EXACT DISTRIBUTION OF WILKS' L_{vc} CRITERION AND ITS PERCENTAGE POINTS IN THE COMPLEX CASE

by

Anita Singh

Mimeograph Series #81-16

Department of Statistics
Division of Mathematical Sciences

May 1981

ABSTRACT

The exact null distribution of the Wilks' L_{vc} criterion for testing the hypothesis H: $\underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\underset{\sim}{e}\underset{\sim}{e}']$, $\sigma > 0$, σ and ρ unknown against the alternative $A \neq H$ where $\underset{\sim}{e}' = (1,1, \ldots, 1)$ in a p-variate complex normal population $CN_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ has been derived in a gamma series form and in an alternate series form using contour integration. Percentage points for $p = 2(1)8$ for various levels of significance and various degrees of freedom have been computed and tabulated.

Keywords and Phrases: Exact null distribution, complex normal population, likelihood ratio criterion, Meijer's G-function, contour integration, Mellin transform, tables of percentage points.

EXACT DISTRIBUTION OF WILKS' L_{vc} CRITERION AND ITS PERCENTAGE POINTS IN THE COMPLEX CASE

BY

ANITA SINGH

1. INTRODUCTION

Let $\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_N$ be independent complex normal random p-vectors with unknown mean vector $\underset{\sim}{\xi}$ and positive definite hermitian (p.d.h.) covariance matrix $\underset{\sim}{\Sigma}$, i.e., $\underset{\sim}{Z}_i \sim CN(\underset{\sim}{\xi}, \underset{\sim}{\Sigma})$. Let $\underset{\sim}{Z} = (\underset{\sim}{Z}_1, \underset{\sim}{Z}_2, \ldots, \underset{\sim}{Z}_N)$. Then $\underset{\sim}{Z} \sim CN(\underset{\sim}{Z}; \underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, (see Goodman [1]), where the complex multivariate normal distribution is defined by

$$CN(\underset{\sim}{Z}; \underset{\sim}{\mu}, \underset{\sim}{\Sigma}) = (\pi)^{-pN}|\underset{\sim}{\Sigma}|^{-N}\exp(-tr\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{Z}-\underset{\sim}{\mu})(\overline{\underset{\sim}{Z}-\underset{\sim}{\mu}})') \tag{1.1}$$

and $\underset{\sim}{\mu} = (\underset{\sim}{\xi}, \underset{\sim}{\xi}, \ldots, \underset{\sim}{\xi})$ is a $p \times N$ complex matrix. Let us define

$$\underset{\sim}{Z}_0 = N^{-1}\sum_{i=1}^{N}\underset{\sim}{Z}_i \quad \text{and} \quad \underset{\sim}{S} = \sum_{i=1}^{N}(\underset{\sim}{Z}_i - \underset{\sim}{Z}_0)\,(\overline{\underset{\sim}{Z}_i - \underset{\sim}{Z}_0})'. \tag{1.2}$$

Then $N^{-1/2}(\underset{\sim}{Z}_0 - \underset{\sim}{\xi}) \sim CN(\underset{\sim}{0}, \underset{\sim}{\Sigma})$ and $\underset{\sim}{S}$ has an independent complex Wishart distribution which is defined by

$$CW(\underset{\sim}{S}; \underset{\sim}{p}, \underset{\sim}{N}, \underset{\sim}{\Sigma}) = [\Gamma_p(n)]^{-1}|\underset{\sim}{\Sigma}|^{-n}|\underset{\sim}{S}|^{n-p}\exp(-tr\underset{\sim}{\Sigma}^{-1}\underset{\sim}{S}) \tag{1.3}$$

with $n = N-1$ and

$$\tilde{\Gamma}_p(n) = (\pi)^{p(p-1)/2}\prod_{i=1}^{p}\Gamma(n-i+1) \tag{1.4}$$

and $\underset{\sim}{S}$ is a p.d.h. matrix of order p. In this paper, we obtain the exact null distribution of Wilks' [6] L_{vc} criterion for testing H: $\underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\underset{\sim}{e}\underset{\sim}{e}']$, $\sigma > 0$, σ and ρ unknown against the alternative $A \neq H$ where $\underset{\sim}{e}' = (1, 1, \ldots, 1)$. In Section 2, we present the distribution of L_{vc} in terms of Meijer's [2] G-function, where as in Section 3, using the methods similar to those of Pillai and Singh [4], obtain the distribution of L_{vc} in two series forms which are useful to compute the percentage points of L_{vc} to a desirable degree of accuracy. The percentage points of L_{vc} have been tabulated for $p = 2(1)8$ and for various values of the significance level in Table (3.1).

2. DERIVATION OF THE DISTRIBUTION OF L_{vc}

In this section, we obtain the null moments and the exact distribution of L_{vc} in terms of Meijer's [2] G-function using Mellin's integral transform.

As in Singh and Pillai [5] the test of H: $\underset{\sim}{\Sigma} = \sigma^2[(1-\rho)\underset{\sim}{I} + \rho\underset{\sim}{e}\underset{\sim}{e}']$ reduces to that of H: $\underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \underset{\sim}{I}_{p_2} \end{bmatrix}$, $\sigma_1 > 0$, $\sigma_2 > 0$

and unknown, against the alternative $A \neq H$; $p_2 = p - 1$. The likelihood ratio criterion for testing H versus A, can be expressed in terms of the following statistic

$$L_{vc} = |\underset{\sim}{S}|/[s_{11}(tr(\underset{\sim}{S}_{22}/p_2))^{p_2}] \qquad (2.1)$$

where

$$\underset{\sim}{S} = \begin{bmatrix} s_{11} & \underset{\sim}{S}_{12} \\ \underset{\sim}{S}_{12}' & \underset{\sim}{S}_{22} \end{bmatrix} \begin{matrix} 1 \\ p_2 \end{matrix} .$$

The following lemmas are direct consequences of theorem (3.2) of Singh and Pillai [5].

Lemma 2.1. The h-th moment of $L_{vc} = |\underset{\sim}{S}|/[s_{11}(tr(S_{22}/p_2))^{p_2}]$ under the null hypothesis H is given by

$$E[L_{vc}]^h = p_2^{p_2 h}\Gamma(np_2) \prod_{i=1}^{p_2}\Gamma(h+n-i)/[\Gamma(p_2(h+n)) \prod_{i=1}^{p_2}\Gamma(n-i)] \qquad (2.2)$$

where h is any complex number.

Lemma 2.2. The null density of L_{vc} is given by

$$p(L_{vc}) = D_1(p_2, n)(L_{vc})^{-(p_2+1)} G_{p_2\ p_2}^{p_2\ 0}\left[L_{vc}\middle|\begin{matrix}a_1, a_2, \ldots, a_{p_2}\\ b_1, b_2, \ldots, b_{p_2}\end{matrix}\right] \qquad (2.3)$$

where

$$D_1(p_2, n) = (2\pi)^{(p_2-1)/2} p_2^{\frac{1}{2}-np_2} \Gamma(np_2)/\prod_{i=1}^{p_2}\Gamma(n-i) \qquad (2.4)$$

and

$$a_i = p_2 + n + (i-1)p_2^{-1},$$
$$b_i = p_2 + n - i \quad ; \quad i = 1, 2, \ldots, p_2 . \qquad (2.5)$$

Special Cases In particular for $p_2 = 1$ and $p_2 = 2$, respectively, we have

$$p(L_{vc}) = (L_{vc})^{n-2}\Gamma(n)/\Gamma(n-1) \qquad (2.6)$$

and

$$p(L_{vc}) = \pi^{3/2} 2^{1-2n}\Gamma(2n)/[\Gamma(n-2)\tilde{\Gamma}_2(n)](L_{vc})^{-3}\ G_{22}^{20}\left[L_{vc}\middle|\begin{matrix}2+n, & n+3/2\\ n, & n+1\end{matrix}\right] \qquad (2.7)$$

Using the duplication formula of gamma functions this can be written as

$$p(L_{vc}) = \frac{\Gamma(n)\Gamma(n+\frac{1}{2})}{\Gamma(n-1)\Gamma(n-2)\Gamma(\frac{7}{2})} (L_{vc})^{\frac{3}{2}} (1 - L_{vc})^{\frac{5}{2}} {}_2F_1(\tfrac{3}{2}, 1, \tfrac{7}{2}; 1 - L_{vc}) \quad (2.8)$$

3. EXACT DISTRIBUTION OF L_{vc} IN TWO SERIES FORMS

This section has two parts (a) and (b). In part (a) we obtain the distribution of L_{vc} using method of contour integration. This form of the density is well suited for the computation of percentage points of L_{vc} for small values of N, the sample size. In part (b), we obtain the distribution of L_{vc} as a gamma series. This form of the density has been used for the computations of percentage points for large values of N.

(a) Distribution of L_{vc} through Contour Integration

Using Mellin Integral transform on (2.2), we have the density of L_{vc} in the form

$$p(L_{vc}) = D(p_2, n)\, p_2^{-p_2 n} (L_{vc})^{n-1} (2\pi i)^{-1} \int_{C-i\infty}^{C+i\infty} (L_1)^{-h} \frac{\prod_{i=1}^{p_2} \Gamma(h-i)}{\Gamma(p_2 h)}\, dh \quad (3.1)$$

where

$$D(p_2, n) = \Gamma(np_2) / \prod_{i=1}^{p_2} \Gamma(n-i) \quad (3.2)$$

and

$$L_1 = L_{vc}/p_2^{p_2} \quad .$$

The poles of the integrand are at points

$$h = -\ell, \ \ell = -p_2, \ldots, -1, 0, 1, 2, \ldots . \quad (3.3)$$

The residue at these points can be obtained by $h = t - \ell$ in (3.1) and then finding the residue at $t=0$. Making this transformation, the integrand in (3.1) can be written as

$$G(t-\ell) = (L_1)^{\ell-t} \prod_{i=1}^{p_2} \Gamma(t-\ell-i)/\Gamma(p_2(t-\ell)), \ell = -p_2, \ldots, -1, 0, 1, \ldots \quad (3.4)$$

Two cases arise (A) $\ell \geq 0$, (B) $\ell < 0$.

CASE A: $\ell \geq 0$. After expanding the gamma function in (3.4), we have

$$G(t-\ell) = a_0(L_1)^{\ell} \cdot (t)^{-(p_2-1)} A(t), \quad (3.5)$$

where

$$a_0 = (-1)^{p_2(p_2+1)/2} \cdot p_2(p_2\ell)!/\prod_{i=1}^{p_2} (\ell+i)! \quad (3.6)$$

and

$$A(t) = (L_1)^{-t}(\Gamma(t+1))^{p_2} \prod_{\delta=1}^{p_2\ell} (1-p_2t/\delta)/[\Gamma(tp_2+1) \prod_{i=1}^{p_2} \prod_{\delta=1}^{\ell+i} (1-t/\delta)]. \quad (3.7)$$

From (3.5), we note that we have a pole of order (p_2-1) at $t=0$. Using (4.36), (4.37), (4.38) of Pillai and Singh [3] we can write log $A(t)$ as

$$\log A(t) = a_1t + a_2t^2/2! + a_3t^3/3! + \ldots , \quad (3.8)$$

where

$$a_1 = \sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta) - \log L_1 - \sum_{\delta=1}^{p_2\ell} (p_2/\delta) \quad (3.9)$$

and for $q \geq 2$

$$a_q = \psi_{q-1}^{(1)} [p_2 - p_2^q] + (q-1)![\sum_{i=1}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q - \sum_{\delta=1}^{p_2\ell} (p_2/\delta)^q]$$

Now using (3.8) in (3.5) and lemma (4.1) of Pillai and Singh [3], we get the residue R_ℓ given by

$$R_\ell = (L_1)^\ell a_0 D_{p_2-2}(L_1;\, a)/\Gamma(p_2-1) \tag{3.10}$$

where D_{p_2-2} is the same as the right hand side of (4.28) of Singh and Pillai [5] with a_q's defined by (3.9).

CASE B. $\ell < 0$. As before, after the expansion of the gamma functions in (3.4), we have

$$G(t-\ell) = b_0'(L_1)^\ell (t)^{-(p_2+\ell+1)} B(t) \tag{3.11}$$

where

$$b_0' = (-1)^{(p_2+\ell)(p_2+\ell+1)/2} \Big/ \prod_{i=-\ell}^{p_2} (\ell+i)! \tag{3.12}$$

and

$$B(t) = (L_1)^{-t}(\Gamma(t+1))^{p_2+\ell+1} \prod_{i=1}^{-\ell-1} \Gamma(t-\ell-i)/[\Gamma(p_2(t-\ell)) \prod_{i=-\ell}^{p_2} \prod_{\delta=1}^{\ell+i} (1-t/\delta)]. \tag{3.13}$$

From (3.11) we notice that we have a pole of order $(p_2+\ell+1)$ at $t=0$ and as before using (4.36), (4.37), and (4.38) of Pillai and Singh [3] $\log B(t)$ can be written as

$$\log B(t) = \log b_0'' + b_1 t + b_2 t^2/2! + \ldots , \tag{3.14}$$

where

$$b_0'' = \prod_{i=1}^{-\ell-1} \Gamma(-\ell-i)/\Gamma(-p_2\ell),$$

$$b_1 = -\log L_1 + (p_2+\ell+1)\psi(1) - p_2\psi(-p_2\ell) + \sum_{i=1}^{-\ell-1} \psi(-\ell-i) + \sum_{i=-\ell}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta) \tag{3.15}$$

and for $q \geq 2$, we have

$$b_q = \sum_{i=1}^{-\ell-1} \psi_{q-1}(-\ell - i) + (p_2 + \ell + 1)\psi_{q-1}(1) - p_2^q \psi_{q-1}(-p_2\ell) + (q-1)! \sum_{i=-\ell}^{p_2} \sum_{\delta=1}^{\ell+i} (1/\delta)^q .$$

Now using (3.14) in (3.11) and appealing to lemma (4.1) of Pillai and Singh [3] we have the residue R_ℓ in the form

$$R_\ell = b_0(L_1)^\ell D_{p_2+\ell}(L_1;\ b)/\Gamma(p_2 + \ell + 1), \tag{3.16}$$

where $b_0 = b_0' \cdot b_0''$ and the determinant $D_{p_2+\ell}(L_1;\ b)$ is of order $p_2 + \ell$ and is the same as the one on the R.H.S. of (4.28) of Singh and Pillai [5] with elements a_q's replaced by b_q's, where b_q's are defined by (3.15). Hence for any $p_2 \geq 1$, we have from (3.1) and (3.2) and Cauchy's residue theorem, the exact distribution of L_{vc} in the form

$$p(L_{vc}) = D(p_2,\ n)p_2^{-np_2}(L_{vc})^{n-1}\left[\sum_{\ell\geq 0} R_\ell + \sum_{\ell=-p_2}^{-1} R_\ell\right], \tag{3.17}$$

where R_ℓ's are given in (3.10) and (3.16) respectively.

(b) Distribution of L_{vc} as a gamma series

We shall now obtain the distribution of L_{vc} in a gamma series form. Let $\lambda = (L_{vc})^n$ and $\lambda^* = -2\rho \log \lambda$, where ρ is chosen so that the rate of convergence of the resulting series can be controlled,

$0 < \rho \leq 1$. Let $\phi(t)$ be the characteristic function of λ^*. Then

$$\phi(t) = E(L_{vc})^{-2it\rho n} \tag{3.18}$$

Now using (2.2), (3.18) can be written as

$$\phi(t) = D(p_2, n)\exp(\log G(t)), \text{ where} \tag{3.19}$$

$$G(t) = p_2^{-2np_2 it} \prod_{\delta=1}^{p_2} \Gamma(n\rho(1-2it) + n(1-\rho) - \delta)/\Gamma(np_2\rho(1-2it) + p_2 n(1-\rho)) \tag{3.20}$$

and $D(p_2, n)$ is given by (3.2).

Taking logarithm on both sides of (3.20) and using the expansion (5.7) of Pillai and Singh [3] for each of the gamma functions involved in (3.20), we obtain,

$$\log G(t) = (p_2 - 1)/2\log 2 - (p_2 n - \tfrac{1}{2})\log p_2 - (p_2 + (p_2^2 - 1)/2)\log(n\rho(1-2it)) + \sum_{r=1}^{m} (\rho n(1-2it))^r w_r + R_{m+1}^0(n, t) \tag{3.21}$$

where the coefficients w_r are given by

$$w_r = \left[B_{r+1}(np_2(1-\rho))/p_2^r - \sum_{\delta=1}^{p_2} B_{r+1}(n(1-\rho) - \delta)\right](-1)^r/(r(r+1)). \tag{3.22}$$

Thus $G(t)$ can be written as

$$G(t) = (2\pi)^{(p_2-1)/2} (n\rho(1-2it))^{-(p_2+(p_2^2-1)/2)} p_2^{-(p_2 n-\frac{1}{2})} \sum_{r=0}^{\infty} W_r((1-2it)\rho n)^{-r} + R_{m+1}'(n, t), \tag{3.23}$$

where W_r is the coefficient of $((1-2it)\rho n)^{-r}$ in the expansion of $\exp(\sum_{r=1}^{m} ((1-2it)\rho n)^{-r} w_r)$.

Let $u = p_2 + p_2^2/2 - ½$. Now from (3.19) and (3.23), we have

$$\phi(t) = D_1(p_2, n) \sum_{r=0}^{\infty} ((1 - 2it)\rho n)^{-(r+u)} W_r + R''_{m+1}(n, t) \tag{3.24}$$

where $D_1(p_2, n) = D(p_2, n)(2\pi)^{(p_2-1)/2} p_2^{(½-np_2)}$.

Now $(1 - i\beta t)^{-\alpha}$ being the characteristic function of the gamma density $g_\alpha(\beta, x)$, we have the density of L_{vc} in the form

$$p(L_{vc}) = D_1(p_2, n) \sum_{r=0}^{\infty} (\rho n)^{-(r+u)} W_r g_{r+u}(2, \lambda^*) + R'''_{m+1}(n) \tag{3.25}$$

Hence the probability that λ^* is larger than any value, say λ_0 is

$$P[\lambda^* > \lambda_0] = D_1(p_2, n) \sum_{r=0}^{\infty} (\rho n)^{-(r+u)} W_r G_{r+u}(2, \lambda_0) + R_{m+1}(n) \tag{3.26}$$

where $$G_{r+u}(2, \lambda_0) = \int_{\lambda_0}^{\infty} g_{r+u}(2, x)dx \quad \text{and} \tag{3.27}$$

$$R_{m+1}(n) = (2\pi)^{-1} D_1(p_2, n) \int_{\lambda_0}^{\infty} \int_{-\infty}^{\infty} e^{-it\lambda^*} \sum_{r=0}^{\infty} W_r(\rho n)^{-(r+u)} (1 - 2\ it)^{-(r+u)}$$

$$[\exp(R'''_{m+1}(n)) - 1]dt\ d\lambda^*. \tag{3.28}$$

The choice of $\rho = 1$ does not give rapid convergence of the series in (3.26) for small values of N. Therefore, we chose ρ such that $w_1 = 0$ which is obtained by taking ρ as

$$\rho = 1 - [(2p_2^2(p_2+1)(p_2+2) + p_2^2 - 1)/(6np_2(p_2^2+2p_2-1))] \ . \qquad (3.29)$$

Thus from (3.26) we obtain the distribution of λ^* as a series of gamma distributions.

4. COMPUTATIONS OF PERCENTAGE POINTS

In this section, we tabulate the .005, .01, .025, .05, .1, and .25 percentage points of $L_1 = L^{2/N}$ for $p = 2(1)8$ and various values of N using (3.17), (3.26) and (3.29). These percentage points have been presented in Table (3.1) upto four significant digits. All the computations were carried out on CDC 6500 computer at the Purdue University Computing Center. The accuracy of the results have been checked by computing the percentage points for the case $p = 3$ in two ways (i) using the exact distribution of L_{vc} given by (3.17) and (ii) using the chi-square series form of the distribution of L_{vc} given by (3.26). The results obtained were in complete agreement at least upto four decimal places.

Table 3.1

Percentage Points of Wilk's L_{VC} Criterion (Complex Case)

p = 2

N \ α	.005	.01	.025	.05	.1	.25
3	$.0^2 5000$	.01000	.02500	.05000	.1000	.2500
4	.07071	.1000	.1581	.2236	.3162	.5000
5	.1710	.2154	.2924	.3684	.4642	.6299
6	.2659	.3162	.3976	.4729	.5623	.7071
7	.3466	.3981	.4782	.5493	.6310	.7579
8	.4135	.4642	.5407	.6070	.6813	.7937
9	.4691	.5180	.5904	.6518	.7197	.8203
10	.5157	.5623	.6306	.6877	.7499	.8409
11	.5551	.5995	.6637	.7169	.7743	.8572
12	.5887	.6310	.6915	.7411	.7943	.8706
13	.6178	.6579	.7151	.7616	.8111	.8816
14	.6431	.6813	.7354	.7791	.8254	.8909
15	.6653	.7017	.7530	.7942	.8377	.8989
16	.6849	.7197	.7684	.8074	.8483	.9057
17	.7024	.7356	.7820	.8190	.8577	.9117
18	.7181	.7499	.7941	.8293	.8660	.9170
19	.7322	.7627	.8049	.8384	.8733	.9217
20	.7450	.7743	.8147	.8467	.8799	.9259
22	.7673	.7943	.8316	.8609	.8913	.9330
24	.7860	.8111	.8456	.8727	.9006	.9389
25	.7942	.8185	.8518	.8779	.9047	.9415
26	.8019	.8254	.8575	.8827	.9085	.9439
28	.8156	.8377	.8677	.8912	.9152	.9481
30	.8276	.8483	.8766	.8985	.9211	.9517
35	.8517	.8697	.8942	.9132	.9326	.9589
40	.8699	.8859	.9075	.9242	.9412	.9642
45	.8841	.8984	.9178	.9327	.9479	.9683
50	.8955	.9085	.9260	.9395	.9532	.9715
55	.9049	.9168	.9328	.9450	.9575	.9742
60	.9127	.9237	.9384	.9497	.9611	.9764
65	.9193	.9295	.9431	.9536	.9641	.9782
70	.9250	.9345	.9472	.9569	.9667	.9798
75	.9300	.9389	.9507	.9598	.9690	.9812
80	.9343	.9427	.9538	.9623	.9709	.9824
85	.9382	.9460	.9565	.9646	.9726	.9834
90	.9416	.9490	.9590	.9665	.9742	.9844
95	.9446	.9517	.9611	.9683	.9755	.9852
100	.9474	.9541	.9631	.9699	.9768	.9860

Table 3.1 (Continued)

p = 3

N \ α	.005	.01	.025	.05	.1	.25
4	$.0^2 1011$	$.0^2 2028$	$.0^2 5128$	.01045	.02167	.06003
5	.02051	.02951	.04824	.07077	.1055	.1869
6	.06311	.08122	.1144	.1497	.1984	.2978
7	.1158	.1407	.1835	.2260	.2814	.3863
8	.1702	.1995	.2479	.2940	.3519	.4566
9	.2222	.2542	.3055	.3532	.4113	.5131
10	.2703	.3039	.3565	.4043	.4615	.5594
11	.3143	.3485	.4013	.4485	.5043	.5978
12	.3541	.3884	.4408	.4871	.5410	.6302
13	.3901	.4243	.4758	.5208	.5729	.6579
14	.4227	.4564	.5069	.5506	.6007	.6817
15	.4523	.4854	.5346	.5769	.6252	.7025
16	.4792	.5116	.5595	.6005	.6469	.7207
17	.5037	.5354	.5820	.6216	.6662	.7369
18	.5261	.5571	.6023	.6406	.6836	.7513
19	.5467	.5769	.6208	.6579	.6993	.7642
20	.5656	.5950	.6377	.6735	.7135	.7758
22	.5992	.6271	.6674	.7010	.7383	.7960
24	.6280	.6546	.6926	.7243	.7591	.8128
25	.6410	.6669	.7039	.7346	.7684	.8203
26	.6531	.6783	.7144	.7442	.7769	.8271
28	.6750	.6991	.7332	.7614	.7923	.8394
30	.6944	.7173	.7498	.7765	.8057	.8500
35	.7341	.7546	.7835	.8070	.8327	.8713
40	.7648	.7832	.8092	.8303	.8531	.8874
45	.7891	.8059	.8294	.8485	.8691	.8998
50	.8089	.8243	.8458	.8632	.8819	.9098
55	.8253	.8395	.8593	.8753	.8925	.9180
60	.8391	.8523	.8707	.8855	.9013	.9248
65	.8509	.8632	.8803	.8941	.9088	.9306
70	.8611	.8727	.8886	.9015	.9152	.9355
75	.8700	.8809	.8959	.9079	.9208	.9398
80	.8778	.8881	.9022	.9136	.9257	.9436
85	.8848	.8945	.9078	.9186	.9300	.9469
90	.8910	.9001	.9129	.9230	.9339	.9499
95	.8965	.9053	.9174	.9270	.9373	.9525
100	.9015	.9099	.9214	.9306	.9404	.9549

Table 3.1 (Continued)

p = 4

N \ α	.005	.01	.025	.05	.1	.25
5	$.0^{3}3016$	$.0^{3}5898$	$.0^{2}1467$	$.0^{2}2989$	$.0^{2}6277$	.01822
6	$.0^{2}6977$	.01016	.01696	.02546	.03915	.07392
7	.02530	.03307	.04778	.06409	.08769	.1400
8	.05264	.06498	.08690	.1097	.1407	.2044
9	.08515	.1014	.1291	.1567	.1929	.2634
10	.1199	.1393	.1713	.2023	.2419	.3162
11	.1551	.1769	.2120	.2453	.2869	.3631
12	.1897	.2131	.2505	.2852	.3280	.4046
13	.2229	.2476	.2864	.3220	.3652	.4414
14	.2544	.2800	.3197	.3557	.3990	.4742
15	.2843	.3104	.3506	.3867	.4297	.5036
16	.3123	.3388	.3792	.4151	.4577	.5300
17	.3387	.3652	.4056	.4413	.4832	.5539
18	.3633	.3899	.4301	.4653	.5065	.5755
19	.3865	.4130	.4527	.4875	.5279	.5952
20	.4082	.4345	.4738	.5080	.5476	.6131
22	.4476	.4734	.5117	.5447	.5826	.6448
24	.4825	.5076	.5447	.5764	.6127	.6717
25	.4984	.5232	.5596	.5908	.6262	.6837
26	.5135	.5379	.5736	.6042	.6388	.6948
28	.5411	.5647	.5992	.6286	.6617	.7150
30	.5659	.5887	.6220	.6502	.6819	.7327
35	.6177	.6388	.6692	.6947	.7233	.7686
40	.6587	.6781	.7060	.7293	.7552	.7960
45	.6919	.7098	.7355	.7569	.7806	.8177
50	.7192	.7359	.7597	.7794	.8012	.8352
55	.7422	.7577	.7799	.7981	.8183	.8496
60	.7617	.7762	.7969	.8139	.8327	.8618
65	.7784	.7921	.8115	.8275	.8450	.8721
70	.7930	.8059	.8242	.8391	.8556	.8810
75	.8058	.8180	.8352	.8494	.8648	.8887
80	.8171	.8287	.8450	.8583	.8730	.8955
85	.8272	.8382	.8536	.8663	.8802	.9015
90	.8362	.8467	.8614	.8735	.8866	.9069
95	.8443	.8543	.8684	.8799	.8924	.9117
100	.8517	.8612	.8747	.8857	.8976	.9160

Table 3.1 (Continued)

$p = 5$

N \ α	.005	.01	.025	.05	.1	.25
6	$.0^{3}1234$	$.0^{3}2238$	$.0^{3}5162$	$.0^{2}1014$	$.0^{2}2091$	$.0^{2}6124$
7	$.0^{2}2510$	$.0^{2}3677$	$.0^{2}6227$	$.0^{2}9493$	.01492	.02947
8	.01029	.01361	.02006	.02743	.03845	.06424
9	.02379	.02975	.04061	.05224	.06865	.1041
10	.04188	.05053	.06563	.08115	.1022	.1454
11	.06322	.07435	.09323	.1120	.1368	.1858
12	.08659	.09990	.1220	.1435	.1712	.2245
13	.1111	.1262	.1510	.1746	.2046	.26088
14	.1359	.1526	.1795	.2049	.2365	.2949
15	.1607	.1787	.2073	.2339	.2668	.3265
16	.1851	.2041	.2340	.2616	.2954	.3558
17	.2089	.2287	.2597	.2879	.3222	.3830
18	.2319	.2524	.2841	.3128	.3474	.4082
19	.2541	.2750	.3073	.3363	.3711	.4316
20	.2754	.2967	.3293	.3585	.3933	.4534
22	.3153	.3371	.3700	.3993	.4337	.4926
24	.3519	.3737	.4067	.4356	.4695	.5268
25	.3689	.3908	.4236	.4523	.4858	.5423
26	.3852	.4070	.4396	.4681	.5013	.5568
28	.4157	.4372	.4694	.4973	.5296	.5834
30	.4435	.4648	.4963	.5236	.5550	.6070
35	.5033	.5237	.5535	.5790	.6083	.6560
40	.5521	.5713	.5994	.6232	.6503	.6943
45	.5924	.6105	.6369	.6592	.6844	.7250
50	.6262	.6433	.6681	.6890	.7125	.7502
55	.6550	.6711	.6944	.7140	.7360	.7711
60	.6797	.6949	.7169	.7353	.7560	.7888
65	.7011	.7156	.7364	.7538	.7732	.8040
70	.7199	.7336	.7533	.7698	.7881	.8172
75	.7364	.7495	.7683	.7839	.8012	.8287
80	.7512	.7637	.7815	.7963	.8128	.8388
85	.7644	.7763	.7933	.8074	.8231	.8478
90	.7762	.7876	.8039	.8174	.8324	.8559
95	.7870	.7979	.8135	.8264	.8407	.8632
100	.7967	.8072	.8222	.8345	.8482	.8697

Table 3.1 (Continued)

p = 6

N \ α	.005	.01	.025	.05	.1	.25
8	$.0^{3}9465$	$.0^{2}1384$	$.0^{2}2354$	$.0^{2}3620$	$.0^{2}5772$	.01177
9	$.0^{2}4201$	$.0^{2}5607$	$.0^{2}8393$	.01165	.01665	.02886
10	.01062	.01342	.01864	.02437	.03266	.05135
11	.02019	.02462	.03254	.04087	.05245	.07720
12	.03249	.03862	.04925	.06009	.07473	.1048
13	.04697	.05476	.06796	.08111	.09847	.1331
14	.06308	.07243	.08798	.1032	.1229	.1613
15	.08035	.09111	.1088	.1257	.1475	.1889
16	.09835	.1104	.1299	.1484	.1718	.2157
17	.1168	.1299	.1510	.1708	.1956	.2414
18	.1354	.1495	.1719	.1928	.2187	.2661
19	.1539	.1689	.1924	.2142	.2410	.2896
20	.1723	.1880	.2125	.2350	.2625	.3119
22	.2081	.2249	.2509	.2745	.3030	.3533
24	.2423	.2598	.2868	.3111	.3401	.3907
25	.2586	.2765	.3038	.3282	.3574	.4079
26	.2745	.2926	.3202	.3447	.3739	.4244
28	.3048	.3232	.3511	.3758	.4049	.4548
30	.3332	.3517	.3797	.4043	.4332	.4842
35	.3961	.4147	.4422	.4662	.4942	.5410
40	.4493	.4674	.4941	.5172	.5438	.5881
45	.4944	.5119	.5376	.5596	.5849	.6266
50	.5331	.5499	.5744	.5954	.6194	.6587
55	.5664	.5825	.6060	.6260	.6487	.6858
60	.5955	.6109	.6333	.6523	.6739	.7089
65	.6210	.6357	.6571	.6752	.6957	.7289
70	.6435	.6576	.6781	.6954	.7149	.7463
75	.6636	.6771	.6967	.7132	.7318	.7617
80	.6815	.6945	.7132	.7290	.7468	.7753
85	.6977	.7101	.7281	.7432	.7602	.7874
90	.7123	.7243	.7415	.7560	.7723	.7983
95	.7255	.7371	.7537	.7676	.7832	.8082
100	.7377	.7488	.7648	.7782	.7932	.8171

Table 3.1 (Continued)

p = 7

N \ α	.005	.01	.025	.05	.1	.25
9	$.0^{3}3774$	$.0^{3}5468$	$.0^{3}9234$	$.0^{2}1420$	$.0^{2}2278$	$.0^{2}4745$
10	$.0^{2}1725$	$.0^{2}2315$	$.0^{2}3504$	$.0^{2}4921$	$.0^{2}7143$	.01275
11	$.0^{2}4689$	$.0^{2}5981$	$.0^{2}8427$	.01117	.01522	.02465
12	$.0^{2}9547$	.01176	.01577	.02008	.02620	.03971
13	.01631	.01957	.02533	.03132	.03959	.05711
14	.02482	.02921	.03678	.04446	.05482	.07611
15	.03485	.04039	.04974	.05906	.07139	.09612
16	.04615	.05280	.06387	.07472	.08887	.1167
17	.05845	.06616	.07884	.09111	.1069	.1374
18	.07153	.08024	.09441	.1080	.1252	.1581
19	.08518	.09481	.1103	.1250	.1436	.1784
20	.09922	.1097	.1264	.1422	.1619	.1984
22	.1279	.1399	.1587	.1761	.1976	.2369
24	.1568	.1699	.1903	.2090	.2319	.2730
25	.1710	.1846	.2057	.2250	.2484	.2901
26	.1852	.1992	.2209	.2405	.2644	.3067
28	.2127	.2275	.2501	.2704	.2949	.3379
30	.2393	.2545	.2778	.2987	.3235	.3669
35	.3006	.3166	.3407	.3621	.3872	.4304
40	.3545	.3708	.3950	.4163	.4411	.4832
45	.4018	.4179	.4419	.4627	.4869	.5275
50	.4433	.4591	.4825	.5027	.5261	.5651
55	.4797	.4952	.5179	.5375	.5600	.5973
60	.5120	.5269	.5490	.5678	.5895	.6252
65	.5406	.5551	.5764	.5946	.6153	.6495
70	.5662	.5802	.6008	.6183	.6382	.6709
75	.5892	.6028	.6226	.6394	.6586	.6899
80	.6099	.6230	.6421	.6584	.6768	.7068
85	.6287	.6414	.6598	.6755	.6932	.7220
90	.6458	.6581	.6759	.6910	.7080	.7358
95	.6614	.6733	.6905	.7050	.7215	.7482
100	.6757	.6872	.7039	.7179	.7338	.7595

Table 3.1 (Continued)

p = 8

N \ α	.005	.01	.025	.05	.1	.25
10	$.0^{3}1597$	$.0^{3}2284$	$.0^{3}3803$	$.0^{3}5810$	$.0^{3}9305$	$.0^{2}1957$
11	$.0^{3}7175$	$.0^{3}9652$	$.0^{2}1470$	$.0^{2}2080$	$.0^{2}3053$	$.0^{2}5577$
12	$.0^{2}2056$	$.0^{2}2641$	$.0^{2}3764$	$.0^{2}5044$	$.0^{2}6970$	.01159
13	$.0^{2}4445$	$.0^{2}5519$	$.0^{2}7500$	$.0^{2}9663$	.01279	.01989
14	$.0^{2}8022$	$.0^{2}9711$	.01273	.01593	.02042	.03021
15	.01282	.01522	.01940	.02373	.02966	.04219
16	.01879	.02195	.02738	.03287	.04027	.05548
17	.02583	.02979	.03648	.04315	.05198	.06976
18	.03383	.03860	.04654	.05435	.06456	.08475
19	.04266	.04822	.05738	.06628	.07779	.1002
20	.05218	.05851	.06884	.07878	.09150	.1160
22	.07283	.08059	.09307	.1049	.1197	.1477
24	.09492	.1039	.1183	.1317	.1483	.1790
25	.1063	.1159	.1310	.1451	.1625	.1944
26	.1178	.1279	.1438	.1585	.1765	.2095
28	.1409	.1519	.1691	.1849	.2041	.2387
30	.1639	.1757	.1940	.2106	.2307	.2665
35	.2195	.2327	.2528	.2708	.2923	.3299
40	.2709	.2849	.3059	.3246	.3467	.3847
45	.3177	.3319	.3534	.3722	.3944	.4322
50	.3598	.3741	.3956	.4144	.4363	.4733
55	.3976	.4119	.4332	.4517	.4731	.5092
60	.4317	.4458	.4667	.4848	.5057	.5408
65	.4624	.4763	.4967	.5144	.5347	.5687
70	.4901	.5037	.5237	.5409	.5607	.5935
75	.5153	.5286	.5481	.5648	.5839	.6156
80	.5383	.5512	.5701	.5863	.6049	.6355
85	.5592	.5718	.5902	.6059	.6239	.6535
90	.5784	.5906	.6085	.6238	.6412	.6699
95	.5960	.6079	.6253	.6402	.6571	.6847
100	.6123	.6239	.6408	.6552	.6716	.6983

BIBLIOGRAPHY

1. Goodman, N.R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction). Ann. Math. Statist. 34, 152-176.

2. Meijer, G. (1946). Nederl, Akad Wetensch. Proc., 49, 344-356.

3. Pillai, K.C.S. and Singh (Jain), A. (1978). On the exact non-null distribution of Wilks' L_{vc} criterion. Mimeograph series no. 525. Department of Statistics, Purdue University, W. Lafayette, Indiana.

4. Pillai, K.C.S. and Singh, A. (1981). On the exact non-null distribution of Wilks' L_{vc} criterion and Power studies. Ann. Inst. Statist. Math. (to appear).

5. Singh, A. and Pillai, K.C.S. (1981). On the exact non-null distribution of Wilks' L_{vc} criterion in the complex case. Mimeograph series no. 80-33, Department of Statistics, Purdue University, West Lafayette, Indiana.

6. Wilks, S.S. (1946). Sample Criteria for testing equality of means, equality of variances and equality of covariances in a normal multivariate distribution. Ann. Math. Statist., 17, 257-281.