ADMISSIBILITY OF GENERALIZED BAYES RULES IN THE CONTROL PROBLEM[1]

by

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ABSTRACT

Let $X = (X_1, \ldots, X_p)^t$ have a p-variate normal distribution with unknown mean θ and identity covariance matrix. The following transformed version of a control problem is considered. Assume the loss incurred in estimating θ by d is $L(d,\theta) = (\theta^t d - 1)^2$. Let $g(|\theta|^2)$ be a generalized prior density. Admissibility of the corresponding generalized Bayes rule δ is considered.

Using techniques of Brown (1979) and Berger (1976b), it is shown that if g is bounded, satisfies certain regularity conditions, and for a constant K and r^2 sufficiently large

$$g(r^2) \leq Kr^{(4-p)} ,$$

then δ is admissible.

A useful asymptotic approximation to δ is obtained. This result, used in conjunction with a theorem of Berger and Zaman (1979), yields a new inadmissibility theorem.

1. Introduction

Recently, interest has been shown in the decision theoretic treatment of a statistical control problem. See, for example, references [5], [10], and [12]. In this problem nonstochastic control (or design) variables are to be chosen in a linear model so that the resulting dependent random variable will be close to a desired, fixed value. For more complete discussions of this problem, see Zaman (1980) and Zellner (1971).

Zaman (1980) considers the following transformed version of the control problem. Let $X = (X_1,\ldots,X_p)^t$ have a p-variate normal distribution with unknown mean $\theta = (\theta_1,\ldots,\theta_p)^t$ and identity covariance matrix. The problem is to choose a decision rule $\delta(X) = (\delta_1(X),\ldots,\delta_p(X))^t$ subject to incurring a loss, $L(\delta,\theta)$, where

(1.1) $$L(\delta , \theta) = (\theta^t\delta - 1)^2.$$

This version of the control problem is the subject of this paper.

Define a spherically symmetric (s.s.) decision rule δ to be of the form

(1.2) $$\delta(x) = \phi(|x|)|x|^{-1}x$$

where $|x|^2 = \sum_{i=1}^{p} x_i^2$. Previous admissibility considerations for such rules include results for the uniform measure, generalized Bayes rule, δ_u, given by

(1.3) $$\delta_u(x) = (1 + |x|^2)^{-1}x.$$

Zaman (1980) has shown that δ_u is admissible when p = 1,2,3. Stein and Zaman (1980) proved that δ_u is admissible when p = 4. The most general, previous inadmissibility results are given in Berger and Zaman (1979). In particular, it is shown that δ_u is inadmissible in more than four dimensions (also see Stein and Zaman (1980)).

It is suggested in the literature, particularly Berger and Zaman (1979), that generalized Bayes rules corresponding to priors with tails as sharp as $|\theta|^{4-p}$ are admissible, while rules corresponding to priors with tails flatter than $|\theta|^{4-p}$ are inadmissible. The results of this paper provide conditions under which these conjectures are true. The importance of such results is twofold. First, a gap in the theory is filled. Second, the identification of the "boundary of admissibility" of generalized Bayes rules (i.e., prior tail behavior of the form $K|\theta|^{4-p}$ for some constant K) is important in applications. Rules corresponding to priors on such a "boundary" frequently display desirable properties (see Berger (1980a), (1980b)). Hence, such rules are often proposed in applications, at least in the absence of complete prior information.

The proof of admissibility given here is essentially an implementation of a general technique given by Brown (1979). Berger (1976b) has carried out Brown's program in a different setting and provides a background drawn on in our proof. The argument employs an approximation to generalized Bayes rules for large $|x|$. Applying this approximation, an extension of an inadmissibility result of Berger and Zaman (1979) is obtained. This approximation may also be of interest beyond its use here.

The next section presents some notation and preliminary results. In Section 3, the admissibility theorem is stated and discussed. An approximation to generalized Bayes rules is developed in Section 4. The proof of admissibility is given in Section 5. This proof is quite long. Hence, some details are either omitted or given in the Appendix.

2. Preliminaries

Define the function $f(\theta,r)$, for r a scalar, by

$$f(\theta,r) = \exp\{-\tfrac{1}{2}(\theta_1 - r)^2\}\exp\{-\tfrac{1}{2}|\theta^*|^2\} \tag{2.1}$$

where $\theta^* = (\theta_2,\ldots,\theta_p)^t$. Also, define the quantities $N(\cdot)$ and $D(\cdot)$ by

$$N(r) = \int \theta_1 f(\theta,r)d\pi(\theta) \tag{2.2}$$

and

$$D(r) = \int \theta_1^2 f(\theta,r)d\pi(\theta) \tag{2.3}$$

for an arbitrary (generalized) s.s. prior, $\pi(\theta)$. Note that in (2.2) and (2.3), as well as the remainder of this paper, the region of integration for θ is assumed to the parameter space, Θ, unless otherwise stated. The parameter space for the control problem is taken to be $\mathbb{R}^p - \{0\}$, where $\{0\}$ denotes the origin. See Berger and Zaman for a discussion of this restriction. Zaman (1980) has proven that if δ is (generalized) Bayes w.r.t. π then δ is unique and $\phi(|x|)$ as defined in (1.2) is given by

$$\phi(|x|) = N(|x|)[D(|x|)]^{-1} . \tag{2.4}$$

Two other previous results play a major role in the analysis here. The first fact, based on a complete class theorem given by Zaman (1980), is that if δ is s.s. and admissible, then

$$0 \le \phi(|x|)|x|^{-1} \le 1, \quad \forall\ |x| . \tag{2.5}$$

Second, Berger and Zaman (1979) have obtained a useful representation of the (finite) Bayes risk of an arbitrary, finite risk, s.s. rule, δ, against a given s.s. prior measure, π. Let $r(\delta,\pi)$ denote the described Bayes risk. Then $r(\delta,\pi)$ is given by

$$(2.6) \qquad r(\delta,\pi) = K_0 \int [\int_0^\infty r^{(p-1)}\{\phi(r)\theta_1 - 1\}^2 f(\theta,r)dr]d\pi(\theta) ,$$

where $K_0 = (2\pi)^{-\frac{p}{2}}$.

Throughout this paper, K (or K', K", etc.) denotes a generic constant; $K_1 = K_0^{-1}$.

3. The Admissibility Theorem

The proof of admissibility is based on a theorem of Stein (1955), but in a form given by Farrell (1968). Their sufficient condition for admissibility is stated for our purposes in the following theorem.

Theorem 3.1. *Let* δ *be the generalized Bayes rule w.r.t. the measure* $\pi(\theta)$ *where*

$$(3.1) \qquad d\pi(\theta) = g(|\theta|^2)d\theta.$$

Suppose that there exists a sequence of finite, non-negative functions $h_n(|\theta|^2)$ *such that*

i) $\int_\Theta g(|\theta|^2)h_n(|\theta|^2)d\theta < \infty \quad \forall_n = 1,2,\ldots$.

ii) $\lim_{n\to\infty} h_n(|\theta|^2) = 1$.

iii) $\lim_{n\to\infty} \int_\Theta [R(\delta,\theta) - R(\delta_n,\theta)]g(|\theta|^2)h_n(|\theta|^2)d\theta = 0.$

where δ_n *denotes the Bayes rule w.r.t. the prior* $\pi_n(\theta)$ *given by*

$$(3.2) \qquad d\pi_n(\theta) = g(|\theta|^2)h_n(|\theta|^2)d\theta$$

and $R(\delta,\theta)$ *denotes the usual risk function of the rule* δ. *Then* δ *is admissible.*

The details of the proof of Theorem 3.1 may be found in Farrell (1968). Also, see Brown (1971). For the control problem, the key elements required are i) the uniqueness of s.s. (generalized) Bayes rules, and ii) the convexity of the control problem loss function in the variable δ.

The hueristic proof of admissibility given in Brown (1979) suggests choices for the functions $h_n(|\theta|^2)$ and proposes methods for approximating the integrands appearing in Conditon iii) of Theorem 3.1. The choice of $h_n(|\theta|^2)$ used below is

$$h_n(|\theta|^2) = \begin{cases} 1 & \text{if } 0 \le |\theta|^2 \le 1 \\ H_n(|\theta|^2) & \text{if } 1 \le |\theta|^2 \le n^2 \\ 0 & \text{if } |\theta|^2 \ge n^2 \end{cases} \tag{3.3}$$

where

$$H_n(|\theta|^2) = [1 - (\ell n\, |\theta|^2)/(\ell n\, n^2)]^{17} \; . \tag{3.4}$$

Some comments are in order. First, our proof is somewhat facilitated by defining g and h_n to be functions of $|\theta|^2$ rather than $|\theta|$, although this appears silly in the definition of H_n. Second, in general, implementation of Brown's argument requires that the functions h_n are "flat". This flatness requirement accounts for the functional form of H_n. Choosing the 17th power as in (3.4) enhances the flatness of H_n, but will also be of further technical importance. We doubt that 17 is the smallest power that can be used in the definition of H_n to prove the admissibility theorem below. However, complicating matters by searching for such a smallest power would not simplify the proof, nor add to its tractability.

Throughout the remainder of this paper, the following notation is used. Define

$$g'(y^2) = \frac{dg(y^2)}{dy^2} .$$

Theorem 3.2. Let δ be the generalized Bayes rule w.r.t. the measure $\pi(\theta)$, defined in (3.1). Suppose that

i) $0 \le g(|\theta|^2) \le B < \infty$.

ii) g is absolutely continuous w.r.t. Lebesgue measure.

Furthermore, suppose that there exists a constant $T > 0$ such that if $r^2 \ge T$ then the following conditions hold:

iii) $g(r^2)$ has a continuous second derivative w.r.t. r^2.

iv) There exists constants, c_1 and c_2 such that

a) $|g'(r^2)| \le c_1 r^{-2} g(r^2)$.

b) $|g''(r^2)| \le c_2 r^{-4} g(r^2)$.

v) $g(r^2) > 0$.

vi) $\sup_{\{y:|y|\le\frac{1}{2} r^2\}} g(y + r^2) \le Kg(r^2)$.

vii) There exists $q > 0$ such that

$$r^{-q} = o(1)g(r^2) .$$

viii) For some constant, c_3,

$$g(r^2) \le c_3 r^{(4-p)} .$$

Then δ is admissible.

Note that Assumption viii) is the conjectured boundary of admissibility as discussed in the Introduction. Note that if $p < 4$, Assumption viii) must be limited by Assumption i). Assumptions iii) and iv) are required for the approximation of the generalized Bayes rule. These conditions essentially require $g(r^2)$ to be flat for large r^2. In particular, sharp tailed priors such as exponentially decreasing priors are eliminated from consideration. However, such priors are proper and therefore yield admissible rules. Finally, Assumptions vi) and vii) are technical, but not very restrictive.

4. Generalized Bayes Rules and Inadmissibility

We begin by presenting an asymptotic approximation of a generalized Bayes rule in terms of the corresponding prior kernel g. For technical reasons it is convenient to consider the function ϕ^* defined by

$$\phi^*(r) = r\phi(r) - 1. \tag{4.1}$$

Theroem 4.1. *Assume that* δ *and* g *are defined as in Theorem 3.2. If* g *satisfies Assumptions* i) *through* vii) *of Theorem 3.2, then*

$$\phi^*(r) = -r^{-2}\{1 + 2r^2 \frac{g'(r^2)}{g(r^2)} + o(1)\}. \tag{4.2}$$

Note that the implied approximation of δ for $|x|$ sufficiently large is

$$\delta(x) = \{1 - [|x|^{-2} + 2 \frac{g'(|x|^2)}{g(|x|^2)}] + o(|x|^{-2})\}|x|^{-2}x . \tag{4.3}$$

Before proceeding with the proof of this theorem, additional notation and preliminary lemmas are given. The first lemma is straightforward computation.

Lemma 4.1. (Berger and Zaman (1979)).

i) $\int\theta_1 f(\theta,r)d\theta = K_1 r$.

ii) $\int\theta_1\{|\theta|^2-r^2\}f(\theta,r)d\theta = K_1(p+2)r$.

iii) $\int \theta_1^2 f(\theta,r)d\theta = K_1(1 + r^2)$.

iv) $\int \theta_1^2\{|\theta|^2 - r^2\}f(\theta,r)d\theta = K_1(K + (p + 4)r^2)$.

Define the quantity N* by

$$N^*(r) = \int (r\theta_1 - \theta_1^2)f(\theta,r)g(|\theta|^2)d\theta \ . \tag{4.4}$$

Then by definition, $\phi^* = N^*D^{-1}$. Also, define the set $\mathcal{G}$ by

$$\mathcal{G} = \{\theta: \ ||\theta|^2 - r^2| \le \tfrac{1}{2} r^2\} \ . \tag{4.5}$$

Note that $\theta \in \mathcal{G}$ if and only if

$$\tfrac{1}{2} r^2 \le |\theta|^2 \le \tfrac{3}{2} r^2 \ . \tag{4.6}$$

Arguments of the type given in the next lemma are used repeatedly below.

Lemma 4.2. For any nonnegative integer M

$$\int_{\mathcal{G}^c} (r\theta_1 - \theta_1^2)f(\theta,r)g(|\theta|^2)d\theta \le Kr^{-2M}(1 + r^{M+1}) \ . \tag{4.7}$$

Proof: Let I denote the integral appearing in (4.7). Note that for $\theta \in \mathcal{G}^c$, $||\theta|^2 - r^2|^M \ge Kr^{2M}$. Hence, a simple Chebyshev argument implies that

$$I \le Kr^{-2M} \int_{\mathcal{G}^c} (r\theta_1 - \theta_1^2)||\theta|^2 - r^2|^M f(\theta,r)g(|\theta|^2)d\theta \ . \tag{4.8}$$

Since g is bounded, it is clear that

$$I \le KBr^{-2M}\int |r\theta_1 - \theta_1^2|\,||\theta|^2 - r^2|^M f(\theta,r)d\theta \ . \tag{4.9}$$

A simple computation now yields the result. ||

Proof of Theorem 4.1: The first step is to approximate N*. Recall that

$$N^* = \int_{\mathcal{G}} (r\theta_1-\theta_1^2)f(\theta,r)g(|\theta|^2)d\theta + \int_{\mathcal{G}^c} (r\theta_1-\theta_1^2)f(\theta,r)g(|\theta|^2)d\theta \ . \tag{4.10}$$

Call these two integrals I_1 and I_2 respectively. I_2 can be bounded by applying Lemma 4.2. In particular, choose M to be equal to $q+1$. Then for r large (i.e., $r > 1$), it is clear that

(4.11) $$I_2 \leq Kr^{-q} .$$

Next, consider I_1. By Taylor's Theorem, g can be written as

(4.12) $$g(|\theta|^2) = g(r^2) + (|\theta|^2 - r^2)g'(r^2) + \tfrac{1}{2}(|\theta|^2 - r^2)^2 g''(r_0^2) ,$$

where r_0^2 is some point contained in the interval $[\frac{1}{2} r^2, \frac{3}{2} r^2]$. Now, substitute this expression for g into I_1 and integrate term by term. Denote the resulting three integrals I_a, I_b, and I_c, respectively. Rewrite I_a as

(4.13) $$I_a = g(r^2)\{\int (r\theta_1 - \theta_1^2) f(\theta,r)d\theta - \int_{G^c} (r\theta_1 - \theta_1^2) f(\theta,r)d\theta\} .$$

The first integral is computed using Lemma 4.1; the second is bounded as above. Therefore, I_a is given by

(4.14) $$I_a = -K_1 g(r^2) + BKr^{-q} .$$

Essentially the same argument, ignoring lower order terms, implies that

(4.15) $$I_b = -K_1 2r^2 g'(r^2) .$$

Finally, consider I_c. Clearly,

(4.16) $$I_c \leq \tfrac{1}{2} \int_G |r\theta_1 - \theta_1^2| (|\theta|^2 - r^2)^2 |g''(r_0^2)| f(\theta,r)d\theta .$$

From Assumptions iv) and vi), it follows that $|g''(r_0^2)| \leq Kr^{-4} g(r^2)$, and hence we have that

(4.17) $$I_c \leq Kr^{-4} g(r^2) \int |r\theta_1 - \theta_1^2| (|\theta|^2 - r^2)^2 f(\theta,r)d\theta .$$

Therefore, as in Lemma 4.2, (4.17) reduces to

(4.18) $$I_c \le K' r^{-1} g(r^2) \ .$$

Then Assumption vii) and lines (4.11), (4.14), (4.15), and (4.18) yield

(4.19) $$N^* = -K_1 g(r^2)\{1 + 2r^2 \frac{g'(r^2)}{g(r^2)} + o(1)\} \ .$$

By arguments similar to those above, it can be shown that

(4.20) $$D = K_1 r^2 g(r^2)\{1 + o(1)\} \ .$$

A standard manipulation then gives the desired result. ||

The final topic of this section is inadmissibility. Employing powerful theoretical techniques, Berger and Zaman (1979) have obtained the following theorems. Suppose δ is of the form (1.2) with

(4.21) $$\phi(r) = r^{-1}(1 - cr^{-2} + o(r^{-2}))$$

where c is a constant. Then δ is inadmissible if $c > 5 - p$. Then applying methods similar to those used in this section, they obtained the following result. If δ is generalized Bayes w.r.t. the prior π given by

$$d\pi(\theta) = |\theta|^{c-1} d\theta$$

where $c > 1 - p$, then δ is of the form (1.2) and (4.21) and, hence, is inadmissible when $c > 5 - p$.

Based on Theorem 4.1 and the results of Berger and Zaman (1979), we have the following generalization.

<u>Theorem 4.2</u>. <u>Assume</u> δ <u>and</u> g <u>satisfy the assumptions of Theorem 4.1. Furthermore, suppose that</u>

$$2r^2 \frac{g'(r^2)}{g(r^2)} = c - 1 + o(1)$$

<u>as</u> $r \to \infty$. <u>If</u> $c > 5 - p$, <u>then</u> δ <u>is inadmissible</u>.

<u>Proof</u>: Obvious. ||

It should be noted that Theorem 4.2 is not the most general result possible based on the theoretical results of Berger and Zaman (1979) and our approximation. However, it includes most cases of genuine interest and is quite simple to present.

5. Admissibility Proof

Recall the definitions of ϕ, N, D, etc. Let the analogous quantities corresponding to δ_n be denoted ϕ_n, N_n, D_n, etc.

The proof of Theorem 3.2 is a verification of the conditions of Theorem 3.1. Clearly the functions $h_n(|\theta|^2)$ defined in (3.3) and (3.4) satisfy Conditions i) and ii). To verify Condition iii), let

$$\mathcal{E}_n = \int [R(\delta,\theta) - R(\delta_n,\theta)] g(|\theta|^2) h_n(|\theta|^2) d\theta \ . \tag{5.1}$$

By formula (2.6), suppressing the dependence of ϕ and ϕ_n on r, $\mathcal{E}_n$ can be written as

$$\mathcal{E}_n = K_o \int [\int_0^\infty r^{(p-r)} \{(\phi\theta_1 - 1)^2 - (\phi_n\theta_1 - 1)^2\} f(\theta,r) g(|\theta|^2) h_n(|\theta|^2) dr] d\theta \ . \tag{5.2}$$

The interchange of order of integration (by Fubini's Theorem, since $\pi_n(\theta)$ is a proper prior) and simplification reduce (5.2) to

$$\mathcal{E}_n = K_o \int_0^\infty r^{(p-1)} (\phi - \phi_n)^2 D_n dr \ . \tag{5.3}$$

The quantity $\mathcal{E}_n$ is partitioned into the following three integrals (ignoring K_o):

$$\mathcal{E}_n^1 = \int_0^{(\ln \ln n^2)^{\frac{1}{2}}} r^{(p-1)} (\phi - \phi_n)^2 D_n dr \ , \tag{5.4}$$

(5.5) $$\varepsilon_n^2 = \int_{(\ln \ln n^2)^{\frac{1}{2}}}^{n-n^{8/9}} r^{(p-1)}(\phi-\phi_n)^2 D_n dr ,$$

and

(5.6) $$\varepsilon_n^3 = \int_{n-n^{8/9}}^{\infty} r^{(p-1)}[\int\{(\phi\theta_1-1)^2 - (\phi_n\theta_1-1)^2\} f(\theta,r)g(|\theta|^2)h_n(|\theta|^2)d\theta]dr .$$

The proof is completed by showing that these quantities all vanish as $n \to \infty$.

Proposition 5.1. $\lim_{n\to\infty} \varepsilon_n^1 = 0$.

Proof: Define the sets A, B, and C by

$$A = \{\theta: 0 \leq |\theta|^2 \leq 1\} ,$$

$$B = \{\theta: 1 \leq |\theta|^2 \leq n^2\} ,$$

and $$C = \{\theta: |\theta|^2 \geq n^2\} .$$

Also, let

(5.7) $$\lambda_n = \int_B \theta_1 f(\theta,r)g(|\theta|^2)(1-H_n(|\theta|^2))d\theta + \int_C \theta_1 f(\theta,r)g(|\theta|^2)d\theta$$

and

(5.8) $$\gamma_n = \int_B \theta_1^2 f(\theta,r)g(|\theta|^2)(1-H_n(|\theta|^2))d\theta + \int_C \theta_1^2 f(\theta,r)g(|\theta|^2)d\theta .$$

In the Appendix it is shown that

(5.9) $$\phi - \phi_n = D^{-1}(\lambda_n-\phi_n\gamma_n) .$$

Hence, ε_n^1 can be written as

$$(5.10) \qquad \varepsilon_n^1 = \int_0^{(\ln \ln n^2)^{\frac{1}{2}}} r^{(p-1)}(\lambda_n^2+\phi_n^2\gamma_n^2-2\lambda_n\phi_n\gamma_n)D^{-2}D_n dr \ .$$

Define

$$(5.11) \qquad I_n^1 = \int_0^{(\ln \ln n^2)^{\frac{1}{2}}} r^{(p-1)}\lambda_n^2 D^{-1} dr$$

and

$$(5.12) \qquad I_n^2 = \int_0^{(\ln \ln n^2)^{\frac{1}{2}}} r^{(p-1)}\phi_n^2\gamma_n^2 D^{-1} dr .$$

Since $h_n(|\theta|^2) \leq 1$, it follows that $D_n D^{-1} \leq 1$. Then the desired result clearly holds if both I_n^1 and I_n^2 tend to zero as $n \to \infty$. We will show that this is the case for I_n^2. The proof for I_n^1 is essentially the same.

Recall that the priors $\pi_n(\theta)$ are proper. Hence, the estimators δ_n are admissible. Expression (2.5) then implies that $\phi_n^2 \leq r^2$, $\forall_n=1,2,\ldots$. Hence, it suffices to show that

$$(5.13) \qquad I_n^3 = \int_0^{(\ln \ln n^2)^{\frac{1}{2}}} r^{(p+1)}\gamma_n^2 D^{-1} dr \to 0 \ .$$

Next, consider γ_n. Since $1 - H_n(|\theta|^2) \leq 1$, it follows that

$$(5.14) \qquad \gamma_n \leq \int_{1 \leq |\theta|^2 \leq \ln n^2} \theta_1^2 f(\theta,r)g(|\theta|^2)(1-H_n(|\theta|^2))d\theta + \int_{\ln n^2 \leq |\theta|^2} \theta_1^2 f(\theta,r)g(|\theta|^2)d\theta \ .$$

Call these two integrals J_1 and J_2, respectively. Since $1 - H_n(|\theta|^2)$ is non-decreasing in $|\theta|^2$, J_1 is bounded by

$$(5.15) \qquad J_1 \leq (1-H_n(\ln n^2)) \int_{1 \leq |\theta|^2 \leq \ln n^2} \theta_1^2 f(\theta,r)g(|\theta|^2)d\theta$$

and, hence, by Lemma 4.1,

(5.16) $$J_1 \leq K(1-H_n(\ell n\ n^2))(1+r^2)\ .$$

For sufficiently large n, it can be shown that

$$1 - H_n(\ell n\ n^2) \leq K\ \frac{\ell n\ \ell n\ n^2}{\ell n\ n^2} \leq K'(\ell n\ n^2)^{-3/4}\ .$$

Hence, we have established that

(5.17) $$J_1 \leq K(1+r^2)(\ell n\ n^2)^{-3/4}\ .$$

To bound J_2, note that on the region of integration of J_2, $|\theta|^2 \geq \ell n\ n^2$. A Chebyshev argument as in the proof of Lemma 4.2 yields the inequality

(5.18) $$J_2 \leq K(1+r^4)(\ell n\ n^2)^{-1}\ .$$

The next step is to bound D. By definition, D is given by

(5.19) $$D = \exp\{-\tfrac{1}{2}r^2\} \int \theta_1^2 \exp\{-\tfrac{1}{2}|\theta|^2\} \exp(\theta_1 r) g(|\theta|^2) d\theta.$$

First, note that since $0 \leq r \leq (\ell n\ \ell n\ n^2)^{\frac{1}{2}}$, it follows that

(5.20) $$\exp(-\tfrac{1}{2}r^2) \geq (\ell n\ n^2)^{-\frac{1}{2}}\ .$$

Let J_3 denote the integral in (5.19). A straightforward argument (see Lemma A.1) demonstrates that

(5.21) $$J_3 \geq L_{T,p}$$

where $L_{T,p}$ is some constant greater than zero. Therefore, (5.20) and (5.21) imply that

(5.22) $$D^{-1} \leq (L_{T,p})^{-1}(\ell n\ n^2)^{\frac{1}{2}}\ .$$

Finally, expressions (5.17), (5.18) and (5.22) imply that

$$(5.23) \qquad I_n^3 \le K \int_0^{(\ell n\,\ell n\, n^2)^{-\frac{1}{2}}} r^{(p+1)} (\ell n\, n^2)^{\frac{1}{2}} \{(1+r^2)(\ell n\, n^2)^{-3/4} + (1+r^4)(\ell n\, n^2)^{-1}\}^2 dr\ .$$

It is clear by inspection of (5.23) that it suffices to show that

$$(5.24) \qquad (\ell n\, n^2)^{-1} \int_0^{(\ell n\,\ell n\, n^2)^{\frac{1}{2}}} r^i dr \to 0$$

where i is any fixed, non-negative integer. Simple integration in (5.24) yields

$$K(\ell n\, n^2)^{-1}(\ell n\,\ell n\, n^2)^{\frac{i+1}{2}} \to 0$$

as was to be shown. ||

Next, consider ε_n^2. The heart of the proof for ε_n^2 is the approximation of both ϕ^* and ϕ_n^*. The arguments used for ϕ_n^* are essentially the same, though more delicate, as those used in Theorem 4.1. The key result is given in the following proposition. The proof is given in the Appendix.

<u>Proposition 5.2.</u> <u>Assume</u> $(\ell n\,\ell n\, n^2)^{\frac{1}{2}} \le r \le n - n^{8/9}$. <u>Then for</u> n <u>sufficiently large</u>

$$(5.25) \qquad \phi^*(r) - \phi_n^*(r) = K\, \frac{H_n'(r^2)}{H_n(r^2)} + o(1).$$

It is necessary to present some notation. Define the function s by

$$(5.26) \qquad s = s(y^2) = \ell n\, n^2 - \ell n\, y^2\ .$$

Note that then

$$H_n(y^2) = [s(y^2)]^{17}[\ell n\, n^2]^{-17}\ .$$

The following information concerning the derivatives of H_n is needed.

Lemma 5.1. *If* $1 < y^2 < n^2$, *then*

i) $|H_n'(y^2)| = 17[s(y^2)]^{16}[y^2(\ln n^2)^{17}]^{-1}$

and ii) $|H_n''(y^2)| \leq K[s(y^2)]^{15}[1+s(y^2)][y^4(\ln n^2)^{17}]^{-1}$.

Proof: Simple computation. ||

Proposition 5.3. $\lim_{n\to\infty} \varepsilon_n^2 = 0.$

Proof: First, note that by definition

$$\phi(r) - \phi_n(r) = r^{-1}(\phi^*(r) - \phi_n^*(r)) \ .$$

Applying Proposition 5.2 and an approximation of D_n (equation (A.10)) obtained in the proof of that proposition, it is sufficient to show that

$$\varepsilon_n^4 = \int_{(\ln\ln n^2)^{\frac{1}{2}}}^{n-n^{8/9}} r^{(p-1)}[r^{-1}\frac{H_n'(r^2)}{H_n(r^2)}]^2[r^2g(r^2)H_n(r^2)]dr \tag{5.27}$$

vanishes as $n \to \infty$. Lemma 5.1 and simplification yield

$$\varepsilon_n^4 = K\int_{(\ln\ln n^2)^{\frac{1}{2}}}^{n-n^{8/9}} r^{(p-5)}[s(r^2)]^{15}[\ln n^2]^{-17}g(r^2)dr \ . \tag{5.28}$$

Now, since $[s(r^2)]^{15}[\ln n^2]^{-15} \leq 1$ and, by Assumption viii), $g(r^2) \leq C_3r^{4-p}$, we have that (for large n)

$$\varepsilon_n^4 \leq K[\ln n]^{-2}\int_{(\ln\ln n^2)^{\frac{1}{2}}}^{n} r^{-1}dr$$

$$= K[\ln n]^{-2}[\ln n - \tfrac{1}{2}\ln\ln\ln n^2]$$

$$\leq K'[\ln n]^{-1} \to 0. \quad ||$$

<u>Proposition 5.4</u>. $\lim_{n\to\infty} \varepsilon_n^3 = 0$.

<u>Proof</u>: Note that

$$(5.29)\quad \varepsilon_n^3 = \int_{n-n^{8/9}}^{\infty} r^{(p-1)}[\int(\phi\theta_1 - 1)^2 f(\theta,r)g(|\theta|^2)h_n(|\theta|^2)d\theta]dr$$

$$- \int_{n-n^{8/9}}^{\infty} r^{(p-1)}[\int(\phi_n\theta_1 - 1)^2 f(\theta,r)g(|\theta|^2)h_n(|\theta|^2)d\theta]dr \,.$$

Call these two integrals ε_n^5 and ε_n^6 respectively.

First, consider ε_n^6. Since δ_n is Bayes w.r.t. $\pi_n(\theta)$, ϕ_n minimizes the inner integral of ε_n^6. Then, clearly,

$$(5.30)\quad \varepsilon_n^6 \le \int_{n-n^{8/9}}^{\infty} r^{(p-1)}[\int(r^{-1}\theta_1-1)^2 f(\theta,r)g(|\theta|^2)h_n(|\theta|^2)d\theta]dr.$$

Define the sets $\Gamma_1 = \{\theta\colon 0 \le |\theta| \le n - 2n^{8/9}\}$ and $\Gamma_2 = \{\theta\colon n - 2n^{8/9} < |\theta| \le n\}$. Since $h_n(|\theta|^2) = 0$ for $|\theta| > n$, it is clear that

$$(5.31)\quad \varepsilon_n^6 \le \int_{n-n^{8/9}}^{\infty} r^{(p-3)}[\int_{\Gamma_1} (\theta_1 - r)^2 f(\theta,r)g(|\theta|^2)h_n(|\theta|^2)d\theta]dr$$

$$+ \int_{n-n^{8/9}}^{\infty} r^{(p-3)}[\int_{\Gamma_2} (\theta_1 - r)^2 f(\theta,r)g(|\theta|^2)h_n(|\theta|^2)d\theta]dr \,.$$

Call these integrals J_4 and J_5 respectively. Consider J_4. Since $r \ge n - n^{8/9}$, if $\theta \in \Gamma_1$, then $r - \theta_1 \ge n^{8/9}$. Hence, a Chebyshev argument and the fact that $h_n(|\theta|)^2 \le 1$ imply that

$$(5.32)\quad J_4 \le n^{-4(8/9)} \int_{n-n^{8/9}}^{\infty} r^{(p-3)}[\int_{\Gamma_1} (\theta_1 - r)^6 f(\theta,r)g(|\theta|^2)d\theta]dr.$$

Case 1: p=1,2. Recall that g is bounded. Interchange of orger of integration in (5.32) yields

(5.33) $$J_4 \leq Bn^{-4(8/9)}\int_{\Gamma_1} [\int_{-\infty}^{\infty} (\theta_1 - r)^6 f(\theta,r)dr]d\theta$$

$$\leq Kn^{-4(8/9)}(n - 2n^{8/9})^2 \to 0.$$

Case 2: p > 2. Clearly, we have

(5.34) $$J_4 \leq n^{-4(8/9)} \int_{\Gamma_1} [\int_{-\infty}^{\infty} |r|^{(p-3)}(\theta_1 - r)^6 f(\theta,r)dr]g(|\theta|^2)d\theta$$

$$\leq Kn^{-4(8/9)}\int_{\Gamma_1} (1 + |\theta_1|^{(p-3)})g(|\theta|^2)\exp\{-\tfrac{1}{2}|\theta^*|^2\}d\theta .$$

For sufficiently large n, partition Γ_1 into the sets $\Gamma_3 = \{\theta: 0 \leq |\theta| \leq (T+1)^{\frac{1}{2}}\}$ and $\Gamma_4 = \{\theta: (T+1)^{\frac{1}{2}} < |\theta| \leq n - 2n^{8/9}\}$. It is clear that

(5.35) $$n^{-4(8/9)} \int_{\Gamma_3} (1 + |\theta_1|^{(p-3)})g(|\theta|^2)\exp\{-\tfrac{1}{2}|\theta^*|^2\}d\theta$$

$$= n^{-4(8/9)}K \to 0.$$

Next, Assumption viii) implies that

(5.36) $$n^{-4(8/9)} \int_{\Gamma_4} (1 + |\theta_1|^{(p-3)})g(|\theta|^2)\exp\{-\tfrac{1}{2}|\theta^*|^2\}d\theta$$

$$\leq n^{-4(8/9)} \int_{\Gamma_4} |\theta_1|^{(p-3)}g(|\theta|^2)\exp\{-\tfrac{1}{2}|\theta^*|^2\}d\theta$$

$$\leq Kn^{-4(8/9)}\int_{\Gamma_4} |\theta_1|^{(p-3)}|\theta|^{(4-p)}\exp\{-\tfrac{1}{2}|\theta^*|^2\}d\theta$$

$$\leq K'n^{-4(8/9)} \int_0^n ydy$$

$$= K''n^{-4(8/9)}n^2 \to 0.$$

Now, consider J_5. For n large and $\theta \in \Gamma_2$, $h_n(|\theta|^2) = H_n(|\theta|^2)$. Since H_n is non-increasing, if $\theta \in \Gamma_2$, then $H_n(|\theta|^2 \leq H_n[(n-2n^{8/9})^2]$. Furthermore, it is easy to see that

$$H_n[(n - 2n^{8/9})^2] = [\ell n\,(1 - 2n^{-1/9})^{-1}]^{-17}[\ell n\; n]^{-17}$$

$$= [\ell n\, n]^{-17}[(1 + o(1))2n^{-1/9}]^{17}$$

$$\leq K[\ell n\, n]^{-17} n^{-17/9} \;.$$

Hence, it can be concluded that

(5.37) $$J_5 \leq K[n^{1/9}\ell n\; n]^{-17} \int_{n-n^{8/9}}^{\infty} r^{(p-3)}$$

$$[\int_{\Gamma_2} (\theta_1 - r)^2 f(\theta,r) g(|\theta|^2) d\theta] dr.$$

<u>Case 1: p=1,2</u>. The argument here is similar to that used for J_4.

<u>Case 2: p > 2</u>. Again employing Assumption viii), (5.37) reduces to

(5.38) $$J_5 \leq K[n^{1/9}\ell n\; n]^{-17} \int_{n-2n^{8/9}}^{n} y dy$$

$$= K'[n^{1/9}\ell n\; n]^{-17}[n^2 - n^2(1 - 2n^{-1/9})^2]$$

$$= K(\ell n\, n)^{-17} n^{-17/9+2-1/9}(1 - n^{-2/9})$$

$$\leq K(\ell n\, n)^{-17} \to 0.$$

Finally, consider $\mathcal{E}_n^5$. By Theorem 4.1, ϕ can be approximated by

$$\phi(r) = r^{-1}(1 + o(1))$$

for large n and, hence, large r. Some simple algebra then implies that

(5.39) $$(\phi\theta_1 - 1)^2 \leq Kr^{-2}[(\theta_1 - r)^2 + |\theta_1 - r| + 1].$$

The same arguments used for ε_n^6 are now used to show that ε_n^5 tends to zero as $n \to \infty$. ||

APPENDIX

Verification of (5.9): Recall that $\phi_n = N_n D_n^{-1}$. Since $A \cup B \cup C = \mathbb{R}^p$, N_n is given by

$$N_n = \int \theta_1 f(\theta,r) g(|\theta|^2) d\theta$$

$$- [\int_B \theta_1 f(\theta,r) g(|\theta|^2)(1 - H_n(|\theta|^2)) d\theta$$

$$+ \int_C \theta_1 f(\theta,r) g(|\theta|^2) d\theta] \ .$$

By the definitions of N and λ_n, N_n can be written as

$$N_n = N - \lambda_n.$$

Similarly, D_n is given by

$$D_n = D - \gamma_n.$$

Now consider the following algebraic manipulation:

$$\phi_n = \frac{N-\lambda_n}{D-\gamma_n} = \frac{N-\lambda_n}{D} [1 - \frac{\gamma_n}{D}]^{-1}$$

$$= \frac{N-\lambda_n}{D} [1 + \frac{\gamma_n}{D-\gamma_n}]$$

$$= \frac{N}{D} - \frac{\lambda_n}{D} + \frac{\gamma_n}{D} [\frac{N-\lambda_n}{D-\gamma_n}]$$

$$= \phi - \lambda_n D^{-1} + \phi_n \gamma_n D^{-1}. \quad ||$$

Lemma A.1. $J_3 \geq L_{T,p} > 0$.

Proof: Let $a_1 = p^{-1}T + 1$ and $a_2 = a_1 + 1$, where T is as given in Theorem 3.2. Define the set $\mathcal{D}$ by

$$\mathcal{D} = \{\theta | \sqrt{a_1} \leq \theta_i \leq \sqrt{a_2}, \ \forall_i = 1,\ldots,p\} \ .$$

First, note that $\mathcal{D}$ is a non-empty, compact subset of Θ. Furthermore, if $\theta \in \mathcal{D}$, then i) $\exp(\theta_1 r) \geq 1$ and ii) by Assumption v), $g(|\theta|^2) > 0$.

Therefore, it is clear that

$$J_3 \geq \int_{\mathcal{D}} \theta_1^2 \exp\{-\tfrac{1}{2}|\theta|^2\} g(|\theta|^2) d\theta .$$

Since the integrand on the R.H.S. of this expression is a bounded, positive, continuous function of θ on $\mathcal{D}$ and $\mathcal{D}$ is compact, the desired conclusion follows. ||

The following lemma is needed in the proof of Proposition 5.2. The proof of this lemma is identical to that given in Berger (1976b), Lemma 3.2.8.

Lemma A.2. Assume that $(\ln \ln n^2)^{\frac{1}{2}} \leq r \leq n - n^{8/9}$. Then uniformly in r,

i) $\lim_{n\to\infty} [H_n(r^2)r^2]^{-1} = 0$

and ii) $\lim_{n\to\infty} [s(r^2)r]^{-1} = 0.$

Proof of Proposition 5.2: The key to the proof is the approximation of $\phi_n^*(r)$. First define the sets $\mathcal{G}_1$ and $\mathcal{G}_2$ by

$$\mathcal{G}_1 = \{\theta: ||\theta|^2 - r^2| \leq \tfrac{1}{2} r^2 \text{ and } |\theta|^2 \leq n^2\}$$

and

$$\mathcal{G}_2 = \{\theta: ||\theta|^2 - r^2| > \tfrac{1}{2} r^2 \text{ and } |\theta|^2 \leq n^2\} .$$

Since $h_n(|\theta|^2) = 0$ if $|\theta|^2 > n^2$, it is clear that

$$\text{(A.1)} \qquad N_n^* = \int_{\mathcal{G}_1} (r\theta_1 - \theta_1^2) f(\theta,r) g(|\theta|^2) h_n(|\theta|^2) d\theta$$
$$+ \int_{\mathcal{G}_2} (r\theta_1 - \theta_1^2) f(\theta,r) g(|\theta|^2) h_n(|\theta|^2) d\theta .$$

Call these two integrals I_1 and I_2 respectively. First, consider I_2. By noting that $h_n(|\theta|^2) \leq 1$ and then applying Lemma 4.1 with $M = q+3$, I_2 can be bounded by

$$I_2 \leq Kr^{-(q+2)} \tag{A.2}$$

for n (and hence r) sufficiently large. Since Lemma A.2,i) implies that $[H_n(r^2)r^2]^{-1} \to 0$ uniformly in r,(A.2)can be written as

$$I_2 \leq K'r^{-q}H_n(r^2) \tag{A.3}$$

for sufficiently large n.

Next, consider I_1. Again, by Taylor's Theorem

(A.4)

$$\begin{aligned} g(|\theta|^2)H_n(|\theta|^2) = \{&g(r^2)H_n(r^2) \\ &+ (|\theta|^2-r^2)g'(r^2)H_n(r^2) + \tfrac{1}{2}(|\theta|^2-r^2)^2 g''(r_1^2)H_n(r_1^2)\} \\ &+ \{(|\theta|^2-r^2)g(r^2)H_n'(r^2)\} + \{(|\theta|^2-r^2)^2[g'(r_1^2)H_n'(r_1^2) \\ &+ \tfrac{1}{2}g(r_1^2)H_n''(r_1^2)]\} \end{aligned}$$

where $r_1^2 \in [\frac{1}{2}r^2, \frac{3}{2}r^2]$.

Let $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ denote the quantities in brackets ({}) in(A.4). We now substitute $\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3$ into I_1 and integrate. Consider $\mathcal{B}_1$. Since H_n is monotone, for all $r_1^2 \in [\frac{1}{2}r^2, \frac{3}{2}r^2]$

$$H_n(r_1^2) \leq H_n(\tfrac{1}{2}r^2) \leq KH_n(r^2).$$

Given this fact, $\mathcal{B}_1$ is treated exactly as in Theorem 4.1 yielding

$$\int_{\mathcal{G}_1} (\theta_1 r - \theta_1^2)f(\theta,r)\mathcal{B}_1 d\theta = -K_1 g(r^2)H_n(r^2)\{1 + 2\frac{g'(r^2)}{g(r^2)} + o(1)\}. \tag{A.5}$$

Note that I_2 has been absorbed by (A.5) by the result of (A.3) and Assumption vii). Note that the bound in (A.5) is equal to $H_n(r^2)$ times the corresponding bound in Theorem 4.1.

Now consider $\mathcal{B}_2$. First, as above and as in Theorem 4.1, ignoring lower order terms, the integral of $\mathcal{B}_2$ is given by

$$\text{(A.6)} \quad g(r^2)H_n'(r^2) \int_{\mathcal{A}_1} (r\theta_1-\theta_1^2)(|\theta|^2-r^2)f(\theta,r)d\theta \approx -K_1 2r^2 q(r^2)H_n'(r^2).$$

Finally, consider $\mathcal{B}_3$. Let

$$I_3 = \int_{\mathcal{A}_1} (r\theta_1 - \theta_1^2)f(\theta,r)\mathcal{B}_3 d\theta .$$

Then, it is clear that

$$\text{(A.7)} \qquad I_3 \le \int |r\theta_1 - \theta_1^2|(|\theta|^2 - r^2)^2 f(\theta,r) K[|g'(r_1^2)||H_n'(r_1^2)| + g(r_1^2)|H_n''(r_1^2)|]d\theta.$$

Applying Assumptions iv) and vi), and Lemma 5.1, we obtain

$$I_3 \le Kr^{-4}g(r^2)[(s(r^2))^{15}(1+s(r^2))(\ell n\, n^2)^{-17}] \int |r\theta_1 - \theta_1^2|(|\theta|^2 - r^2)^2 f(\theta,r)d\theta,$$

and so

$$\text{(A.8)} \qquad I_3 \le K'r^{-1}g(r^2)[(s(r^2))^{15}(1+s(r^2))(\ell n\, n^2)^{-17}] .$$

Combining the results of (A.5), (A.6), and (A.8), the desired approximation for N_n^* is

$$\text{(A.9)} \qquad N_n^* = -K_1 g(r^2)H_n(r^2)\{2r^2 \frac{H_n'(r^2)}{H_n(r^2)} + [1 + 2\frac{g'(r^2)}{g(r^2)} + o(1)] + [Kr^{-1}(s(r^2))^{-1}(1+(s(r^2))^{-1}]\} .$$

The next step is to approximate D_n. By the same arguments as above, but using one less term in the Taylor series representation, it can be shown that

(A.10) $$D_n = K_1 r^2 g(r^2) H_n(r^2)\{1 + K(rs(r^2))^{-1} + o(1)\} .$$

Finally, since $(rs(r^2))^{-1} \to 0$ uniformly in r, a simple manipulation yields

$$\phi_n^*(r) = -r^{-2}\{1 + 2\left[\frac{g'(r^2)}{g(r^2)}\right] + o(1)\} - 2\left[\frac{H_n'(r^2)}{H_n(r^2)}\right] + o(1).$$

Now, recalling the result for $\phi^*(r)$ in Theorem 4.1, the proof is complete. ||

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