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FILTERING WITH SINGULAR CUMULATIVE SIGNALS

by

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Summary

Let $X = W + \alpha L$, where W is a Wiener process, L is the local time at 0 of either W or a Wiener process independent of W, and α is a constant. We show that if $|\alpha| \geq 16$ the minimal filtration of X contains the minimal filtration of W, that is, the "signal" αL can be completely recovered from observation of X. If α is small the problem is unsolved.

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1. Introduction

A typical problem in filtering theory is that one observes a process $Y_t = W_t + \int_0^t h_s ds$, where W is a standard Wiener process representing the intergral of white noise, and one wants to estimate the "signal" h_t. In this paper we consider $Y^\alpha = W + \alpha L$ where L is local time at zero of W and α is a constant. We show in Theorem 3.2 that one has equality of the filtrations of Y^α and of W for $|\alpha| \geq 16$, and hence one need not make estimates at all, since αL can be completely recovered from observation of Y^α. We actually prove the stronger result that, for any $a < b$ and $|\alpha| \geq 16$,

$$\sigma(L_t - L_a,\ a \leq t \leq b) \subseteq \sigma(Y_t^\alpha - Y_a^\alpha,\ a \leq t \leq b). \tag{1.1}$$

With minor modifications our proof shows that (1.1) holds as well if L is local time at 0 of a Wiener process independent of W. By complicating the proof we can lower somewhat the bound 16, but we cannot get close to 1, and we conjecture that (1.1) does not hold for all positive α.

We thank J. Pitman for introducing us to these kinds of questions. He asked one of us ([3]) whether the filtrations of W + L and W are identical, a question we have, evidently, not answered.

2. Theorems and Proofs

Throughout this paper we will take W to be a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. $(\mathcal{F}_t)_{t\geq 0}$ will denote the minimal completed filtration of W, and L will denote its local time at 0. We take L to be normalized so that

$$(2.1) \qquad \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(-\varepsilon,\varepsilon)}(W_s)ds = L_t$$

holds.

Define Y^α by

$$Y_t^\alpha = W_t + \alpha L_t$$

and let $(\mathcal{G}_t^\alpha)_{t\geq 0}$ be the minimal completed filtration for Y^α. Clearly $\mathcal{G}_t^\alpha \subseteqq \mathcal{F}_t$, $t \geq 0$.

(2.2) THEOREM. <u>With the above notations</u> $\mathcal{G}^\alpha = \mathcal{F}$ <u>for</u> $|\alpha| \geq 16$.

First note we only need show $\mathcal{F}_t \subseteqq \mathcal{G}_t$, $t \geq 0$. We consider only the case $\alpha = 16$; from the proof it is clear the same technique works for $|\alpha| > 16$. We let $\mathcal{G} = \mathcal{G}^\alpha$ and $Y = Y^\alpha$.

First a few facts needed later are collected. A standard estimate gives

$$(2.3) \qquad P(W_1 > 5) < e^{-15},$$

and the following lemma is a standard large deviation result

(2.4) Lemma. <u>If</u> $X_1,\ldots,X_n$ <u>are</u> iid <u>zero-one valued random variables with</u> $P(X = 1) \leq e^{-15}$, <u>then</u> $P(\Sigma_{i=1}^n X_i > n/10) = o(2^{-n})$.

To prove this lemma we just take t = 10 in the inequality $P(\Sigma X_i > n/10) \leq E(e^{t\Sigma X_i})/e^{t(n/10)}$.

Next fix $t > 0$ and let H be an open interval in $[0,t]$. We will construct a zero-one valued r.v. T, $\mathcal{G}_t$ - measurable, such that $\{T = 0\} = \{\exists s: B_s = 0 \text{ in } H\}$ a.s. This will allow us to recover L in a $\mathcal{G}$-optional way.

Divide H into m equal disjoint subintervals $(\Delta_p)_{1\leq p\leq m}$ so that

$$H = \bigcup_{p=1,m} \Delta_p.$$

Fix $p < m$ and write $\Delta = \Delta_p = (x, x + \gamma)$. Divide Δ into 2^n equal disjoint subintervals $(J_q)_{1\leq q\leq 2^n}$ so that:

$$\Delta_p = \Delta = \bigcup_{q=1,2^n} J_q.$$

The length of $J_q = |J_q| = \gamma/2^n$, and letting $\varepsilon = \gamma/2^n$ we form the following intervals for each $J_q = (a,b)$:

$$I_1^q = (a,\ a + \varepsilon) = (a_1^q,\ b_1^q) = (a_1,\ b_1)$$

$$\vdots$$

$$I_n^q = (a + 2^{n-1}\varepsilon,\ a + 2^n\varepsilon) = (a_n^q,\ b_n^q) = (a_n,\ b_n).$$

Recalling $\Delta = (x, x + \gamma)$, note that each $J_i \subseteq (x, x + 2\gamma) \subseteq H$, for Δ_p with $p < m$. Finally, we define r.v.'s $Z(J_q,n)$ by:

$$(2.5) \qquad Z(J_q,n) = \begin{cases} 1 & \text{if } \sum_{i=1,n} 1_{\{Y_{b_i} - Y_{a_i} > 5\sqrt{b_i - a_i}\}} > n/10 \\ 0 & \text{otherwise.} \end{cases}$$

(2.6) LEMMA. $P(Z(J_q,n) = 1 \cap \{\exists s: W_s = 0 \text{ in } (a_1^q, b_n^q)\}) = o(2^{-n})$.

<u>Proof of Lemma 2.6</u>: Let $\Lambda_q = \{\exists s: W_s = 0 \text{ in } (a_1^q, b_n^q)\}$. Note that L is constant on Λ_q; hence

$$P(Z(J_q,n) = 1 \cap \Lambda_q) \geq P(\textstyle\sum_{i=1,n} 1(W_{b_i} - W_{a_i} > 5\sqrt{b_i - a_i}) > \frac{n}{10})$$

$$= P(\textstyle\sum_{i=1,n} X_i > n/10)$$

and an application of Lemma 2.4 completes the proof.

(2.7) LEMMA. $P(Y_{s+1/2} - Y_s > 5\frac{1}{\sqrt{2}}) \geq .11$, _for any_ s, $0 \leq s < 1/2$.

Proof of Lemma 2.7: Suppose we can show $P(L_{s+1/2} - L_s > .37) \geq .112$ for any s, $0 \leq s < 1/2$. Then

$$P(Y_{s+1/2} - Y_s > 5\frac{1}{\sqrt{2}})$$

$$\geq P(16L_{s+1/2} - 16L_s > 5.8) - P(W_{1/2} < -2.2)$$

$$\geq .112 - .002 = .11.$$

Let $R = \inf\{t > s: W_t = 0\}$. Then

$$P(L_{s+1/2} - L_s > .37) \geq P(L_{R+1/6} - L_R > .37 | R \leq s + 1/3) \cdot P(R \leq s + 1/3)$$

$$\geq .35P(R \leq s + 1/3),$$

since $P(L_{R+1/6} - L_R > .37 | R \leq s + 1/3) \geq P(L_{1/6} > .37) = 2P(W_{1/6} > .37)$. It remains to show $P(R \leq s + 1/3) \geq .32$. Then $P(R \leq s + 1/3) =$ $E\{P(B \text{ hits } 0 \text{ in } (s, s + 1/3) | B_s)\}$

$$= E\{2(1 - \Phi_{1/3}(|B_s|))\}$$

$$= 4\int_0^\infty (1 - \Phi_{1/3}(y))\varphi_s(y)dy \qquad [0 \leq s < 1/2]$$

$$\geq 4\int_0^\infty (1 - \Phi_{1/3}(y))\varphi_{1/2}(y)dy$$

$$\geq .32,$$

where Φ and φ are the usual Gaussian cdf and pdf.

□

Set $\Gamma_q = \{\exists s: B_s = 0 \text{ in} \cup I_n^q\}$. The last lemma immediately implies

(2.8) $P(Y_{b_i} - Y_{a_i} > 5\sqrt{b_i - a_i} | \Gamma_q) > .11$, for $1 \le i \le n$

Using (2.8) observe that:

$$(.11)n \le E\{\Sigma_{i=1,n} 1(Y_{b_i} - Y_{a_i} > 5\sqrt{b_i - a_i}) | \Gamma_q\}$$

$$\le nP(\Sigma_{i=1,n} 1(Y_{b_i} - Y_{a_i} > 5\sqrt{b_i - a_i}) > n/10) + n/10;$$

hence

(2.9) $$P(Z(J_{q,n}) = 1 | \Gamma_q) > .01.$$

Recalling that p is fixed, define

(2.10) $$R_{p,n} = \sup_{1 \le q \le 2^n} Z(J_{q,n}).$$

Let

(2.11) $$A_p = \{\exists s: W_s = 0 \text{ in } \Delta_p\}, \text{ and}$$

$$A_{p,q_0} = \{J_{q_o,n} \text{ is the first } J_q \text{ such that } \exists s \text{ with } W_s = 0 \text{ in } J_{q,n}\}.$$

Then

(2.12) $$P(R_{p,n} = 1 | A_p) \ge \sum_{q=1}^{n} P(Z(J_{q,n}) = 1 | A_{p,q}) P(A_{p,q} | A_p)$$

$$> (.01) \sum_{q=1}^{2^n} P(A_{p,q} | A_p) = .01$$

where we have used (2.9). Next, letting

(2.13) $$D_p = \{\nexists s: W_s = 0 \text{ in } \Delta_p \cup \Delta_{p+1}\}$$

we have that

$$P(R_{p,n} = 1 \cap D_p) \le \sum_{q=1,2^n} P(Z(J_{q,n}) = 1 \cap D_p)$$

$$= 2^n o(2^{-n}) = o(1)$$

by Lemma 2.6, hence

$$c_n = P(R_{p,n} = 1 \cap D_p) = o(1).$$

Choose a subsequence $\{n_i\}$ such that $\Sigma_{i=1,\infty} c_{n_i} < \infty$, for c_n as above.

Define

(2.14) $$R_p = \limsup_{i\to\infty} R_{p,n_i},$$

with R_p zero-one valued. By the definition of R_p and Borel-Cantelli we have

(2.15) $$P(R_p = 1 \cap D_p) = 0$$

while

(2.16) $$P(R_p = 1 | A_p) \geq .01$$

from (2.12) and Fatou's lemma. Note that R_p depends only on the values of Y in $\Delta_p \cup \Delta_{p+1}$. We now define

(2.17) $$T_m = \sup_{1\leq p\leq m-1} R_p.$$

Let

(2.18) $$\begin{cases} A = \{\exists s: \text{ such that } W_s = 0 \text{ in } H\} \\ D = \{\nexists s: \text{ such that } W_s = 0 \text{ in } H\}. \end{cases}$$

Then it follows easily from (2.15) and (2.16) that

(2.19) $$\begin{cases} P(T_m = 1 \cap D) = 0 \\ P(T_m = 1|A) \geq .01. \end{cases}$$

Next we set

(2.20) $$T_H' = \limsup_{m\to\infty} T_m.$$

Using (2.19) note that Borel-Cantelli and Fatou's lemma respectively yield

$$P(\{T_H' = 1\} \cap D) = 0$$
$$P(T_H' = 1|A) \geq .01.$$

We note that $\{T_H' = 1\}$ is measurable $\sigma(Y_t - Y_a,\ a \leq t \leq b + \varepsilon)$ for any ε. Secondly, the proof just given immediately generalizes to show that

(2.21) $$P(T'_H = 1 | A \cap B) \geq .01,$$

where B is any set in $\sigma(Y_t, t \leq a)$. Now let (c_n, d_n), $n \geq 1$, be an ordering of all the open intervals contained in (a,b) which have rational endpoints. Define $T_H = T = \sup_n T'_{(c_n,d_n)}$. Then (2.21) immediately implies

(2.22) $$P(T_H = 1 | A) = 1,$$

since if W_t hits 0 in H it is possible to find with probability 1, an arbitrarily large finite disjoint collection of intervals (c_k, d_k) such that W_t hits 0 in each of the intervals.

At the beginning of the proof (following Lemma 2.4) we fixed an open interval H in [0,t]. For each such H with rational endpoints, $\{T_H = 0\} = \{\nexists s\colon W_s = 0 \text{ in } H\}$ a.s., and $\{T_H = 0\} \in \mathcal{G}_t$. Thus the complement of the zero set of W is a $\mathcal{G}^\alpha$-optional set. Then $H_s = 1_Z(s)$, the indicator of the zero set of W, is $\mathcal{G}^\alpha$-predictable and $W_t = \int_0^t \frac{1}{\alpha} H_s dY_s$. Alternatively, a theorem due to Lévy allows us to express the local time L as a limit of these random intervals (cf [2, p. 730]). Thus L is a $\mathcal{G}^\alpha$-optional process and this completes the proof of Theorem 3.2.

□

(2.23) COMMENT. Filtering theory is usually concerned with cumulative signal processes that have absolutely continuous paths. If $Y_t = W_t + \int_0^t h(s, W_s)ds$ for bounded, Borel h, then using Girsanov's theorem one can find a new probability law Q that makes Y into a Brownian motion. W is then a solution of the differential equation

$$dW_s = dY_s - h(s, W_s)ds$$

and W is a strong solution by Zvonkin's theorem [5]. Thus in this case as well the filtrations of Y and of W are the same and one need not make estimates of $h_s = h(s, W_s)$. Indeed, we do not know of any additive functionals A of W such that if Y = W + A then the filtrations of Y and W are different; we conjecture, however, that if $A = \varepsilon L$ for small enough ε then the filtrations are in fact different.

REFERENCES

1. Jacod, J. (1979). Calcul Stochastique et Problēmes de Martingales. Springer Lecture Notes in Math. 714.

2. Kingman, J.F.C. (1973). An Intrinsic Description of Local Time. J. London Math. Soc. 6 725-731.

3. Pitman, J. and Freedman, D. Personal Communication.

4. Wong, E. (1971). Stochastic Processes in Information and Dynamical Systems. McGraw-Hill, New York.

5. Zvonkin, A.K. (1974). A Transformation of the Phase Space of a Diffusion Process that Removes the Drift. Math. of the USSR-Sbornik (English Translation). 22 129-149.