I(q) = \int (\sigma^2 \frac{f'}{f} - q)^2 f dx.$$

If ξ is improper with $\xi(A) = \xi(A+1)$, and $\xi[0,1) = 1$, the same holds for the average risk with the integral being restricted to an interval of length 1.

1. Case of a possibly fixed mean. Suppose $d\xi(\theta) = c_1\delta\{0\} + c_2\eta(\theta)$, where η has a density. We shall assume that the scale factor is so adjusted that for θ near 0, $d\xi(\theta) \sim d\delta\{0\} + d\theta$. Thus to this approximation

(5) $$f(x) = 1 + \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} ,$$

and the correction to the Bayes estimator is $Xh(X)$, where

(6) $$h(x) = \frac{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}}{1 + \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}} .$$

A possible simple attempt to approximate h is to use X if $h(X) \leq \frac{1}{2}$, and 0 otherwise, i.e., to estimate θ by 0 if $\theta = 0$ has posterior probability at least 1/2, and to estimate θ by X otherwise.

If we let $g(\sigma,a)$ be the disutility of not knowing whether $\theta = 0$ or not, the crude procedure can be easily seen to have a posterior disutility at most twice that of the Bayes procedure, and numerical calculations show that its disutility is less than 1.5 times that of the Bayes procedure.

The crude procedure comes close to being Bayes for small or large σ, but the approach for small σ is only logarithmic.

2. Case of possibly integer mean. If we know the mean is an integer we may try using the improper prior distribution with $\xi\{n\} = 1$ for all integers n. Thus

$$f(x) = \sum_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(x-n)^2} = \sum_{-\infty}^{\infty} e^{-2m^2\pi^2\sigma^2+2m\pi i x} \tag{7}$$

by the Poisson summation formula. For any σ, one of the series must converge rapidly.

The behavior of f and the resulting estimator

σ. For $\sigma^2 = .5$, f is almost constant. Even for $\sigma^2 = .125$, $\frac{f(.5)}{f(0)} > .7$, and the Bayes risk is only 7% less than that of X. However, the Bayes estimator clearly differs from X, but nowhere near the extent by which one value of θ dominates the posterior (for X = .4, the posterior probability of 0 is nearly .69, and for X = 0, the posterior probability is more than .96). The results indicate that attenuation of the dominant term is needed. When $\sigma^2 = .03$, the dominant term is now so strong that the nearest integer to X is already a good estimator. However, the marginal function f can probably be locally approximated, even for σ^2 as large as .5, by a combination of a multiple of the dominant term and a uniform term, and the method of the preceding section used. In the case where the prior is a mixture of a distribution on the integers and a density, this method of approximating by an attenuated dominant term corresponding to the most likely integer and an augmented locally flat prior should be even better.

3. Case of a possibly rational mean. If we assume the mean is rational, a possible prior is given by

(8) $$\xi^*(\tfrac{m}{n}) = \sum_{k=1}^{\infty} \frac{a_{kn}}{k} ,$$

with $\Sigma a_m = 1$. With a_m proportional to $m^{-1.5}$, this makes $\xi^*(i) \sim .5$ if i is an integer. The marginal density $f = \Sigma a_m f_m$, where

(9) $$f_m(x) = \frac{1}{m} \sum \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(- \frac{(x-\frac{j}{m})^2}{2\sigma^2}\right)$$

$$= \sum e^{-2\pi^2 m^2 k^2 \sigma^2 + 2\pi i k m x} .$$

For any fixed x, the practical computational problem is not difficult. However, in this investigation it was necessary to obtain f and f' for many points for the purpose of numerical integration to evaluate the risk. Consequently, the last expression in (9) was used for f and differentiated for f' (notice that the coefficient of $e^{2\pi i j x}$ in (9) only depends on j and on whether m divides j). Since f is even and f' odd, this was done by a single application of the Fast Fourier Transform.

Examination of the detailed output for selected values of σ^2 indicate that the reduction in the Bayes risk form that of the estimator X is mainly due to the effect of moving the estimator to a salient nearby rational number. However, we have not been able to quantify this. For example the contribution of $\theta = \frac{1}{50}$, which is clearly separated from that of other multiples of $\frac{1}{50}$ for $\sigma^2 = 5 \times 10^{-5}$, does not show as a distinct component for the above model for $\sigma^2 = 5 \times 10^{-7}$. The reason is that values of θ like $\frac{1}{48}$, $\frac{1}{49}$, $\frac{1}{51}$, $\frac{1}{52}$, $\frac{2}{97}$, $\frac{2}{99}$, etc., cause the marginal density of x in a neighborhood of $\frac{1}{50}$ to appear to come from a smooth density rather than a discrete distribution. This effect is even greater if there is a continuous component. We give a short table of the average Bayes risk relative to σ^2 for mixtures of the prior ξ^* above and the uniform prior.

	Proportion Discrete			
$\log_2 \sigma^2$	1.0	.8	.5	.2
-3	.990	.992	.997	1.000
-4	.938	.961	.985	.998
-5	.885	.925	.970	.995
-6	.820	.882	.952	.992
-7	.749	.833	.929	.987
-8	.685	.786	.905	.982
-9	.622	.739	.880	.975
-10	.565	.695	.855	.968
-11	.514	.655	.830	.960
-12	.467	.617	.806	.951
-13	.426	.583	.784	.942
-14	.389	.553	.763	.933
-15	.355	.525	.744	.925
-16	.325	.499	.726	.916
-17	.297	.476	.709	.908
-18	.272	.455	.694	.901
-19	.249	.435	.680	.894
-20	.228	.417	.667	.887
-21	.209	.401	.655	.881

4. Summary. Although our investigations on the estimation of a possibly rational mean with squared error loss have concentrated on a few specific priors, the results indicate that the Bayes procedure is approximately one which adjusts the sample mean (or possibly the Bayes estimate from a highly smoothed prior) to include the special contribution of a salient specific candidate for the true value of θ. It seems very likely that such a procedure will be robust.

One may ask whether there should be some component of the loss function to favor the estimate being a rational number. An additive component will do this even if the sample is so large that this is strongly contraindicated by the data. In many situations this is appropriate, (see, for example Rubin [1970]). However, there cannot be a consistent procedure under those circumstances, and we cannot reasonably compare the performance with that of $\bar{X}$.

Reference

Rubin, H. Decision theoretic approach to some multivariate problems, Multivariate Analysis II (Proc. Second Internat. Symp., Dayton, Ohio 1968) pp. 507-513. Academic Press, New York 1969.