SIMULTANEOUS DIAGONALIZATION OF RECTANGULAR MATRICES[1]

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1. INTRODUCTION

Let A, B be matrices of order $m\times n$ with elements from a field $\mathfrak{F}$. The vectorspace spanned by such matrices is denoted by $\mathfrak{F}^{m\times n}$. A matrix $D\in\mathfrak{F}^{m\times n}$ is said to be diagonal if $(D)_{ij}$, the element in the (i,j)th position of D = 0 whenever $i \neq j$. We ask ourselves the following question:

'Given a pair of matrices $A,B\in\mathfrak{F}^{m\times n}$ does there exist nonsingular matrices $S\in\mathfrak{F}^{m\times m}$ and $T\in\mathfrak{F}^{n\times n}$ such that

$$SAT = D_a \ , \ SBT = D_b \tag{1.1}$$

where D_a and D_b are diagonal matrices in $\mathfrak{F}^{m\times n}$?'

If A and B represent linear transformations from an n dimensional vectorspace $V_n(\mathfrak{F})$ to an m-dimensional vectorspace $V_m(\mathfrak{F})$ with reference to chosen bases in $V_m(\mathfrak{F})$ and $V_n(\mathfrak{F})$ we are thus essentially seeking changes in bases so that the transformations could be described in simpler terms through diagonal matrices.

Theorem 3.1 provides necessary and sufficient conditions for (1.1) to hold. Simultaneous diagonability of a set $\{A_\theta\}$ of matrices in $\mathfrak{F}^{m\times n}$ is studied in Theorem 4.1. We note here that since the vectorspace $\mathfrak{F}^{m\times n}$

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is finitedimensional one may without any loss of generality assume that set $\{A_\theta\}$ so studied consists of only a finite number of such matrices.

Williamson [12] showed that complex matrices A and B can be simultaneously diagonalized as in (1.1) through unitary matrices S and T iff AB* is normal, where * on a matrix indicates its complex conjugate transpose. Necessary and sufficient conditions for the existence of unitary matrices S and T such that

$$SA_\theta T = D_\theta$$

is diagonal for each A_θ in a set $\{A_\theta\}$ of complex matrices are given by Gibson [3]. The reader is referred to Gibson [3] for a bibliography on other related work in this area.

2. SOME OTHER NOTATIONS AND PRELIMINARY RESULTS

$\mathfrak{F}^m$ denotes the vector space of m-tuples with elements in $\mathfrak{F}$. Lower case letters a, b indicate column vector representations of such m-tuples For a matrix A, $\mathcal{M}(A)$ denotes its column span and $\mathcal{N}(A)$ its null space. A' denotes the transpose of A. A^-, a generalized inverse (g-inverse) of A, is a matrix A^- satisfying the equation $AA^-A = A$ [11]. The class of all possible g-inverses of A is denoted by $\{A^-\}$. Two subspaces of a vector space are said to be virtually disjoint if they have only the null vector in common.

<u>Definition 2.1</u>: Given a matrix $A \in \mathfrak{F}^{m\times n}$ and subspaces $\mathcal{S} \subset \mathfrak{F}^m, \mathcal{T} \subset \mathfrak{F}^n$, the shorted matrix $S(A|\mathcal{S},\mathcal{T})$ is a matrix $C \in \mathfrak{F}^{m\times n}$ such that

$$\mathcal{M}(C) \subset \mathcal{S}, \quad \mathcal{M}(C') \subset \mathcal{T} \tag{2.1}$$

and if E is any matrix $\in \mathfrak{F}^{m\times n}$ that satisfies (2.1) then

$$\text{Rank }(A-E) \geq \text{Rank }(A-C) \tag{2.2}$$

This definition extends the notion of a shorted positive operator studied by Krein [6], Anderson and Trapp [1] and Mitra and Puri [8]. Shorted matrices are studied in greater detail elsewhere [9].

Let $X \in \mathfrak{F}^{m\times p}, Y \in \mathfrak{F}^{q\times n}$ be such that

$$\mathcal{S} = \mathcal{M}(X), \quad \mathcal{T} = \mathcal{M}(Y')$$

and 0 be the null matrix in $\mathfrak{F}^{q\times p}$. We consider the bordered matrix

$$F = \begin{pmatrix} A & X \\ Y & 0 \end{pmatrix} \tag{2.3}$$

and let

$$G = \begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix} \in \{F^-\} \tag{2.4}$$

where $C_1 \in \mathfrak{F}^{n\times m}, C_2 \in \mathfrak{F}^{n\times q}, C_3 \in \mathfrak{F}^{p\times m}$, and $C_4 \in \mathfrak{F}^{p\times q}$.

Theorem 2.1 gives a set of necessary and sufficient conditions for the existence of an unique shorted matrix $S(A|\mathcal{S},\mathcal{T})$ and provides an explicit expression for the same.

Theorem 2.1. (a) The shorted matrix $S(A|\mathcal{S},\mathcal{T})$ exists and is unique iff the matrix F satisfies the rank addivity conditions

$$\text{Rank } F = \text{Rank } (A\vdots X) + \text{Rank } Y = \text{Rank } \binom{A}{Y} + \text{Rank } X. \tag{2.5}$$

(b) When (2.5) is satisfied,

(i) $C_2 \in \{Y^-\}, C_3 \in \{X^-\}$,

(ii) AC_2Y, XC_3A and XC_4Y are invariant under the choice of G in (2.4) and further

$$AC_2Y = XC_3A = XC_4Y = A - AC_1A \text{ (say)}. \tag{2.6}$$

(iii) The matrix C in (2.6) is the unique shorted matrix $S(A|\mathcal{S},\mathcal{T})$.

Proof. The 'if' part of (a) and the (b) part of Theorem 2.1 are proved for complex matrices in [7]. (See Theorems 1 and 2 and Remark 1 following Theorem 2 in [7]. The transition from the complex field to arbitrary field $\mathfrak{F}$ presents no special difficulties. Theorem 1 in [7] is a generalization of similar theorems due to Khatri [5] and Rao [10].) To prove the 'only if' part of (a) assume now that $A_0 = S(A|\mathcal{S},\mathcal{T})$ is the unique shorted matrix. Write $A = A_0 + A_1$ and observe that the uniqueness of the shorted matrix $S(A|\mathcal{S},\mathcal{T})$ implies that $\mathcal{M}(A_1)$ is virtually disjoint with $\mathcal{S}$ and $\mathcal{M}(A_1')$ with $\mathcal{T}$. If $\mathcal{M}(A_1)$ is not virtually disjoint with $\mathcal{S}$, let ℓ_1 be a nonnull m-tuple in $\mathcal{M}(A_1) \cap \mathcal{S}$. Let A_1 be of rank s. Consider a rank factorization of A_1

$$A_1 = LR$$

where $L = (\ell_1 \vdots \ell_2 \vdots \ldots \vdots \ell_s)$, $R' = (r_1 \vdots r_2 \vdots r_s)$. For any nonnull n-tuple t_1 in $\mathcal{T}$, the matrix $E = A_0 + \ell_1 t_1'$ satisfies condition (2.1) and further $A - E$ has the same rank as $A - A_0 = A_1$. This contradicts the uniqueness of the shorted matrix $S(A|\mathcal{S},\mathcal{T})$. A similar argument shows that $\mathcal{M}(A_1')$ is virtually disjoint with $\mathcal{T}$. If $\mathcal{M}\binom{A}{Y}$ is not virtually disjoint with $\mathcal{M}\binom{X}{0}$ let vectors $a \in \mathfrak{F}^n$, $b \in \mathfrak{F}^m$ be such that

$$Aa = Xb \neq 0\,,$$
$$Ya = 0\,. \tag{2.7}$$

(2.7) $\Rightarrow$ $A_1 a = Xb \neq 0$ which contradicts the assumption that $\mathcal{M}(A_1)$ is virtually disjoint with $\mathcal{S}$. The other part of (2.5) is similarly established.

Q.E.D.

We also need an explicit representation of a g-inverse of F, given in Theorem 2.2. The proof is by direct computation. The complex version of Theorem 2.2 appears as Theorem 3 in [7]. This generalizes a theorem of Hall and Meyer [4].

Theorem 2.2. For any choice of the g-inverses of X, Y and E_XAF_Y,

$$\begin{pmatrix} 0 & Y^- \\ X^- & -X^-AY^- \end{pmatrix} + \begin{pmatrix} I \\ -X^-A \end{pmatrix} Q\ (I\ -AY^-) \tag{2.8}$$

is a g-inverse of F, where $Q = F_Y(E_XAF_Y)^-E_X, E_X = I - XX^-$ and $F_Y = I - Y^-Y$.

3. SIMULTANEOUS DIAGONALIZATION OF A PAIR OF MATRICES

Theorem 3.1. Let $A,B \in \mathfrak{F}^{m\times n}$. There exists a pair of nonsingular matrices satisfying (1.1) iff.

(a) $\text{Rank}\begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \text{Rank}\ (A \vdots B) + \text{Rank}\ B = \text{Rank}\ \binom{A}{B} + \text{Rank}\ B$ (3.1)

and

(b) AC_2BC_2 is semisimple (or equivalently C_3BC_3A is semisimple) (3.2)

where $\begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix}$ is any g-inverse of $F = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}$.

Proof. ('only if' part) We assume here that nonsingular S and T exist such that

$$SAT = D_a\ ,\ SBT = D_b$$

where D_a and D_b are diagonal matrices. It is easily seen that

$$\text{Rank}\begin{pmatrix} D_a & D_b \\ D_b & 0 \end{pmatrix} = \text{Rank}\ (D_a \vdots D_b) + \text{Rank}\ D_b = \text{Rank}\begin{pmatrix} D_a \\ D_b \end{pmatrix} + \text{Rank}\ D_b$$

Hence (3.1) follows.

Further the matrix $\begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix}$ is a g-inverse of F iff $C_1 = T\bar{C}_1S$,

$C_2 = T\bar{C}_2S$, $C_3 = T\bar{C}_3S$, $C_4 = T\bar{C}_4S$ where $\begin{pmatrix} \bar{C}_1 & \bar{C}_2 \\ \bar{C}_3 & -\bar{C}_4 \end{pmatrix}$ is a g-inverse of

$\begin{pmatrix} D_a & D_b \\ D_b & 0 \end{pmatrix}$.

We now show that there exists a choice of a g-inverse of $\begin{pmatrix} D_a & D_b \\ D_b & 0 \end{pmatrix}$ such that $\bar{C}_2$ and $\bar{C}_3$ are both diagonal. For this we use formula (2.8) and substitute for D_b^- and Q the matrices defined as follows:

$$(D_b^-)_{ii} = 1/(D_b)_{ii} \text{ if } (D_b)_{ii} \neq 0,\ (D_b^-)_{ij} = 0, \text{ otherwise,} \tag{3.3}$$

$$\begin{aligned} &(Q)_{ii} = 1/(D_a)_{ii} \text{ if } (D_a)_{ii} \neq 0 \text{ and } (D_b)_{ii} = 0 \\ &(Q)_{ij} = 0, \text{ otherwise.} \end{aligned} \tag{3.4}$$

Since D_b^- and Q are diagonal matrices

$$\bar{C}_2 = D_b^- - QD_aD_b^-$$

is diagonal and

$$AC_2BC_2 = S^{-1}D_aT^{-1}T\bar{C}_2SS^{-1}D_bT^{-1}T\bar{C}_2S = S^{-1}D_1S,$$

where $D_1 = D_a\bar{C}_2D_b\bar{C}_2 \in \mathfrak{F}^{m\times m}$, and is diagonal. This establishes the fact that AC_2BC_2 is semisimple. We now show that if (3.1) holds the semisimplicity of AC_2BC_2 is equivalent to semisimplicity of AC_2BB^- for any choice of B^-. This follows from the fact that if x is an eigenvector of AC_2BC_2 for a nonnull eigenvalue λ,

$$AC_2BC_2x = \lambda x \Rightarrow AC_2x = \lambda x, \tag{3.5}$$

since $x \in \mathcal{M}(AC_2B) = \mathcal{M}(BC_3A) \subset \mathcal{M}(B)$ and $C_2 \in \{B^-\}$. For the same reason

$$AC_2BB^-x = AC_2x = \lambda x. \tag{3.6}$$

This shows that x is an eigenvector of AC_2BB^- for the same eigenvalue λ

and vice versa. Since Rank (AC_2BC_2) = Rank (AC_2BB^-) = Rank(AC_2B), the equivalence of the two statements follows.

Since AC_2B is invariant under choice of a g-inverse of F, if AC_2BC_2 is semisimple for one choice of this g-inverse it is so for every other choice.

('if' part): Let B be of rank r. Consider a rank factorization of B,

$$B = UV,$$

where $U \in \mathfrak{F}^{m\times r}, V \in \mathfrak{F}^{r\times n}$.

Since $\mathcal{M}(AC_2B) \subset \mathcal{M}(B)$, $\mathcal{M}(B'C_2'A') \subset \mathcal{M}(B')$

$$AC_2B = UKV$$

for some $K \in \mathfrak{F}^{r\times r}$. Choose and fix a g-inverse of B, $B^- = V_R^{-1}U_L^{-1}$ where U_L^{-1} and V_R^{-1} are respectively left and right inverses of U and V. Semi-simplicity of AC_2BC_2 implies semisimplicity of $AC_2BB^- = UKU_L^{-1}$ which in turn implies semisimplicity of K. Put $K = WDW^{-1}$ where $W, D \in \mathfrak{F}^{r\times r}$ and D is diagonal. Then

$$AC_2B = UKV = UWDW^{-1}V = S_1DT_1$$

where $S_1 = UW$, $T_1 = W^{-1}V$. Check that $B = S_1T_1$. Also, let S_2T_2 be a rank factorization of $A - AC_2B$. $\mathcal{M}(A-AC_2B) \cap \mathcal{M}(B) = \{0\}$ and $\mathcal{M}(A'-B'C_2'A') \cap \mathcal{M}(B') = \{0\}$ follows from (3.1) and the proof of Theorem 2 of [7]. Hence $\mathcal{M}(S_2)$ is virtually disjoint with $\mathcal{M}(S_1)$ and $\mathcal{M}(T_2')$ with $\mathcal{M}(T_1')$. Let S_3 and T_3 be so chosen that $(S_1 \vdots S_2 \vdots S_3)$ and $(T_1' \vdots T_2' \vdots T_3')$ are nonsingular. Put $S^{-1} = (S_1 \vdots S_2 \vdots S_3)$, $(T')^{-1} = (T_1' \vdots T_2' \vdots T_3')$ and check that

$$SAT = D_a \text{ and } SBT = D_b$$

where $D_a = \mathrm{diag}(D,I,0)$, $D_b = \mathrm{diag}(I,0,0)$ are clearly diagonal matrices. This completes the proof of the 'if' part and of Theorem 3.1. Q.E.D.

4. SIMULTANEOUS DIAGONALIZATION OF SEVERAL MATRICES

Without any loss of generality let us assume here that $m \leq n$. We shall further assume here that the field $\mathfrak{F}$ contains more than m distinct nonnull elements.

We need the following result.

Lemma 4.1.(2) If matrices A and B satisfy condition (3.1) there exists a nonnull scalar k such that

$$\mathcal{M}(A) \subset \mathcal{M}(A+kB), \ \mathcal{M}(A') \subset \mathcal{M}(A'+kB') \tag{4.1a}$$

or equivalently

$$\mathcal{M}(B) \subset \mathcal{M}(A+kB), \ \mathcal{M}(B') \subset \mathcal{M}(A'+kB') \tag{4.1b}$$

and

$$\text{Rank}\{B(A+kB)^{-}B\} = \text{Rank } B. \tag{4.1c}$$

Conversely (4.1a) or (4.1b) and (4.1c) imply (3.1).

Proof. Assume now that (3.1) holds and let

$$\begin{pmatrix} C_1 & C_3 \\ C_2 & -C_4 \end{pmatrix} \in \left\{ \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}^{-} \right\} .$$

Let k be so chosen that $k \neq 0$ and

$$\det(BC_4+kI) \neq 0.$$

Clearly

$$\mathcal{M}(BC_4B) \subset \mathcal{M}(B) = \mathcal{M}(BC_4B+kB), \ \mathcal{M}(B'C_4'B') \subset \mathcal{M}(B') = \mathcal{M}(B'C_4'B'+kB'). \tag{4.2}$$

Since $\mathcal{M}(A-BC_4B) \cap \mathcal{M}(B) = \{0\}$ and $\mathcal{M}(A'-B'C_4'B') \cap \mathcal{M}(B') = \{0\}$ follows from (3.1) and the proof of Theorem 2 of [7]. Hence

$$\begin{aligned} \mathcal{M}(A) &= \mathcal{M}(A-BC_4B+BC_4B) = \mathcal{M}(A-BC_4B) + \mathcal{M}(BC_4B) \\ &\subset \mathcal{M}(A-BC_4B) + \mathcal{M}(BC_4B+kB) \\ &= \mathcal{M}(A-BC_4B+BC_4B+kB) = \mathcal{M}(A+kB) \end{aligned}$$

and similarly $\mathcal{M}(A')\subset\mathcal{M}(A'+kB')$. This establishes (4.1a). (4.1b) is trivial.

If (4.1a) holds, the matrix $\begin{pmatrix} A+kB & B \\ 0 & 0 \end{pmatrix}$ can be reduced to $\begin{pmatrix} A+kB & 0 \\ 0 & B(A+kB)^{-}B \end{pmatrix}$ through sweep out operations on its rows and columns.

Hence

$$\mathrm{Rank}\begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \mathrm{Rank}\begin{pmatrix} A+kB & B \\ B & 0 \end{pmatrix} = \mathrm{Rank}(A+kB) + \mathrm{Rank}\ B(A+kB)^{-}B$$

$$= \mathrm{Rank}\begin{pmatrix} A \\ B \end{pmatrix} + \mathrm{Rank}\ B(A+kB)^{-}B = \mathrm{Rank}(A\vdots B) + \mathrm{Rank}\ B(A+kB)^{-}B,$$

and (3.1) implies (4.1c). Conversely the same argument shows that (4.1c) imply (3.1). Q.E.D.

Theorem 4.1. Let $A_1,A_2,\ldots,A_p \in \mathfrak{F}^{m\times n}$. The following two statements are equivalent.

(a) There exists nonsingular matrices $S\in\mathfrak{F}^{m\times m}$, $T\in\mathfrak{F}^{n\times n}$ such that

$$SA_iT = D_i, \quad i = 1,2,\ldots,p \tag{4.3}$$

where each D_i is a diagonal matrix in $\mathfrak{F}^{m\times n}$.

(b) There exists nonnull scalars $k_2,\ldots,k_p$ in $\mathfrak{F}$ such that if

$$A_0 = A_1 + k_2A_2 + \ldots + k_pA_p, \tag{4.4}$$

then for $i = 1,2,\ldots,p$; $j = 1,2,\ldots,p$,

(i) $\mathcal{M}(A_i)\subset\mathcal{M}(A_0)$, $\mathcal{M}(A_i')\subset\mathcal{M}(A_0')$, (4.5)

(ii) $A_iA_0^{-}$ is semisimple, (4.6)

(iii) $A_iA_0^{-}A_j = A_jA_0^{-}A_i$. (4.7)

(2) Lemma 4.1 is false if the field contains only m distinct nonnull elements on less.

Proof. (a) $\Rightarrow$ (b): Since A_1 and A_2 are simultaneously reducible to diagonal matrices using Theorem 3.1 and then Lemma 4.1, a nonnull scalar k_2 can be determined so that if

$$A_{(2)} = A_1 + k_2 A_2$$

then

$$\mathcal{M}(A_1) \subset \mathcal{M}(A_{(2)}), \mathcal{M}(A_1') \subset \mathcal{M}(A_{(2)}')$$

$$\mathcal{M}(A_2) \subset \mathcal{M}(A_{(2)}), \mathcal{M}(A_2') \subset \mathcal{M}(A_{(2)}').$$

Since $A_{(2)}$ and A_3 are simultaneously reducible to diagonal matrices, the same argument can be repeated and the nonnull scalars $k_2, k_3, \ldots, k_p$ can be recursively determined so as to satisfy (4.5).

Let $D_0 = D_1 + k_2 D_2 + \ldots + k_p D_p$. Then D_0 is diagonal and

$$SA_0T = D_0 .$$

As in the proof of Theorem 3.1 it is seen that if $A_i A_0^-$ is semisimple for some choice of A_0^- it is so for every other choice. Choose for D_0^- the following diagonal matrix in $\mathfrak{F}^{n\times m}$

$$(D_0^-)_{ii} = 1/(D_0)_{ii} \text{ if } (D_0)_{ii} \neq 0$$

$$(D_0^-)_{ij} = 0, \text{ otherwise.}$$

It is seen that $TD_0^-S \in \{A_0^-\}$ and with this choice of A_0^- the truth of (4.6) and (4.7) are easily verified. We note that on account of (4.5), $A_i A_0^- A_j$ is invariant under choice of A_0^-.

(b) $\Rightarrow$ (a): Consider a rank factorization of A_0,

$$A_0 = UV,$$

where $U \in \mathfrak{F}^{m\times r}$, $V \in \mathfrak{F}^{r\times n}$ and $r = \text{Rank } A_0$. Choose and fix a g-inverse A_0^- where

$$A_0^- = V_R^{-1} U_L^{-1}$$

and U_L^{-1} and V_R^{-1} are respectively left and right inverses of U and V (4.5) $\Rightarrow$

$$A_i = UB_iV$$

for some matrix $B_i \in \mathcal{F}^{r\times r}$

$$A_iA_0^- = UB_iU_L^{-1}.$$

Since on account of (4.6) and (4.7) the matrices $A_iA_0^-$ commute and are semisimple, it follows that the matrices B_i commute and are semisimple. Hence there exists a nonsingular matrix $W \in \mathcal{F}^{r\times r}$ such that

$$W^{-1}BW = D_i, \quad i = 1,2,\dots,p$$

where $D_1,D_2,\dots,D_p$ are diagonal matrices. The rest of the proof of Theorem 4.1 can be completed on the same lines as in the proof of the 'if' part of Theorem 3.1. Q.E.D.

Theorem 4.2 is an extension of Theorem 6 of Bhimasankaram [2].

Theorem 4.2. Let $A_1,A_2,\dots,A_p$ be complex hermitian matrices of order n×n. Then there exists a nonsingular matrix T such that T^*A_iT is diagonal for each i iff there exists nonnull real scalars $k_2,k_3,\dots,k_p$ such that if

$$A_0 = A_1 + k_2A_2 + \dots + k_pA_p$$

then for $i = 1,2,\dots,p$; $j = 1,2,\dots,p$

(a) $\mathcal{M}(A_i) \subset \mathcal{M}(A_0)$

(b) $A_iA_0^-$ is semisimple with real eigenvalues for some g-inverse A_0^- of A_0.

(c) $A_iA_0^-A_j = A_jA_0^-A_i$.

Proof: The 'only if' part follows from the corresponding part of Theorem 4.1 since here without any loss of generality one can restrict the scalar k_i to be real. The 'if' part follows from Theorem 6 of Bhimasankaram [2].

Q.E.D.

REFERENCES

1. W. N. Anderson, Jr. and G. Trapp, Shorted Operators II, SIAM J. Appl. Math. 28: 60-71 (1975).

2. P. Bhimasankaram, Simultaneous reduction of several hermitian forms, Sankhyā A, 33: 417-422 (1971).

3. P. M. Gibson, Simultaneous diagonalization of rectangular complex matrices, Linear Algebra Appl. 9: 45-53 (1974).

4. F. J. Hall and C. D. Meyer, Generalized inverses of the fundamental bordered matrix used in linear estimation, Sankhyā A 37: 428-438 (1975).

5. C. G. Khatri, Some results for the singular multivariate regression models, Sankhyā A 30: 268-280 (1968).

6. M. G. Krein, The theory of self-adjoint extensions of semi-bounded hermitian operators and its applications, Mat Sbornik N.S. 20(62): 431-495 and 21(63): 365-404 (1947).

7. S. K. Mitra, Properties of the fundamental bordered matrix used in linear estimations, Statistics and Probability: Essays in Honor of C. R. Rao (ed. G. Kallianpur et al), North Holland Publishing Co., New York (1982).

8. S. K. Mitra and M. L. Puri, Shorted operators and generalized inverses of matrices, Linear Algebra Appl. 25: 45-56 (1979).

9. S. K. Mitra and M. L. Puri, Shorted matrices an extended concept and some applications, to appear in Linear Algebra Appl.

10. C. R. Rao, Unified theory of linear estimation, Sankhyā A 33: 371-394 (1971).

11. C. R. Rao and S. K. Mitra, Generalized inverse of matrices and its applications, Wiley, New York (1971).

12. J. Williamson, Note on a principal axis transformation for nonhermitian matrices, Bull. Amer. Math. Soc. 45: 920-922 (1939).