Empirical Bayes Rules for Selecting Good Populations*

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Abstract

A problem of selecting populations better than a control is considered. When the populations are uniformly distributed, empirical Bayes rules are derived for a linear loss function for both the known control parameter and the unknown control parameter cases. When the priors are assumed to have bounded supports, empirical Bayes rules for selecting good populations are derived for distributions with truncation parameters (i.e. the form of the pdf is $f(x|\theta) = p_i(x)c_i(\theta)I_{(0,\theta)}(x)$). Monte Carlo studies are carried out which determine the minimum sample sizes needed to make the relative errors less than ε for given ε-values.

AMS Subject Classification: Primary 62F07, Secondary 62C10.

Key Words: Empirical Bayes; Asymptotically Optimal; Selection and Ranking; Truncation Parameter; Better than a Control.

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1. Introduction

We assume that G is an unknown prior distribution on Θ, and denote the minimum Bayes risk in a decision problem by r(G). Robbins, in his pioneering papers (1955), (1964), proposed sequences of decision rules, based on data from n independent repetitions of the same decision problem, whose (n+1)st stage Bayes risk converges to r(G) as $n \to \infty$. Such sequences of rules are called empirical Bayes rules. Empirical Bayes rules have been derived for multiple decision problems by Deely (1965), Van Ryzin (1970), Huang (1975), Van Ryzin and Susarla (1977), and Singh (1977). However, the forms of densities of the populations that these authors considered are either $c(\theta)h(x)e^{\theta x}$, for continuous case or $c(\theta)h(x)\theta^{x}$, for discrete case, and the loss functions are either squared error or merely $\max_{1 \le j \le k} \theta_j - \theta_i$ type. Fox (1978) discussed some estimation problems under the squared error loss, in which empirical Bayes rules were derived for uniform distributions for the first time. Barr and Rizvi (1966), and McDonald (1974) also considered selection problems related to uniform distribution by subset selection approach.

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The problem considered in this paper is related to truncation parameters and can be illustrated by the following example. Suppose that there are k treatments for a certain disease, and the effect of the treatment i follows an unknown distribution G_i, $1 \leq i \leq k$. The effectiveness of the treatment i has been tested on n subjects (for different treatments, different groups of subjects are used. If the same subject has to be used for more than one test, let there be a wash-out period between tests, so the effects of different treatments are independent.). Let θ_{ij} be the parameter which represents the effectiveness of the treatment i on the subject j. The treatment i is good for the subject j if $\theta_{ij} > \theta_0$ and hence is called a good treatment, otherwise it is called a bad treatment. θ_0 is called the control parameter. Let Y_{ij} be an observable result of the treatment i on the subject j and assume that θ_{ij} is underestimated by Y_{ij}. The pdf of Y_{ij} is $f(x|\theta_{ij}) = P_i(x)c_i(\theta_{ij})I_{(0,\theta_{ij})}(x)$. Our purpose is to find an empirical Bayes rule which decides on the quality (better or worse than the control) of the treatment i based on Y_{ij} $(1 \leq i \leq k,\ 1 \leq j \leq n)$ and X_i $(1 \leq i \leq k)$, where X_i is the endpoint result of the treatment i on the present patient. In Section 2, a general formulation is given and empirical Bayes rules are derived for selecting populations better than a known control when the populations are uniformly distributed (i.e. $p_i(x) = 1$). In Section 3, the same problem is considered except that the control parameter is unknown. In Section 4, empirical Bayes rules are found for truncation parameters (that is the densities are of the form $p_i(x)c_i(\theta_i)I_{(0,\theta_i)}(x)$). Rate of convergence is also discussed. Monte Carlo studies are carried out for the priors $G(\theta) = \frac{\theta^2}{c^2} I_{(0,c)}(\theta)$. The smallest sample size N is determined to guarantee that the relative error is less than ε.

2. θ_0 known

Assume that $\pi_1, \pi_2, \ldots, \pi_k$ are k populations and X_i is a random observation for a certain characteristic of π_i. Assume that $X_i \sim U(0,\theta_i)$, where θ_i is unknown for $1 \leq i \leq k$. Let θ_0 be a known control parameter, we define π_i

to be a good population according to the specified characteristic if $\theta_i > \theta_0$ and to be a bad population if $\theta_i \leq \theta_0$. Let $\Theta = \{\underset{\sim}{\theta} = (\theta_1,\ldots,\theta_k) | \theta_i > 0$ for all $1 \leq i \leq k\}$. For any $\underset{\sim}{\theta} \in \Theta$, let $A(\underset{\sim}{\theta}) = \{i | \theta_i > \theta_0\}$ and $B(\underset{\sim}{\theta}) = \{i | \theta_i \leq \theta_0\}$, then $A(\underset{\sim}{\theta})$ $(B(\underset{\sim}{\theta}))$ is the set of indices of good (bad) populations. Our goal is to select all the good populations and reject the bad ones. We formulate the problem in the empirical Bayes framework as follows:

(1) Let $\mathcal{G} = \{S | S \subseteq \{1,2,\ldots,k\}\}$ be the action space.

When we take action S, we say π_i is good if $i \in S$ and π_i is bad if $i \notin S$.

(2) $L(\underset{\sim}{\theta},S) = L_1 \sum_{i\in A(\underset{\sim}{\theta})\setminus S} (\theta_i-\theta_0) + L_2 \sum_{i\in B\cap S} (\theta_0-\theta_i)$

is the loss function. (2.1)

(3) Let $dG(\underset{\sim}{\theta}) = \prod_{i=1}^{k} dG_i(\theta_i)$ be an unknown prior distribution on Θ, where G_i has a continuous pdf g_i with respect to the Lebesgue measure.

(4) Let $(\theta_{i1},Y_{i1}),\ldots,(\theta_{in}, Y_{in})$ be pairs of random variables from π_i and $Y_{ij}|\theta_{ij} \sim U(0,\theta_{ij})$ for $1 \leq i \leq k$, $1 \leq j \leq n$. Let $\underset{\sim}{Y}_j = (Y_{1j},\ldots,Y_{kj})$, then $\underset{\sim}{Y}_j$ denotes the previous j-th observations from $\pi_1,\ldots,\pi_k$.

(5) Let $\underset{\sim}{X} = (X_1,\ldots,X_k)$ be the present observation and $f(\underset{\sim}{x}|\underset{\sim}{\theta}) = \prod_{i=1}^{k} \frac{1}{\theta_i} I_{(0,\theta_i)}(x_i)$. Note that $X_i = Y_{i,n+1}$ and $\theta_i = \theta_{i,n+1}$. Since we are interested in Bayes rules, we can restrict our attention to the non-randomized rules.

(6) Let $D = \{\delta | \delta\colon \mathcal{X} \to \mathcal{G}$ is measurable$\}$, then $r(\underset{\sim}{G}) = \inf_{\delta\in D} r(\underset{\sim}{G},\delta)$ is the minimum Bayes risk.

The decision rules $\{\delta_n(\underset{\sim}{x};\ \underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n)\}_{n=1}^{\infty}$ are said to be asymptotically optimal (a.o.) or empirical Bayes (e.B.) relative to G if $r_n(\underset{\sim}{G},\delta_n) = \int_{\mathcal{X}} E \int_{\Theta} L(\underset{\sim}{\theta},\delta_n(\underset{\sim}{x};\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n))f(\underset{\sim}{x}|\underset{\sim}{\theta})dG(\underset{\sim}{\theta})d\underset{\sim}{x} \to r(\underset{\sim}{G})$ as $n \to \infty$. For simplicity,

$\delta_n(\underset{\sim}{x};\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n)$ will be denoted by $\delta_n(\underset{\sim}{x})$.

Let $m_i(x)$ be the marginal pdf of X_i and $M_i(x)$ be the marginal distribution of X_i. Then we have

$$m_i(x) = \int_x^\infty \frac{1}{\theta}\, dG_i(\theta) \quad \text{for all } x > 0, \tag{2.2}$$

and

$$M_i(x) = \int_0^x \int_t^\infty \frac{1}{\theta}\, dG_i(\theta)dt = \int_x^\infty \int_0^x \frac{1}{\theta}\, dtdG_i(\theta) + \int_0^x \int_0^\theta \frac{1}{\theta}\, dtdG_i(\theta)$$

$$= xm_i(x) + G_i(x).$$

Hence, $G_i(x) = M_i(x) - xm_i(x)$. $\quad$ (2.3)

Now, the loss function defined in (2.1) can be expressed as

$$L(\underset{\sim}{\theta},S) = \sum_{i\in S}[L_2(\theta_0-\theta_i)I_{(0,\theta_0]}(\theta_i)-L_1(\theta_i-\theta_0)I_{(\theta_0,\infty)}(\theta_i)]$$

$$+ \sum_{i=1}^{k} L_1(\theta_i-\theta_0)I_{(\theta_0,\infty)}(\theta_i). \tag{2.4}$$

Since the second sum in (2.4) does not depend on the action S, we can omit it and need only to consider the first sum as our loss for finding an empirical Bayes rules from now on. Therefore,

$$r(\underset{\sim}{G},\delta) = \int_{\mathcal{X}} \sum_{i\in\delta(\underset{\sim}{x})} [\int_{\{\theta_i\leq\theta_0\}} L_2(\theta_0-\theta_i)f(\underset{\sim}{x}|\underset{\sim}{\theta})d\underset{\sim}{G}(\underset{\sim}{\theta})$$

$$- \int_{\{\theta_i>\theta_0\}} L_1(\theta_i-\theta_0)f(\underset{\sim}{x}|\underset{\sim}{\theta})d\underset{\sim}{G}(\underset{\sim}{\theta})]d\underset{\sim}{x}.$$

So, if $\delta_B(\underset{\sim}{x}) = S^*$ is the Bayes rule wrt $\underset{\sim}{G}$, one finds $i \in S^*$ if

$$\int_{(0,\theta_0]\cap(x_i,\infty)} L_2(\theta_0-\theta_i)\,\frac{1}{\theta_i}\, dG_i(\theta_i)$$

$$\leq \int_{\theta_0\vee x_i}^{\infty} L_1(\theta_i-\theta_0)\,\frac{1}{\theta_i}\, dG_i(\theta_i).$$ Hence,

$S^* = \{i | x_i \geq \theta_0\} \cup \{i | x_i < \theta_0 \text{ and } H_i(x_i) \leq c_i(\theta_0)\}$, where

$$H_i(x_i) = L_2\theta_0 \int_{x_i}^{\theta_0} \frac{1}{\theta_i} dG_i(\theta_i) + L_2G_i(x_i) \quad \text{and}$$

$$c_i(\theta_0) = L_2G_i(\theta_0)+L_1(1-G_i(\theta_0))-L_1\theta_0 \int_{\theta_0}^{\infty} \frac{1}{\theta_i} dG_i(\theta_i).$$

Since $H_i(x_i)$ is decreasing in x_i for $x_i < \theta_0$ and $H(\theta_0) \leq c_i(\theta_0)$, so $S^* = \{i | x_i \geq \theta_0 - b_i\}$, where $b_i \geq 0$ satisfies $H(\theta_0-b_i) = c_i(\theta_0)$. This shows for any $\underset{\sim}{G}$, the above type rules are Bayes rules [see Gupta and Sobel (1958) and Gupta (1963,1965)]. Now, $\underset{\sim}{G}$ is unknown; the Bayes rules are not obtainable. We wish to find a sequence of rules $\{\delta_n(\underset{\sim}{x})\}_{n=1}^{\infty}$ to be a.o. Let

$$\Delta_{G_i}(x_i) = H_i(x_i) - c_i(\theta_0)$$

and

$$S_B(\underset{\sim}{x}) = \{i | x_i < \theta_0, \Delta_{G_i}(x_i) \leq 0\}.$$

Also, for any i $(1 \leq i \leq k)$, let $\Delta_{i,n}(x_i) = \Delta_i(x_i, Y_{i1},\ldots,Y_{in})$ for all $n = 1,2,\ldots$, be a sequence of real-valued measurable functions; we define

$$S_n(\underset{\sim}{x}) = \{i | x_i < \theta_0 \text{ and } \Delta_{i,n}(x_i) \leq 0 \} \tag{2.5}$$

and

$$\delta_n(\underset{\sim}{x}) = \{i | x_i \geq \theta_0\} \cup S_n(\underset{\sim}{x}). \tag{2.6}$$

One can show that

Theorem 2.1. If $\int_0^{\infty}\theta \, dG_i(\theta) < \infty$, $i = 1,2,\ldots,k$, and $\Delta_{i,n}(x_i) \to \Delta_{G_i}(x_i)$ in (p) for almost all $x_i < \theta_0$. Then $\{\delta_n(\underset{\sim}{x})\}_{n=1}^{\infty}$ defined by (2.6) is empirical Bayes.

Proof: For all $S \in \mathcal{G}$, let

$$\mathcal{X}_S = \{\underset{\sim}{x} | x_i \geq \theta_0 \text{ if } i \in S \text{ and } x_i < \theta_0 \text{ if } i \notin S\}.$$

Now, for any $\underset{\sim}{x} \in \mathcal{X}_S$, $\delta_B(\underset{\sim}{x}) = S \cup S_B(\underset{\sim}{x})$. Therefore

$$\int_{\Theta} L(\underset{\sim}{\theta}, \delta_B(\underset{\sim}{x})) f(\underset{\sim}{x}|\underset{\sim}{\theta}) dG(\underset{\sim}{\theta})$$

$$= \sum_{i \in \delta_B(\underset{\sim}{x})} \Big[\int_{\{\theta_i \leq \theta_0\}} L_2(\theta_0 - \theta_i) f(\underset{\sim}{x}|\underset{\sim}{\theta}) dG(\underset{\sim}{\theta}) - \int_{\{\theta_i > \theta_0\}} L_1(\theta_i - \theta_0) f(\underset{\sim}{x}|\underset{\sim}{\theta}) dG(\underset{\sim}{\theta})\Big]$$

$$= \sum_{i \in S} (-\, Q(\underset{\sim}{x})) + \sum_{i \in S_B(\underset{\sim}{x})} \Delta_{G_i}(x_i) \prod_{j \neq i} m_j(x_j),$$

where $\quad Q(\underset{\sim}{x}) = \int_{\{\theta_i > \theta_0\}} L_1(\theta_i - \theta_0) f(\underset{\sim}{x}|\underset{\sim}{\theta}) dG(\underset{\sim}{\theta}).$

Similarly, for $x \in \mathcal{X}_S$, we have

$$\int_{\Theta} L(\underset{\sim}{\theta}, \delta_n(\underset{\sim}{x})) f(\underset{\sim}{x}|\underset{\sim}{\theta}) dG(\underset{\sim}{\theta})$$

$$= \sum_{i \in S} (-\, Q(\underset{\sim}{x})) + \sum_{i \in S_n(\underset{\sim}{x})} \Delta_{G_i}(x_i) \prod_{j \neq i} m_j(x_j).$$

Hence, if $\Delta_{i,n}(x_i) \to \Delta_{G_i}(x_i)$ in (p), then

$$0 \leq \int_{\Theta} [L(\underset{\sim}{\theta}, \delta_n(\underset{\sim}{x})) - L(\underset{\sim}{\theta}, \delta_B(\underset{\sim}{x}))] f(\underset{\sim}{x}|\underset{\sim}{\theta}) dG(\underset{\sim}{\theta})$$

$$\leq \sum_{i \in S_n(\underset{\sim}{x})} |\Delta_{G_i}(x_i) - \Delta_{i,n}(x_i)| \prod_{j \neq i} m_j(x_j)$$

$$+ \Big(\sum_{i \in S_n(\underset{\sim}{x})} - \sum_{i \in S_B(\underset{\sim}{x})}\Big) \Delta_{i,n}(x_i) \prod_{j \neq i} m_j(x_j) \tag{2.7}$$

$$+ \sum_{i \in S_B(\underset{\sim}{x})} |\Delta_{i,n}(x_i) - \Delta_{G_i}(x_i)| \prod_{j \neq i} m_j(x_j)$$

$$\leq 2\varepsilon \sum_{i=1}^{k} \prod_{j \neq i} m_j(x_j)$$

with probability near 1 for large n. Note that (2.7) is non-positive by the definition of $S_n(\underset{\sim}{x})$. Thus, we have proved

$$\int_{\Theta} L(\underset{\sim}{\theta},\delta_n(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})d\underset{\sim}{G}(\underset{\sim}{\theta}) \to \int_{\Theta} L(\underset{\sim}{\theta},\delta_B(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})d\underset{\sim}{G}(\underset{\sim}{\theta})$$

in (p) for all most all x. By Corollary 1 of Robbins (1964), $\{\delta_n(\underset{\sim}{x})\}_{n=1}^{\infty}$ is empirical Bayes. This completes the proof.

In view of (2.2) and (2.3), we have

$$\Delta_{G_i}(x_i) = L_2 m_i(x_i)(\theta_0-x_i) + L_2[M_i(x_i)-M_i(\theta_0)] + L_1[M_i(\theta_0)-1].$$

Hence, if we define

$$\Delta^*_{i,n}(x_i) = L_2 m_{i,n}(x_i)(\theta_0-x_i) + L_2[M_{i,n}(x_i)-M_{i,n}(\theta_0)] + L_1[M_{i,n}(\theta_0)-1], \tag{2.8}$$

where $M_{i,n}(x) = \frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,x]}(Y_{ij})$

and $m_{i,n}(x) = \frac{1}{h}[M_{i,n}(x+h) - M_{i,n}(x)],\ h > 0,$ (2.9)

then $\Delta^*_{i,n}(x_i) \to \Delta_{G_i}(x_i)$ in (p) a.e. in $\underset{\sim}{x}$, if $h = h(n) \to 0$ and $nh \to \infty$ as $n \to \infty$. So, by Theorem 1, $\delta^*_n(\underset{\sim}{x}) = \{i|x_i \geq \theta_0\} \cup \{i|x_i < \theta_0, \Delta^*_{i,n}(x_i) \leq 0\}$ is empirical Bayes.

Remark: In (2.8), $M_{i,n}(x)$ and $m_{i,n}(x)$ can be defined as any functions such that $M_{i,n}(x) \to M_i(x)$ in (p) and $m_{i,n}(x) \to m_i(x)$ in (p) for almost all x. For example, let $m^0_{i,n}(x) = \frac{1}{nh}\sum_{j=1}^{n} w(\frac{x-Y_{ij}}{h})$ where $w(\cdot) \geq 0$ satisfies

(i) $\sup_{-\infty<x<\infty} w(x) \leq K$ for some constant K,

(ii) $\int_{-\infty}^{\infty} w(x)dx = 1$

(iii) $\lim_{x\to\infty} xw(x) = 0$

and $h = h(n)$ satisfies $h \to 0$, $nh \to \infty$ as $n \to \infty$ then $m^0_{i,n}(x)$ is a consistent estimator of $m_i(x)$ (see Parzen (1962)).

3. θ_0 unknown

Let π_0 be the control population and assume that X_0, a certain observable characteristic of π_0, follows $U(0,\theta_0)$. Let $Y_{01},\ldots,Y_{0n}$ be the past data collected from π_0. Based on this further information, we will search for empirical Bayes rules for selecting populations better than the control. Note that now $\underset{\sim}{\theta} = (\theta_0,\theta_1,\ldots,\theta_k)$, $\underset{\sim}{x} = (x_0,x_1,\ldots,x_k)$ and $G(\underset{\sim}{\theta}) = \prod_{i=0}^{k} G_i(\theta_i)$. Under the loss function in (2.4), the Bayes rule δ_B is: $i \in \delta_B(\underset{\sim}{x})$ if

$$L_2 \int_{x_0}^{\infty} \frac{1}{\theta_0} \int_{(0,\theta_0]\cap(x_i,\infty)} \frac{1}{\theta_i} (\theta_0-\theta_i)dG_i(\theta_i)dG_0(\theta_0)$$

$$\leq L_1 \int_{x_0}^{\infty} \frac{1}{\theta_0} \int_{(\theta_0,\infty)\cap(x_i,\infty)} \frac{1}{\theta_i} (\theta_i-\theta_0)dG_i(\theta_i)dG_0(\theta_0).$$

Hence, $i \in \delta_B(\underset{\sim}{x})$ if

(i) $x_i \geq x_0$ and $\Delta^1_{G_0,G_i}(x_0,x_i) \leq 0$, where

$$\Delta^1_{G_0,G_i}(x_0,x_i) = (L_1-L_2)[\int_{x_i}^{\infty} m_i(\theta_0)dG_0(\theta_0) + \int_{x_i}^{\infty} m_0(\theta_i)dG_i(\theta_i)]$$

$$- L_1[1-G_i(x_i)]m_0(x_0)+m_i(x_i)[L_2+(L_1-L_2)G_0(x_i)-L_1G_0(x_0)] \tag{3.1}$$

or

(ii) $x_i < x_0$ and $\Delta^2_{G_0,G_i}(x_0,x_i) \leq 0$, where

$$\Delta^2_{G_0,G_i}(x_0,x_i) = (L_1-L_2)[\int_{x_0}^{\infty} m_i(\theta_0)dG_0(\theta_0) + \int_{x_0}^{\infty} m_0(\theta_i)dG_i(\theta_i)]$$

$$- m_0(x_0)[L_1+(L_2-L_1)G_i(x_0)-L_2G_i(x_i)] + L_2m_i(x_i)(1-G_0(x_0)). \tag{3.2}$$

When $L_1 = L_2 = L$, the Bayes rule is greatly simplified. We find $i \in \delta_B(\underset{\sim}{x})$ if

$$\Delta_{G_0,G_i}(x_0,x_i) = m_0(x_0)[1-G_i(x_i)]-m_i(x_i)[1-G_0(x_0)] \geq 0.$$

Let $\delta_n(\underset{\sim}{x}) = \{i|\Delta_{i,n}(x_i,x_0) \geq 0\}$, where

$$\Delta_{i,n}(x_i,x_0) = m_{0,n}(x_0)[1-G_{i,n}(x_i)] - m_{i,n}(x_i)[1-G_{0,n}(x_0)],$$

$M_{i,n}(x_i)$ and $m_{i,n}(x_i)$ are defined in (2.9), and $G_{i,n}(x_i) = M_{i,n}(x_i) - x_i m_{i,n}(x_i)$. Then, $\{\delta_n(\underset{\sim}{x})\}_{n=1}^{\infty}$ is e.B. by Theorem 3.2. When $L_1 \neq L_2$, one needs to find consistent estimators of $\int_a^\infty m_i(\theta_0)dG_0(\theta_0)$ and $\int_a^\infty m_0(\theta_i)dG_i(\theta_i)$.

<u>Theorem 3.1</u>. Let $M_{i,n}(x)$ and $m_{i,n}(x)$ be defined by (2.9) with $h = h(n)$ satisfying $h \to 0$, $nh^2 \to \infty$ as $n \to \infty$. If $\int_0^\infty \theta\, dG_i(\theta) < \infty$ for all $i = 0,1,\ldots,k$, then $-\int_a^\infty xm_{i,n}(x)dm_{0,n}(x) \to \int_a^\infty m_i(x)dG_0(x)$ in (p) for any $a > 0$.

<u>Proof</u>: See Appendix A.

<u>Theorem 3.2</u>. Assume that $\int_0^\infty \theta\, dG_i(\theta) < \infty$ for all $0 \leq i \leq k$. If for all $1 \leq i \leq k$, $\Delta^1_{i,n}(x_0,x_i) \to \Delta^1_{G_i,G_0}(x_0,x_i)$ in (p) for $x_i \geq x_0$, and $\Delta^2_{i,n}(x_0,x_i) \to \Delta^2_{G_i,G_0}(x_0,x_i)$ in (p) for $x_i < x_0$. Then

$$\begin{aligned}\delta^*_n(\underset{\sim}{x}) &= S^1_n(\underset{\sim}{x}) \cup S^2_n(\underset{\sim}{x}) \\ &= \{i|x_i \geq x_0 \text{ and } \Delta^1_{i,n}(x_0,x_i) \leq 0\} \cup \\ &\quad \{i|x_i < x_0 \text{ and } \Delta^2_{i,n}(x_0,x_i) \leq 0\} \end{aligned} \tag{3.3}$$

defines an empirical Bayes rule.

<u>Proof</u>: $\int_{\Theta} L(\underset{\sim}{\theta},\delta_B(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})d\underset{\sim}{G}(\underset{\sim}{\theta})$

$$= \sum_{i\in S_1^*(\underset{\sim}{x})} \Delta^1_{G_i,G_0}(x_0,x_i)\prod_{j\neq i} m_j(x_j) + \sum_{i\in S_2^*(\underset{\sim}{x})} \Delta^2_{G_i,G_0}(x_0,x_i)\prod_{j\neq i} m_j(x_j),$$

where $S_1^*(\underset{\sim}{x}) = \{i|x_i \geq x_0 \text{ and } \Delta^1_{G_i,G_0}(x_0,x_i) \leq 0\}$

$S_2^*(\underset{\sim}{x}) = \{i|x_i < x_0 \text{ and } \Delta^2_{G_i,G_0}(x_0,x_i) \leq 0\}$,

and $\int_{\Theta} L(\underset{\sim}{\theta},\delta_n^*(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})d\underset{\sim}{G}(\underset{\sim}{\theta})$

$$= \sum_{i\in S_n^1(\underset{\sim}{x})} \Delta^1_{G_i,G_0}(x_0,x_i)\prod_{j\neq i} m_j(x_j) + \sum_{i\in S_n^2(\underset{\sim}{x})} \Delta^2_{G_i,G_0}(x_0,x_i)\prod_{j\neq i} m_j(x_j).$$

Now, following the same method as in the proof of Theorem 2.1, we can show

$$\sum_{i\in S_n^\ell(\underset{\sim}{x})} \Delta^\ell_{G_i,G_0}(x_0,x_i)\prod_{j\neq i} m_j(x_j) \to \sum_{i\in S_\ell^*(\underset{\sim}{x})} \Delta^\ell_{G_i,G_0}(x_0,x_i)\prod_{j\neq i} m_j(x_j)$$

in (p) for $\ell = 1,2$. Hence $\{\delta_n^*(\underset{\sim}{x})\}_{n=1}^\infty$ is empirical Bayes. This completes the proof.

Now, let

$$\Delta^1_{i,n}(x_0,x_i) = (L_2-L_1)\{\int_{x_i}^\infty xm_{i,n}(x)dm_{0,n}(x) + \int_{x_i}^\infty xm_{0,n}(x)dm_{i,n}(x)\}$$
$$- L_1[1-G_{i,n}(x_i)]m_{0,n}(x_0)+m_{i,n}(x_i)[L_2+(L_1-L_2)$$
$$G_{0,n}(x_i)-L_1G_{0,n}(x_0)], \tag{3.4}$$

and

$$\Delta^2_{i,n}(x_0,x_i) = (L_2-L_1)\{\int_{x_0}^\infty xm_{i,n}(x)dm_{0,n}(x) + \int_{x_0}^\infty xm_{0,n}(x)dm_{i,n}(x)\}$$
$$+ L_2[1-G_{0,n}(x_0)]m_{i,n}(x_i)-m_{0,n}(x_0)[L_1+(L_2-L_1)G_{i,n}(x_0),$$
$$- L_2G_{i,n}(x_i)],$$

where $G_{i,n}(x) = M_{i,n}(x) - xm_{i,n}(x)$. (3.5)

Then, by Theorem 3.1 and Theorem 3.2,(3.3), (3.4), and (3.5) define an empirical Bayes rule.

4. Generalization and Simulation

Let $p_i(x)$ be a positive continuously differentiable function which is defined over $(0,\infty)$ for $1 \le i \le k$. Let $c_i(\theta)^{-1} = \int_0^\theta p_i(x)dx$ for $\theta > 0$, then $f_i(x|\theta) = p_i(x)c_i(\theta)I_{(0,\theta)}(x)$ is a density function and θ is a truncation parameter. In this section, we assume that π_i is $f_i(x|\theta_i)$ for $1 \le i \le k$. Under the formulation of Section 2, we wish to find empirical Bayes rules for these more general density functions. For simplicity, we assume that $L_1 = L_2 = L$ and that θ_0 is known. Also we assume $G_i(\theta)$ has a continuous density $g_i(\theta)$ with a bounded support $[0,\alpha_i]$ with a known α_i for all $1 \le i \le k$. We find

$$m_i(x) = \int_0^{\alpha_i} f_i(x|\theta)dG_i(\theta) = p_i(x)\int_x^{\alpha_i} c_i(\theta)dG_i(\theta).$$

If we follow the same discussion as in Section 2, we can show that the Bayes rule δ_B is $i \in \delta_B(\underset{\sim}{x})$ iff

(i) $x_i \ge \theta_0$, or

(ii) $x_i < \theta_0$ and $\theta_0 \int_{x_i}^{\alpha_i} c_i(x)dG_i(x) \le \int_{x_i}^{\alpha_i} xc_i(x)dG_i(x)$.

Hence, $\delta_B(\underset{\sim}{x}) = \{i|x_i \ge \theta_0-d_i\}$, where $d_i \ge 0$ satisfies $\int_{d_i}^{\alpha_i} (\theta_0-x)c_i(x)dG_i(x) = 0$.

Let $d_{i,n} = d_{i,n}(Y_{i1},\ldots,Y_{in})$ be a consistent estimation of d_i, then $\delta_n^0(\underset{\sim}{x}) = \{i|x_i \ge \theta_0-d_{i,n}\}$ is e.B. and they are (weak) admissible in the sense that $\delta_n^0(\cdot,\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n)$ is an admissible rule for the non-empirical problem for all $\underset{\sim}{Y}_1,\ldots,\underset{\sim}{Y}_n$ and n (see Houwelingen (1976). Meeden (1972)). However, to find such a sequence $\{d_{i,n}\}_{n=1}^{\infty}$ is very difficult. In view of Theorem 2.1, a more practical way to find empirical Bayes rules is to estimate

$$\int_{x_i}^{\alpha_i} xc_i(x)dG_i(x).$$

Theorem 4.1. Let $p_i(x)$ and $G_i(x)$ be defined as above. If $m_{i,n}(x)$ is defined by (2.9) with $h \to 0$, $nh \to \infty$, then

$$\int_{x_i}^{\alpha_i} \frac{xp_i'(x)}{p_i^2(x)} m_{i,n}(x)dx - \int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} dm_{i,n}(x)$$
$$\to \int_{x_i}^{\alpha_i} xc_i(x)dG_i(x) \text{ in } (p).$$

Proof: See Appendix B.

Now, let

$$\Delta_{i,n}^*(x_i) = \frac{\theta_0 m_{i,n}(x_i)}{p_i(x_i)} + \int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} dm_{i,n}(x) - \int_{x_i}^{\alpha_i} \frac{xp_i'(x)}{p_i^2(x)} m_{i,n}(x)dx, \quad (4.1)$$

then $\delta_n^*(\underset{\sim}{x}) = \{i|x_i \geq \theta_0\} \cup \{i|x_i < \theta_0 \text{ and } \Delta_{i,n}^*(x_i) \leq 0\}$ (4.2)

defines an empirical Bayes rule.

The following lemma is a direct result of Lemma 3 of Van Ryzin and Susarla (1977).

Lemma 4.2. Let $\Delta_{G_i}(x) = \int_{x}^{\alpha_i} (\theta_0 - t)c_i(t)dG_i(t)\, I_{(0,\alpha_i)}(x)$,

then $0 \leq r_n(\underset{\sim}{G},\delta_n^*) - r(\underset{\sim}{G}) = \sum_{i=1}^{k} \{\int_{H_i^1} |\Delta_{G_i}(x)|p_i(x)|P[\Delta_{i,n}^*(x) < 0]dx$

$$+ \int_{H_i^2} |\Delta_{G_i}(x)|p_i(x)P[\Delta_{i,n}^*(x) \geq 0]dx\},$$

where $\Delta^*_{i,n}(x)$ and δ^*_n are defined by (4.1) and (4.2) respectively, and $H^1_i = \{x | x \leq \theta_0 \text{ and } \Delta_{G_i}(x) > 0\}$ and $H^2_i = \{x | x \leq \theta_0 \text{ and } \Delta_{G_i}(x) < 0\}$.

Now, let $O(\alpha_n)$ denote a quantity such that $0 \leq \lim_{n\to\infty} \frac{O(\alpha_n)}{\alpha_n} < \infty$. Then since $|\Delta_{G_i}(x)| p_i(x) \leq M_i$ for all $x \leq \theta_0$ for some constant M_i, so

$$r_n(\underset{\sim}{G},\delta^*_n) - r(\underset{\sim}{G}) \leq \sum_{i=1}^{k} M_i \{\int_{H^1_i} P[\Delta^*_{i,n}(x) < 0]dx$$

$$+ \int_{H^2_i} P[\Delta^*_{i,n}(x) \geq 0]dx\}.$$

Therefore, if for all $x \leq \theta_0$

$$P[|\Delta^*_{i,n}(x) - \Delta_{G_i}(x)| > |\Delta_{G_i}(x)|] = O(\alpha_n) \text{ as } n \to \infty$$

then

$$r_n(\underset{\sim}{G},\delta^*_n) - r(\underset{\sim}{G}) = O(\alpha_n).$$

Now, by the inequality

$$P[|\Delta^*_{i,n}(x) - \Delta_{G_i}(x)| > |\Delta_{G_i}(x)|] \leq \frac{\mathrm{Var}[\Delta^*_{i,n}(x)]}{[|\Delta_{G_i}(x)| - |\Delta_{G_i}(x) - E\Delta^*_{i,n}(x)|]^2},$$

we conclude that if $\mathrm{Var}[\Delta^*_{i,n}(x)] = O(\alpha_n)$ for all $x \leq \theta_0$ then $r_n(\underset{\sim}{G},\delta^*_n) - r(\underset{\sim}{G}) = O(\alpha_n)$. Note that if $\alpha_n \to 0$, then δ^*_n is empirical Bayes.

In the following, we have carried out some Monte Carlo study to see how fast the derived empirical Bayes rules converge. We let $X_i \sim U(0,\theta_i)$ for $i = 0,1$. θ_0 is treated as unknown. Assume that $g_i(\theta) = \frac{2\theta}{c^2} I_{(0,c)}(\theta)$ for $i = 0,1$ and $L_1 = L_2 = 1$. The smallest sample size N such that

$$\text{Relative error: } \frac{|r_m(\underset{\sim}{G},\delta^*_m) - r(\underset{\sim}{G})|}{r(\underset{\sim}{G})} \leq \varepsilon$$

for $N-4 \leq m \leq N$ is determined. Here $r(\underset{\sim}{G}) = P_{\underset{\sim}{G}}[(\theta_1 > \theta_0, X_1 < X_0) \cup (\theta_1 < \theta_0, X_1 > X_0)] = \frac{c}{15}$. The Monte Carlo studies are repeated for 55 times and the values of N and the associated standard deviations corresponding to selected ε and c are shown in the next table for $h = n^{-1/4}$, for $h = n^{-1/5}$ and for $h = n^{-1/6}$, where h is used to define (2.9).

Table 1

Upper entries are the Lists of values of the smallest N such that $\frac{|r_m(\underset{\sim}{G},\delta_m^*)-r(\underset{\sim}{G})|}{r(\underset{\sim}{G})} \leq \varepsilon$ for $N-4 \leq m \leq N$, lower entries are the list of associated standard deviations for each corresponding case where the density of the priors is $g_i(\theta) = \frac{2\theta}{c^2} I_{(0,c)}(\theta)$ for i = 0,1.

	$h = n^{-1/4}$					$h = n^{-1/5}$					$h = n^{-1/6}$				
c \ ε	.25	.20	.15	.10	.05	.25	.20	.15	.10	.05	.25	.20	.15	.10	.05
1/3	8 .0129	10 .0136	11 .0133	14 .0153	15 .0126	8 .0133	10 .0156	11 .0142	14 .0136	23 .0128	7 .0132	8 .0137	10 .0133	11 .0143	48 .0128
1/2	8 .0210	10 .0181	11 .0178	14 .0233	30 .0204	7 .0210	8 .0192	10 .0210	16 .0204	33 .0258	8 .0198	9 .0318	10 .0225	11 .0216	71 .0202
1	10 .0496	11 .0470	14 .0568	23 .0465	38 .0483	8 .0537	10 .0531	11 .0458	15 .0378	26 .0490	7 .0464	8 .0495	14 .0571	22 .0477	29 .0488
2	42 .1015	46 .1262	117 .0913	224 .1144	270 .1072	19 .1230	21 .1196	22 .1366	159 .1035	-	25 .0895	26 .1199	29 .1066	84 .1031	146 .0884
3	104 .1642	183 .1988	322 .1596	-	-	58 .1657	124 .1622	125 .1991	154 .1569	-	61 .1916	70 .1707	104 .1637	123 .1532	352 .1325

Note: "-" means that N > 400 (Monte Carlo study was curtailed because of limited resources)

Appendix A

Proof of Theorem 3.1.

For i fixed, $\int_0^\infty x m_{i,n}(x) dm_{0,n}(x)$

$$= \frac{1}{n^2}\frac{1}{h^2}\sum_{j=1}^{n}\sum_{\ell=1}^{n}\int_a^\infty x\, I_{(x,x+h]}(Y_{ij})\, dI_{[Y_{0\ell}-h,Y_{0\ell})}(x)$$

$$= \frac{1}{n^2}\frac{1}{h^2}\sum_{j=1}^{n}\sum_{\ell=1}^{n}(U_{j\ell}-V_{j\ell}), \text{ where}$$

$$U_{j\ell} = (Y_{0\ell}-h)I_{(a,\infty)}(Y_{0\ell}-h)I_{(Y_{0\ell}-h,Y_{0\ell}]}(Y_{ij})$$

$$V_{j\ell} = Y_{0\ell}I_{(a,\infty)}(Y_{0\ell})I_{(Y_{0\ell},Y_{0\ell}+h]}(Y_{ij}).$$

Since $Y_{0\ell} \sim M_0(x)$ and $Y_{ij} \sim M_i(x)$ for $1 \leq j, \ell \leq n$, so

$$E\int_a^\infty x m_{i,n}(x) dm_{0,n}(x) = \frac{1}{h^2} E[U_{11}-V_{11}]$$

$$= \int_a^\infty x \frac{1}{h}\int_x^{x+h} dM_i(y)\, \frac{1}{h}[m_0(x+h)-m_0(x)]dx.$$

Now, by (2.2) $m_i(x)$ is decreasing in x, hence

$$\frac{1}{h}\int_x^{x+h} dM_i(y) \leq m_i(x) \leq \frac{1}{x}[1-G_i(x)]. \tag{A.1}$$

Then $\left|x\cdot \frac{1}{h}\int_x^{x+h} dM_i(y)\, \frac{1}{h}[m_0(x+h)-m_0(x)]\right|$

$$\leq [1-G_i(x)]\,\frac{1}{h}\int_x^{x+h}\frac{1}{\theta}\,dG_0(\theta) \leq \frac{1}{x}\,g_0(x+\delta h), \text{ for some } \delta \in [0,1].$$

The last term is integrable over (a,∞), then by Lebesgue Dominated Convergence Theorem (LDCT),

$$E\int_a^\infty xm_{i,n}(x)dm_{0,n}(x) \to \int_a^\infty xm_i(x)m_0'(x)dx$$

$$= -\int_a^\infty m_i(x)dG_0(x) \text{ in (p) if } h \to 0 \text{ as } n \to \infty. \tag{A.2}$$

Now, $\text{Var} \int_a^\infty xm_{i,n}(x)dm_{0,n}(x) = \text{Var} \frac{1}{n^2}\frac{1}{h^2}\sum_{j,\ell}(U_{j\ell}-V_{j\ell})$

$$= \frac{1}{n^2h^4}\text{Var}(U_{11}-V_{11}) + \frac{2(n-1)}{n^2h^4}\text{Cov}(U_{11}-V_{11},\ U_{12}-V_{12}). \tag{A.3}$$

But $\text{Var}(U_{11}-V_{11}) \le E[(U_{11}-V_{11})^2] = E(U_{11}^2) + E(V_{11}^2)$ [because $U_{11}V_{11} = 0$], and $\frac{1}{h}E(U_{11}^2)$

$$= \int_a^\infty x^2\cdot\frac{1}{h}\int_x^{x+h} dM_i(y)dM_0(x+h)$$

$$\le \int_a^\infty x^2\cdot\frac{1}{x}(1-G_i(x))dM_0(x+h) \le \int_a^\infty x\, dM_0(x+h)$$

$$\le E^{M_0}[X] = E^{G_0}[E[X|\Theta_0]] = \frac{1}{2}E^{G_0}[\Theta_0] < \infty,$$

hence $\frac{1}{h}\text{Var}(U_{11}-V_{11}) \le E^{G_0}[\Theta_0]$ for all $h > 0$. (A.4)

Meanwhile, $\text{Cov}(U_{11}-V_{11},\ U_{12}-V_{12}) = \text{Cov}(U_{11},U_{12}) + \text{Cov}(V_{11},V_{12}) - \text{Cov}(U_{11},V_{12}) - \text{Cov}(V_{11},U_{12})$, and $|\frac{1}{h^2}\text{Cov}(U_{11},U_{12})| \le \frac{1}{h^2}[E(U_{11}U_{12}) + E(U_{11})E(U_{12})]$ because $U_{j\ell} > 0$ for all $1 \le j,\ \ell \le n$.

Now, $\frac{1}{h^2}E(U_{11}U_{12}) = \frac{1}{h^2}\int_0^\infty [\int_{(a,\infty)\cap[x-h,x)} ydM_0(y+h)]^2 dM_i(x)$

$$= \frac{1}{h^2}\int_{a+h}^\infty [\int_{x-h}^x ydM_0(y+h)]^2 dM_i(x) + \frac{1}{h^2}\int_a^{a+h}[\int_a^x ydM_0(y+h)]^2 dM_i(x).$$

Because $\int_{x-h}^x ydM_0(y+h) = \int_{x-h}^x y\int_{y+h}^\infty \frac{1}{\theta}dG_0(\theta)dy$

$$\le \int_{x-h}^x y\cdot\frac{1}{y+h}\cdot dy \le h, \text{ and, similarly,}$$

$$\int_a^x y\,dM_0(y+h) \le \int_a^{a+h} y\,dM_0(y+h) \le h \quad \text{for} \quad a < x < a+h,$$

we get $\frac{1}{h^2} E(U_{11}U_{12}) \le 1-M_i(a+h) + M_i(a+h) - M_i(a) = 1-M_i(a)$.
The same argument shows that $\frac{1}{h} E(U_{11}) \le 1-M_i(a)$

$$\frac{1}{h} E(V_{11}) \le 1-M_i(a),$$

hence $|\frac{1}{h^2} \mathrm{Cov}(U_{11},U_{12})| \le 2[1-M_i(a)]$. This implies that

$$\frac{1}{h^2} |\mathrm{Cov}(U_{11}-V_{11},U_{12}-V_{12})| \le 8\,[1-M_i(a)] \text{ for any } h > 0. \tag{A.5}$$

By (A.3), (A.4) and (A.5)

$$\mathrm{Var} \int_a x m_{i,n}(x) dm_{0,n}(x) \to 0 \quad \text{if} \quad nh^2 \to 0 \text{ and } h \to 0. \tag{A.6}$$

Now, (A.2) and (A.6) implies that

$$\int_a^\infty x m_{i,n}(x) dm_{0,n}(x) \to - \int_a^\infty m_i(x) dG_0(x) \text{ in } (p).$$

This finishes the proof.

Appendix B

Proof of Theorem 4.1.

First, $E \int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} dm_{i,n}(x) = \int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} \frac{1}{h} [m_i(x+h)-m_i(x)]dx$

$$\to \int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} dm_i(x) \quad \text{by LDCT.}$$

Now, $\mathrm{Var} \int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} dm_{i,n}(x) = \mathrm{Var}[\frac{1}{nh} \sum_{j=1}^{n} (U_j-V_j)]$,

where $U_j = \frac{Y_{ij}-h}{p_i(Y_{ij}-h)} I_{[x_i,\alpha_i]}(Y_{ij}-h)$, and

$$V_j = \frac{Y_{ij}}{p_i(Y_{ij})} I_{[x_i,\alpha_i]}(Y_{ij}).$$

Hence, $\mathrm{Var} \int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} dm_{i,n}(x) = \frac{1}{nh^2} \mathrm{Var}(U_1-V_1)$

$$\leq \frac{1}{nh^2} E[(U_1-V_1)^2] = \frac{1}{n} \int_{x_i+h}^{\alpha_i} [\frac{1}{h} (\frac{x}{p_i(x)} - \frac{x-h}{p_i(x-h)})]^2 dM_i(x)$$

$$+ \frac{1}{nh} \int_{\alpha_i}^{\alpha_i+h} \frac{1}{h} [\frac{x-h}{p_i(x-h)}]^2 dM_i(x) + \frac{1}{nh} \int_{x_i}^{x_i+h} \frac{1}{h} \frac{x^2}{p_i^2(x)} dM_i(x)$$

$$\leq \frac{1}{n} \max_{x\in[x_i,\alpha_i]} [\frac{d}{dx} \frac{x}{p_i(x)}]^2 + \frac{2}{nh} \max_{x\in[x_i,\alpha_i]} [\frac{x}{p_i(x)}]^2$$

$$\to 0 \quad \text{if} \quad nh \to \infty.$$

We see that

$$\int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} dm_{i,n}(x) \to \int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} dm_i(x) \quad \text{in (p).}$$

Similarly $\int_{x_i}^{\alpha_i} \frac{xp_i'(x)}{p_i^2(x)} m_{i,n}(x)dx \to \int_{x_i}^{\alpha_i} \frac{xp_i'(x)}{p_i^2(x)} m_i(x)dx$ in (p).

Since $\int_{x_i}^{\alpha_i} xc_i(x)dG_i(x) = \int_{x_i}^{\alpha_i} -x \frac{d}{dx} [\frac{m_i(x)}{p_i(x)}]$

$$= \int_{x_i}^{\alpha_i} \frac{xp_i'(x)}{p_i^2(x)} m_i(x)dx - \int_{x_i}^{\alpha_i} \frac{x}{p_i(x)} dm_i(x),$$

the proof is completed.

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A problem of selecting populations better than a control is considered. When the populations are uniformly distributed, empirical Bayes rules are derived for a linear loss function for both the known control parameter and the unknown control parameter cases. When the priors are assumed to have bounded supports, empirical Bayes rules for selecting good populations are derived for distributions with truncation parameters (i.e. the form of the pdf is $f(x|\theta) = p_i(x)c_i(\theta)I_{(0,\theta)}(x)$). Monte Carlo studies are carried out which determine the

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) minimum sample sizes needed to make the relative errors less than ε for given ε-values.