INADMISSIBILITY IN THE CONTROL PROBLEM

by

James O. Berger
Purdue University

and

Asad Zaman
University of Pennsylvania and Stanford University

Department of Statistics
Division of Mathematical Sciences
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Summary

Let $\underset{\sim}{X} = (X_1, \ldots, X_p)^t$ be an observation from a p-variate normal distribution with unknown mean $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_p)^t$ and identity covariance matrix. We consider a control problem which, in canonical form, is the problem of estimating $\underset{\sim}{\theta}$ under the loss

$$L(\underset{\sim}{\theta},\underset{\sim}{\delta}) = (\underset{\sim}{\theta}^t \underset{\sim}{\delta}-1)^2 ,$$

where $\underset{\sim}{\delta}(\underset{\sim}{x}) = (\delta_1(\underset{\sim}{x}), \ldots, \delta_p(\underset{\sim}{x}))^t$ is the estimate of $\underset{\sim}{\theta}$ for a given $\underset{\sim}{x}$. A general theorem is given for establishing inadmissibility of estimators in this problem. As an application, it is shown that estimators of the form

$$\underset{\sim}{\delta}(\underset{\sim}{x}) = (|\underset{\sim}{x}|^2+c)^{-1}\underset{\sim}{x} + |\underset{\sim}{x}|^{-4} w(|\underset{\sim}{x}|)\underset{\sim}{x} ,$$

where $w(|\underset{\sim}{x}|)$ is $o(1)$ as $|\underset{\sim}{x}| \to \infty$, are inadmissible if $c > 5 - p$.

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1. Introduction

The control problem basically deals with a situation in which it is desired to choose the levels of certain factors in a system so that the "output" of the system is at the desired control level. The system could be an economic system, a production system, or a biological system. (As an example of the latter, it might be desired to achieve and maintain certain hormonal levels or certain chemical concentrations in a patient.)

Zaman (1980) considers a standard normal model of the control problem, in which the output, z, occurs as

$$z = \underset{\sim}{\theta}^t \underset{\sim}{y} + \varepsilon ,$$

where $\underset{\sim}{\theta}$ is a p-vector of unknown coefficients of the system, ε is a normally distributed error, and $\underset{\sim}{y}$ is a p-vector of nonstochastic control variables to be chosen so as to achieve some desired output z^*. Suppose that the loss in achieving output z is $(z-z^*)^2$, and that an estimate $\underset{\sim}{\delta}(\underset{\sim}{x}) = (\delta_1(\underset{\sim}{x}), \ldots, \delta_p(\underset{\sim}{x}))^t$ of $\underset{\sim}{\theta}$ is available, from, say, past data, $\underset{\sim}{x}$, on the system. Zaman (1980) (see also Basu (1974) for a very general development) then shows that the problem can be reduced (with suitable redefinitions of variables) to the following problem. Suppose $\underset{\sim}{X} = (X_1, X_2, \ldots, X_p)^t$ is a p-variate normal random variable with unknown mean $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_p)^t$ and identity covariance matrix, and that it is desired to estimate $\underset{\sim}{\theta}$ under loss

$$L(\underset{\sim}{\theta},\underset{\sim}{\delta}) = (\underset{\sim}{\theta}^t \underset{\sim}{\delta}-1)^2 .$$

The estimator $\underset{\sim}{\delta}$ is allowed to assume any value in R^p, but the parameter space is restricted to be $\Theta = R^p - \{0\}$. Zero is excluded from the parameter space, because $\underset{\sim}{\theta} = \underset{\sim}{0}$ would correspond to a control system in which the inputs have no effect. (From a decision theoretic viewpoint, it is necessary to exclude zero, to prevent every estimator from being a Bayes estimator with respect to the prior distribution which puts mass one at zero.)

As usual, an estimator will be evaluated in terms of its risk function $R(\underset{\sim}{\theta},\underset{\sim}{\delta})$, which is simply the expected loss $E_{\underset{\sim}{\theta}}[L(\underset{\sim}{\theta}, \underset{\sim}{\delta}(\underset{\sim}{X}))]$. A decision rule $\underset{\sim}{\delta}$ is <u>inadmissible</u> if there exists another decision rule $\underset{\sim}{\delta}^*$ with $R(\underset{\sim}{\theta},\underset{\sim}{\delta}^*) \leq R(\underset{\sim}{\theta},\underset{\sim}{\delta})$ with strict inequality for some $\underset{\sim}{\theta}$. Otherwise, $\underset{\sim}{\delta}$ is <u>admissible</u>.

In this paper, attention will be restricted to nonrandomized spherically symmetric estimators. It will be shown in Section 2 that the nonrandomized estimators form a complete class (i.e., any randomized estimator can be improved upon by some nonrandomized estimator), so the restriction to nonrandomized estimators is without loss of generality. The restriction to spherically symmetric estimators is quite natural for this problem. Results for nonsymmetric estimators appear to be very difficult to obtain. Note that, since the problem is invariant under the orthogonal group (which is compact), admissibility within the class of invariant (i.e., spherically symmetric) estimators implies overall admissibility.

It will prove convenient to write a spherically symmetric estimator $\underset{\sim}{\delta}$ as

(1.1) $$\underset{\sim}{\delta}(\underset{\sim}{x}) = \phi(|\underset{\sim}{x}|)|\underset{\sim}{x}|^{-1}\underset{\sim}{x} ,$$

where $|\underset{\sim}{x}|^2 = \Sigma_{i=1}^{p} x_i^2$. For rules of this form, we develop, in Section 2, a very useful representation for the Bayes risk of an estimator. Using this representation, and the technique for proving inadmissibility developed in Berger (1979), we derive in Section 3 conditions under which an estimator of the form (1.1) is inadmissible. For example, it is shown that an estimator of the form

(1.2) $$\underset{\sim}{\delta}(\underset{\sim}{x}) = (|\underset{\sim}{x}|^2+c)^{-1}\underset{\sim}{x} + |\underset{\sim}{x}|^{-4}\, w(|\underset{\sim}{x}|)\underset{\sim}{x} ,$$

where $w(|\underset{\sim}{x}|) = o(1)$ (as $|\underset{\sim}{x}| \to \infty$), is inadmissible if $c > 5 - p$. This class of estimators contains virtually all estimators that have been proposed for the control problem. Previous inadmissibility results have dealt only with the case $w(|\underset{\sim}{x}|) \equiv 0$, and have established inadmissibility only for $c > 0$ and $p \geq 6$ (Kei Takeuchi) and for $c = 1$ and $p = 5$ (Stein and Zaman (1980)). It is also shown that for a generalized prior of the form $\pi(d\theta) = |\theta|^{(c-1)}d\theta$, the generalized Bayes rule is of the form (1.2). (This was shown for $c = 1$, i.e., the uniform prior, in Zellner (1971).) This suggests that generalized priors with tails flatter than $|\theta|^{(4-p)}$ have inadmissible generalized Bayes rules.

2. Preliminaries.

We first establish, as promised, that the nonrandomized estimators form a complete class.

Theorem 1. If $\delta^*(\underset{\sim}{x},\cdot)$ is a randomized estimator (i.e., $\delta^*(\underset{\sim}{x},A)$ is the probability of choosing an estimate in $A \subset \mathcal{A} = R^p$), then the estimator

$$\underset{\sim}{\delta}(\underset{\sim}{x}) = \int_{\mathcal{A}} \underset{\sim}{a}\, \delta^*(\underset{\sim}{x},d\underset{\sim}{a})$$

dominates δ^* in terms of risk, unless δ^* is degenerate (i.e., equivalent to a nonrandomized estimator).

Proof: It is easy to check that the loss function, $L(\underset{\sim}{\theta},\underset{\sim}{a})$, is convex in $\underset{\sim}{a}$, so that, by Jensen's inequality,

$$(2.1) \qquad L(\underset{\sim}{\theta}, \delta^*(\underset{\sim}{x},\cdot)) \equiv \int_{\mathcal{A}} L(\underset{\sim}{\theta},\underset{\sim}{a})\, \delta^*(\underset{\sim}{x},d\underset{\sim}{a}) \geq L(\underset{\sim}{\theta}, \underset{\sim}{\delta}(\underset{\sim}{x})) .$$

Hence

$$R(\underset{\sim}{\theta},\delta^*) = E_{\underset{\sim}{\theta}}^{\underset{\sim}{X}}\, L(\underset{\sim}{\theta}, \delta^*(\underset{\sim}{X},\cdot)) \geq E_{\underset{\sim}{\theta}}^{\underset{\sim}{X}}\, L(\underset{\sim}{\theta}, \underset{\sim}{\delta}(\underset{\sim}{X})) = R(\underset{\sim}{\theta}, \underset{\sim}{\delta}(\underset{\sim}{X})) .$$

For a given $\underset{\sim}{\theta}$, equality holds above only if, for a set of $\underset{\sim}{x}$ which has probability one, $\delta^*(\underset{\sim}{x},\cdot)$ gives probability one to a hyperplane of the form

$$(2.2) \qquad H(\underset{\sim}{x}) = \{\underset{\sim}{a} : \underset{\sim}{\theta}^t\, \underset{\sim}{a} = c(\underset{\sim}{x})\} .$$

(Jensen's inequality gives strict inequality in (2.1) unless, with probability one, $L(\underset{\sim}{\theta},\underset{\sim}{a})$ is a linear function of $\underset{\sim}{a}$. This can occur only if $\underset{\sim}{\theta}^t \underset{\sim}{a}$ is a constant.) Of course, δ^*, and hence $H(\underset{\sim}{x})$, cannot depend upon $\underset{\sim}{\theta}$. Choosing $\underset{\sim}{\theta}$ along each coordinate axis in turn allows one to conclude, from (2.2), that $H(\underset{\sim}{x})$ must be a point for $\underset{\sim}{x}$ in a set of probability one. This means that $\delta^*(\underset{\sim}{x},\cdot)$ is degenerate with probability one, completing the proof.||

In Zaman (1980) it is shown that, if an estimator of the form (1.1) is admissible, then, for some probability measure μ on $\Gamma = [0,\infty)$,

$$\phi(v) = \frac{\int_0^\infty \gamma^{-1} \sinh(\gamma v)\ \mu(d\gamma)}{\int_0^\infty \cosh(\gamma v)\ \mu(d\gamma)}, \tag{2.3}$$

where $[\gamma^{-1} \sinh(\gamma v)]$ is defined to be v when $\gamma = 0$. Since $[y \sinh(y)] \le \cosh(y)$, $\lim_{v\to 0} \cosh(\gamma v) = 1$, and $\lim_{v\to 0}[\gamma^{-1} \sinh(\gamma v)]/v = 1$, it is easy to see from (2.3) that, if δ is admissible, then

$$0 \le \phi(v)/v \le 1 \text{ and } \lim_{v\to 0}[\phi(v)/v] = 1 . \tag{2.4}$$

Let Θ^* be the class of spherically symmetric σ-finite measures on Θ. (Thus if $\pi \in \Theta^*$, $A \subset \Theta$, and $\mathfrak{G}$ is a $p \times p$ orthogonal matrix, then $\pi(A) = \pi(\mathfrak{G}A)$.) Define

$$r(\pi,\delta) = \int_\Theta R(\underset{\sim}{\theta},\underset{\sim}{\delta})\ \pi(d\underset{\sim}{\theta}) .$$

When π is a probability measure (or more generally a finite measure) this is called the Bayes risk of $\underset{\sim}{\delta}$. A very useful formula for the Bayes risk of a spherically symmetric estimator is given in the following theorem. For use in this theorem, define the measure $\tilde{\pi}$ on $(0,\infty)$ by

$$(2.5) \qquad \tilde{\pi}(A) = \int_A \int_{R^{p-1}} S_p (2\pi)^{-p/2} \exp\{-\tfrac{1}{2}|\underset{\sim}{\theta}|^2\} \, \pi(d\underset{\sim}{\theta}) \ ,$$

where S_p is the surface area of the unit p-sphere. (Thus $\tilde{\pi}$ is the marginal distribution of the first component of the measure $S_p (2\pi)^{-p/2} \exp\{-\frac{1}{2}|\underset{\sim}{\theta}|^2\} \, \pi(d\underset{\sim}{\theta})$.)

<u>Theorem 2</u>. Suppose $\pi \in \Theta^*$ is a finite measure, and $\underset{\sim}{\delta}$ is an estimator of the form (1.1) with $\phi(r)$ being continuous and piecewise differentiable on $[0,\infty)$ and satisfying

(i) $0 \le \phi(r)/r \le K_0 < \infty$; and

(ii) $\lim_{r \to 0} [r^{(p-1)} \phi(r)^2] = 0$.

Then

$$(2.6) \qquad r(\pi,\underset{\sim}{\delta}) = \pi(\Theta) + 2 \int_0^\infty \int_0^\infty g(r) \, \gamma \sinh(\gamma r) dr \, \tilde{\pi}(d\gamma) \ ,$$

where

$$(2.7) \quad g(r) = r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\} \phi(r) \, [-2 + \{r - (p-1)r^{-1}\} \phi(r) - 2\phi'(r)] \ .$$

<u>Proof</u>: Clearly

$$(2.8)\quad r(\pi,\underset{\sim}{\delta}) = \int_{\Theta}\int_{\mathfrak{X}} (\underset{\sim}{\theta}^t\ \underset{\sim}{\delta}(\underset{\sim}{x})-1)^2(2\pi)^{-p/2}\exp\{-\tfrac{1}{2}|\underset{\sim}{x}-\underset{\sim}{\theta}|^2\}d\underset{\sim}{x}\ \pi(d\underset{\sim}{\theta})$$

$$= \int_{\mathfrak{X}}\int_{\Theta} (\phi(|\underset{\sim}{x}|)|\underset{\sim}{x}|^{-1}\ \underset{\sim}{\theta}^t\ \underset{\sim}{x}-1)^2(2\pi)^{-p/2}\exp\{-\tfrac{1}{2}|\underset{\sim}{x}-\underset{\sim}{\theta}|^2\}\pi(d\underset{\sim}{\theta})d\underset{\sim}{x}\ .$$

Defining $\underset{\sim}{y} = \Theta^{-1}\ \underset{\sim}{x}$ and $\underset{\sim}{\eta} = \Theta^{-1}\ \underset{\sim}{\theta}$, where Θ is a $p \times p$ orthogonal matrix, the spherical symmetry of π implies that

$$T(\underset{\sim}{x}) = \int(\phi(|\underset{\sim}{x}|)|\underset{\sim}{x}|^{-1}\ \underset{\sim}{\theta}^t\ \underset{\sim}{x}-1)^2(2\pi)^{-p/2}\exp\{-\tfrac{1}{2}|\underset{\sim}{x}-\underset{\sim}{\theta}|^2\}\ \pi(d\underset{\sim}{\theta})$$

$$= \int(\phi(|\underset{\sim}{y}|)|\underset{\sim}{y}|^{-1}\ \underset{\sim}{\eta}^t\ \underset{\sim}{y}-1)^2(2\pi)^{-p/2}\exp\{-\tfrac{1}{2}|\underset{\sim}{y}-\underset{\sim}{\eta}|^2\}\ \pi(d\underset{\sim}{\eta}) = T(\underset{\sim}{y})\ .$$

Since this is true for all orthogonal Θ, it is clear that $T(\underset{\sim}{x})$ is itself spherically symmetric. Hence

$$T(\underset{\sim}{x}) = T((|\underset{\sim}{x}|, 0, \ldots, 0)^t)\ ,$$

which, when used in (2.8), gives that

$$(2.9)\quad r(\pi,\underset{\sim}{\delta}) = \int_{R^p} T((|\underset{\sim}{x}|, 0, \ldots, 0)^t)d\underset{\sim}{x}$$

$$= \int_{R^p}\int_{\Theta} (\phi(|\underset{\sim}{x}|)|\underset{\sim}{x}|^{-1}\ \theta_1|\underset{\sim}{x}|-1)^2(2\pi)^{-p/2}\exp\{-\tfrac{1}{2}|\underset{\sim}{x}|^2\}$$

$$\times \exp\{|\underset{\sim}{x}|\theta_1\}\exp\{-\tfrac{1}{2}|\underset{\sim}{\theta}|^2\}\ \pi(d\underset{\sim}{\theta})d\underset{\sim}{x}$$

$$= \int_0^\infty\int_{\Theta}(\phi(r)\theta_1-1)^2\ r^{(p-1)}\ S_p(2\pi)^{-p/2}\exp\{-\tfrac{1}{2}\ r^2\}\exp\{r\theta_1\}$$

$$\times \exp\{-\tfrac{1}{2}|\underset{\sim}{\theta}|^2\}\ \pi(d\underset{\sim}{\theta})dr\ ,$$

the last step following from the change of variables $r = |\underset{\sim}{x}|$. Using (2.5) in this last expression, it can be concluded that

$$(2.10)\quad r(\pi,\underset{\sim}{\delta}) = \int_{-\infty}^{\infty}\int_{0}^{\infty} r^{(p-1)}\{\phi(r)\theta_1-1\}^2 \exp\{-\tfrac{1}{2}r^2\} \exp\{r\,\theta_1\}dr\,\tilde{\pi}(d\theta_1) .$$

We now need the following lemma.

Lemma 1. Suppose $h(r)$ is a continuous and piecewise differentiable function on $(0,\infty)$, which satisfies, for all $\theta_1 \in R^1$,

(i) $\int_0^\infty |h(r)| \exp\{r\,\theta_1\}dr < \infty$;

(ii) $\lim_{r\to 0} h(r) = 0$; and

(iii) $\lim_{r\to\infty} [h(r) \exp\{r\,\theta_1\}] = 0$.

Then

$$(2.11)\quad \int_0^\infty h(r)\,\theta_1 \exp\{r\,\theta_1\}dr = -\int_0^\infty h'(r) \exp\{r\,\theta_1\}dr ,$$

where h' denotes the first derivative of h .

The proof of this lemma is a simple integration by parts, and will be omitted. Setting

$$h(r) = r^{(p-1)}\,\phi(r)^2 \exp\{-\tfrac{1}{2}\,r^2\} ,$$

it is easy to check that the conditions of Theorem 2 imply that $h(r)$ satisfies the conditions of Lemma 1. Hence, letting $\phi'(r) = \frac{d}{dr}\phi(r)$, (2.11) implies that

$$\int_0^\infty r^{(p-1)} \phi(r)^2 \exp\{-\tfrac{1}{2} r^2\}\theta_1 \exp\{r\ \theta_1\}dr$$

$$= \int_0^\infty r^{(p-1)} \phi(r) \exp\{-\tfrac{1}{2} r^2\} [-(p-1)r^{-1} \phi(r) - 2\phi'(r)+r\phi(r)] \exp\{r\ \theta_1\}dr .$$

Expanding $\{\phi(r)\theta_1-1\}^2$ in (2.10), and using the above result, shows that

$$(2.12)\quad r(\pi,\underset{\sim}{\delta}) = \int_{-\infty}^\infty \int_0^\infty r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\}\{1 - 2\ \theta_1\ \phi(r) + \theta_1\ \phi(r)$$

$$\times [-(p-1)r^{-1} \phi(r) - 2\phi'(r) + r\phi(r)]\} \exp\{r\ \theta_1\}dr\ \tilde{\pi}(d\ \theta_1) .$$

Now, if $\phi(r) \equiv 0$, then clearly

$$r(\pi,\underset{\sim}{\delta}) = \int_\Theta (1)\ \pi(d\underset{\sim}{\theta}) = \pi(\Theta) .$$

Setting $\phi(r) \equiv 0$ in (2.12), this gives that

$$\int_{-\infty}^\infty \int_0^\infty r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\} \exp\{r\ \theta_1\}dr\ \tilde{\pi}(d\ \theta_1) = \pi(\Theta) .$$

Using this and (2.7) in (2.12) shows that

$$(2.13)\qquad r(\pi,\underset{\sim}{\delta}) = \pi(\Theta) + \int_{-\infty}^\infty \int_0^\infty \theta_1\ g(r) \exp\{r\ \theta_1\}dr\ \tilde{\pi}(d\ \theta_1) .$$

Now, since zero was excluded from the parameter space and π is symmetric, $\tilde{\pi}$ cannot give positive mass to $\theta_1 = 0$. Also, $\tilde{\pi}$ is symmetric, so that (2.13) can be written

$$r(\pi,\underset{\sim}{\delta}) = \pi(\Theta) + \int_{-\infty}^{0}\int_{0}^{\infty}\theta_1\ g(r)\ \exp\{r\theta_1\}dr\ \tilde{\pi}(d\theta_1) + \int_{0}^{\infty}\int_{0}^{\infty}\theta_1\ g(r)\ \exp\{r\theta_1\}dr\ \tilde{\pi}(d\theta_1)$$

$$= \pi(\Theta) + \int_{0}^{\infty}\int_{0}^{\infty} g(r)[\gamma\ \exp\{r\gamma\} - \gamma\ \exp\{-r\gamma\}]dr\ \tilde{\pi}(d\gamma)$$

$$= \pi(\Theta) + 2\int_{0}^{\infty}\int_{0}^{\infty} g(r)\ \gamma\ \sinh(\gamma r)dr\ \tilde{\pi}(d\gamma)\ ,$$

completing the proof.||

3. Inadmissibility.

To prove that an estimator $\underset{\sim}{\delta}^0$ is inadmissible, we will make use of the technique developed in Berger (1979). (The heuristic basis of this technique was given in Brown (1979). Brown (1980) is also closely related.) The technique is basically to find an estimator $\underset{\sim}{\delta}^*$ which has smaller risk than $\underset{\sim}{\delta}^0$ for large $|\underset{\sim}{\theta}|$, and then to argue that this leads to a violation of Stein's necessary condition for admissibility (Stein (1955)).

For use in the analysis, define, for any estimator $\underset{\sim}{\delta}$ which satisfies the conditions of Theorem 2, the function, on $\Gamma = [0,\infty)$,

$$R^*(\gamma,\underset{\sim}{\delta}) = 2\int_{0}^{\infty} g(r)\ \gamma\ \sinh(\gamma r)dr\ ,$$

where $g(r)$ is defined in (2.7). Furthermore, if $\underset{\sim}{\delta}^0$ and $\underset{\sim}{\delta}^*$ both satisfy the conditions of Theorem 2, define

$$\Delta(\gamma) = R^*(\gamma,\underset{\sim}{\delta}^0) - R^*(\gamma,\underset{\sim}{\delta}^*) \tag{3.1}$$

$$= 2\int_{0}^{\infty} r^{(p-1)}\ \exp\{-\tfrac{1}{2}r^2\}\ \Delta^*(r)\ \gamma\ \sinh(\gamma r)dr\ ,$$

where

$$(3.2)\quad \Delta^*(r) = -2[\phi^0(r) - \phi^*(r)] + [r - (p-1)r^{-1}][\phi^0(r)^2 - \phi^*(r)^2]$$

$$+ 2[\phi^*(r)\,\phi^{*\prime}(r) - 2\,\phi^0(r)\,\phi^{0\prime}(r)]\,.$$

Note that, for any finite measure $\pi \in \Theta^*$ for which either $r(\pi,\underset{\sim}{\delta}^0)$ or $r(\pi,\underset{\sim}{\delta}^*)$ is finite, it follows from Theorem 2 that

$$(3.3)\quad r(\pi,\underset{\sim}{\delta}^0) - r(\pi,\underset{\sim}{\delta}^*) = \int_0^\infty \Delta(\gamma)\,\tilde{\pi}(d\gamma)$$

$$= 2\int_0^\infty \int_0^\infty r^{(p-1)} \exp\{-\tfrac{1}{2}r^2\}\,\Delta^*(r)\,\gamma\sinh(\gamma r)dr\,\tilde{\pi}(d\gamma)\;.$$

<u>Theorem 3</u>. Suppose that $\underset{\sim}{\delta}^0$ is of the form (1.1), with $\phi^0(r)$ being continuous and piecewise differentiable, and that $\underset{\sim}{\delta}^*$ is another estimator of the form (1.1) with ϕ^* satisfying

(i) $\phi^*(r)$ is continuous and piecewise differentiable;

(ii) there exists a constant $K_1 \geq 0$ such that $\phi^*(r) = \phi^0(r)$ for $r \leq K_1$;

(iii) $0 \leq \phi^*(r)/r \leq K_0 < \infty$ for all $r \in (0,\infty)$;

(iv) there exist $\varepsilon > 0$, $\alpha > 0$, and $0 < K_2 < \infty$ such that $\Delta^*(r) \geq \varepsilon\, r^{-\alpha}$ for $r > K_2$; and

(v) $\displaystyle\int_{K_1}^\infty r^{(p-1)} \exp\{-\tfrac{1}{2}r^2\}\,\Delta^*(r)\,\exp\{\psi(r)\}dr > 0$,

where

$$\psi(r) = \int_{K_1/2}^{r} [\phi^0(v)]^{-1}\, dv \ .$$

Then $\underset{\sim}{\delta}^0$ is inadmissible.

Proof: The proof will be by contradiction. Thus assume that $\underset{\sim}{\delta}^0$ is admissible, and is hence of the form (2.3), with ϕ^0 satisfying (2.4). It is then clear that the conditions of Theorem 2 are satisfied by $\underset{\sim}{\delta}^0$. Furthermore, conditions (i), (ii), and (iii) of this theorem show that $\underset{\sim}{\delta}^*$ satisfies the conditions of Theorem 2.

Now let $\mathcal{D}$ be the class of all spherically symmetric estimators such that $0 \le \phi(|\underset{\sim}{x}|)/|\underset{\sim}{x}| \le 1$. From Zaman (1980), it follows that $\mathcal{D}$ is a complete class of estimators (i.e., any estimator not in this class is dominated, in terms of risk, by an estimator in this class). The problem has now been put in the framework of Berger (1979), and so, to prove inadmissibility of $\underset{\sim}{\delta}^0$, it is only necessary to verify that the conditions of the theorem in Berger (1979) are satisfied. These conditions are that there exists a sequence $\{\pi_n\}$ of finite measures in Θ^*, with corresponding Bayes rules $\underset{\sim}{\delta}^n$ such that $r(\pi_n, \underset{\sim}{\delta}^n) < \infty$, and a nonnegative function $h(\gamma)$, which is strictly positive on the interior of Γ, such that

(a) $\tilde{\pi}_n(C) \ge 1$ $(n = 1, 2, \ldots)$, for some compact set C in the interior of Γ;

(b) $\lim_{n\to\infty} \int_{\Theta} [R(\underset{\sim}{\theta}, \underset{\sim}{\delta}^0) - R(\underset{\sim}{\theta}, \underset{\sim}{\delta}^n)]\, \pi_n(d\underset{\sim}{\theta}) = 0$;

(c) the measures $\mu_n(d\gamma) = h(\gamma)\, \tilde{\pi}_n(d\gamma) / \int_{\Gamma} h(\gamma)\, \tilde{\pi}_n(d\gamma)$ converge weakly to a probability measure μ on Γ; and

(d) the function $g(\gamma) = [h(\gamma)]^{-1}\, \Delta(\gamma)$ (see (3.1)) is continuous on Γ and is positive outside some compact set $B \subset \Gamma$;

The theorem in Berger (1979) states that, if these conditions hold, then

$$\int_\Gamma g(\gamma)\,\mu(d\gamma) \le 0 \ . \tag{3.4}$$

The existence of finite measures $\pi_n \in \Theta^*$ which satisfy conditions (a) and (b) follows from Stein's necessary condition for admissibility (Stein (1955)). The verification that this necessary condition applies, and that the π_n can be chosen to be in Θ^*, is given in the Appendix. (That the π_n have finite Bayes risks is obvious for this problem.)

Zaman (1980) shows that if $h(\gamma) = \gamma^2$, then the measures

$$\mu_n(d\gamma) = \frac{\gamma^2\,\tilde{\pi}_n(d\gamma)}{\int_0^\infty \gamma^2\,\tilde{\pi}_n(d\gamma)}$$

are probability measures on Γ, and that a subsequence of these measures converges weakly to a probability measure μ on Γ. Since this subsequence still satisfies conditions (a) and (b), we can assume that (a), (b), and (c) all hold.

<u>Verification of condition (d)</u>.

It is clear from (3.1) that $\Delta(\gamma)$, and hence $g(\gamma) = \gamma^{-2}\Delta(\gamma)$, is continuous on $(0,\infty)$. Also, the function $\gamma^{-1}\sinh(\gamma r)$ is, for all $r > 0$, monotonically increasing in γ. (This can be checked by differentiating with respect to γ, and using the inequality $y\cosh(y) \ge \sinh(y)$, for $y > 0$). Hence, defining

$$f(r) = 2r^{(p-1)}\exp\{-\tfrac{1}{2}r^2\}\,\Delta^*(r)$$

and

$$\beta = \int_0^\infty f(r) r \, dr$$

(which is easily seen to be finite for $\underset{\sim}{\delta}^0$ and $\underset{\sim}{\delta}^*$ satisfying the conditions of Theorem 2), it follows, from (3.1), the monotone convergence theorem, and the fact that $\sinh(y) > y$ for $y > 0$, that

$$\begin{aligned} \lim_{\gamma \to 0} |\gamma^{-2} \Delta(\gamma) - \beta| &\leq \lim_{\gamma \to 0} \int_0^\infty |f(r)| (\gamma^{-1} \sinh(\gamma r) - r) dr \\ &= \int_0^\infty |f(r)| \lim_{\gamma \to 0} (\gamma^{-1} \sinh(\gamma r) - r) dr = 0 \ . \end{aligned}$$

Hence $g(\gamma)$ is continuous at $\gamma = 0$ if we define $g(0) = \beta$.

The remaining part of condition (d) is verified in the following lemma.

<u>Lemma 2</u>. There exists a $K_3 < \infty$ such that $g(\gamma) > 0$ for $\gamma > K_3$. (Thus $g(\gamma)$ is positive outside the compact set $B = [0, K_3)$.)

<u>Proof</u>: From (3.1), it clearly suffices to prove that, for $\gamma > K_3$,

$$\text{(3.5)} \qquad I \equiv \int_0^\infty r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\} \Delta^*(r) \gamma \sinh(\gamma r) dr > 0 \ .$$

Note first that, from condition (iv) of the theorem,

$$I_1 \equiv \int_{K_2}^\infty r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\} \Delta^*(r) \gamma \sinh(\gamma r) dr \geq \varepsilon \int_{K_2}^\infty r^{(p-1-\alpha)} \exp\{-\tfrac{1}{2} r^2\} \times \gamma \sinh(\gamma r) dr \ .$$

Hence, for large enough γ , say $\gamma > K_4$,

$$(3.6) \qquad I_1 \geq \frac{\varepsilon}{3} \int_{K_2}^{\infty} r^{(p-1-\alpha)} \exp\{-\tfrac{1}{2} r^2\} \gamma \exp\{\gamma r\} dr$$

$$= \frac{\varepsilon}{3} \gamma \exp\{\tfrac{1}{2} \gamma^2\} \int_{K_2}^{\infty} r^{(p-1-\alpha)} \exp\{-\tfrac{1}{2}(r-\gamma)^2\} dr$$

$$\geq \frac{\varepsilon}{4} \gamma \exp\{\tfrac{1}{2} \gamma^2\} \gamma^{(p-1-\alpha)} = \frac{\varepsilon}{4} \exp\{\tfrac{1}{2} \gamma^2\} \gamma^{p-\alpha} ,$$

the last inequality following from a standard Taylors series argument.

An integration by parts, as in the proof of Theorem 2, shows that

$$(3.7) \qquad I_2 = \int_0^{K_2} r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\} \Delta^*(r) \gamma \sinh(\gamma r) dr$$

$$= -K_2^{(p-1)} [\phi^0(K_2)^2 - \phi^*(K_2)^2] \exp\{-\tfrac{1}{2} K_2^2\} \gamma \sinh(\gamma K_2)$$

$$+ \int_0^{K_2} r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\} [\phi^0(r)^2 - \phi^*(r)^2] \gamma^2 \cosh(\gamma r) dr$$

$$- \int_0^{K_2} 2[\phi^0(r) - \phi^*(r)] r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\} \gamma \sinh(\gamma r) dr .$$

Applying condition (iii) of the theorem, and recalling that $0 \leq \phi^0(r)/r \leq 1$, it follows that

$$|I_2| \leq (K_0^2 + 1) K_2^{(p+1)} \exp\{-\tfrac{1}{2} K_2^2\} \gamma \exp\{K_2 \gamma\}$$

$$+ (K_0^2 + 1) \int_0^{K_2} r^{(p+1)} \exp\{-\tfrac{1}{2} r^2\} \gamma^2 \exp\{\gamma r\} dr$$

$$+ (K_0 + 1) \int_0^{K_2} r^p \exp\{-\tfrac{1}{2} r^2\} \gamma \exp\{\gamma r\} dr$$

$$\leq (K_0^2 + 2)(K_2 + 1)^{p+1} \exp\{\tfrac{1}{2} \gamma^2\}\gamma \Big[\exp\{- \tfrac{1}{2}(K_2-\gamma)^2\} + (\gamma+1) \int_0^{K_2} \exp\{- \tfrac{1}{2}(r-\gamma)^2\}dr\Big] .$$

Since, for $\gamma > K_2$,

$$\int_0^{K_2} \exp\{- \tfrac{1}{2}(r-\gamma)^2\}dr \leq K_2 \exp\{- \tfrac{1}{2}(K_2-\gamma)^2\} ,$$

it can be concluded that, for $\gamma > K_2$,

$$(3.8) \qquad |I_2| \leq (K_0^2+2)(K_2+1)^{p+2} \exp\{\tfrac{1}{2} \gamma^2\} \gamma(\gamma+1) \exp\{- \tfrac{1}{2} (K_2-\gamma)^2\} .$$

Choosing $K_3(> \max\{K_2,K_4\})$ so that,

$$\exp\{- \tfrac{1}{2}(K_2-K_3)^2\} \leq \tfrac{\varepsilon}{8} \gamma^{(p-\alpha-1)}(\gamma+1)^{-1}(K_0^2+2)^{-1}(K_2+1)^{-(p+2)} ,$$

it follows from (3.5), (3.6), and (3.8) that, for $\gamma > K_3$,

$$I = I_1 + I_2 \geq \tfrac{\varepsilon}{8} \exp\{\tfrac{1}{2} \gamma^2\} \gamma^{(p-\alpha)} > 0 ,$$

completing the proof of the lemma.||

We have thus verified all the conditions of the theorem in Berger (1979), and so can conclude that (3.4) must be satisfied. Note, however, from (3.1), that

$$\int_\Gamma g(\gamma)\ \mu(d\gamma) = \int_0^\infty \gamma^{-2}\ \Delta(\gamma)\ \mu(d\gamma)$$

$$= 2\int_0^\infty \int_0^\infty r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\}\ \Delta^*(r)\ \gamma^{-1} \sinh(\gamma r) dr\ d\mu(\gamma)\ .$$

Since $\phi^*(r) = \phi^0(r)$ for $r \le K_1$ (condition (ii) of the theorem), it is clear from (3.2) that

$$\int_\Gamma g(\gamma)\ \mu(d\gamma) = 2\int_0^\infty \int_{K_1}^\infty r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\}\ \Delta^*(r)\ \gamma^{-1} \sinh(\gamma r) dr\ d\mu(\gamma)\ .$$

Because $\Delta^*(r)$ is positive for $r > K_2$, orders of integration can be interchanged above to give

$$(3.9)\quad \int_\Gamma g(\gamma)\ \mu(d\gamma) = 2\int_{K_1}^\infty r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\}\ \Delta^*(r) \int_0^\infty \gamma^{-1} \sinh(\gamma r) d\mu(\gamma) dr\ .$$

Since we are assuming that $\underset{\sim}{\delta}^0$ is admissible, ϕ^0 must satisfy (2.3), which can be rewritten

$$\phi^0(v) = \left[\frac{d}{dv} \log \int_0^\infty \gamma^{-1} \sinh(\gamma v) d\ \mu(\gamma)\right]^{-1}\ .$$

Hence, for $r > K_1/2$,

$$\psi(r) = \int_{K_1/2}^{r} [\phi^0(v)]^{-1} dv = \log\left[\int_0^\infty \gamma^{-1} \sinh(\gamma r) d\ \mu(\gamma)\right] - \rho\ ,$$

where

$$\rho = \int_0^\infty \gamma^{-1} \sinh(K_1\ \gamma/2) d\ \mu(\gamma)\ .$$

It follows that

$$\int_0^\infty \gamma^{-1} \sinh(\gamma r) d\,\mu(\gamma) = \exp\{\psi(r) + \rho\} \quad ,$$

which, when used in (3.9), gives

$$\int_\Gamma g(\gamma)\ \mu(d\gamma) = 2\ e^\rho \int_{K_1}^\infty r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\}\ \Delta^*(r)\ \exp\{\psi(r)\} dr \ .$$

By condition (v) of the theorem, this is positive, contradicting (3.4). The conclusion is that $\underset{\sim}{\delta}^0$ cannot be admissible, completing the proof of the theorem.||

4. Applications.

As mentioned in the Introduction, virtually all estimators studied have been of the form (1.2). We can obtain the following inadmissibility result for estimators of this form.

Theorem 4. Assume that $\underset{\sim}{\delta}^0$ is of the form (1.1) with

$$\phi^0(r) = \frac{r}{r^2 + c} - \frac{w(r)}{r^3} \ , \tag{4.1}$$

where $w(r) = o(1)$ (as $r \to \infty$). Then $\underset{\sim}{\delta}^0$ is inadmissible if $c > 5 - p$.

Proof: For convenience, define $\varepsilon = c - (5\text{-}p)$ and $\beta = -\varepsilon^2/8$. Assume that $c > 5 - p$, so that $\varepsilon > 0$. Define, for $K_1 > 0$ (K_1 to be chosen later),

$$\phi^*(r) = \begin{cases} \phi^0(r) & \text{if } r \le K_1 \\ \dfrac{r}{r^2+5-p} - \dfrac{w(r)}{r^3}\left(1 - \dfrac{\varepsilon}{r^2+c} + \dfrac{\beta}{r^2(r^2+c)}\right) & \text{if } r \ge 2K_1 \\ \left(2 - \dfrac{r}{K_1}\right)\phi^0(K_1) + \left(\dfrac{r}{K_1} - 1\right)\phi^*(2K_1) & \text{if } K_1 < r < 2K_1 . \end{cases},$$

Assuming $0 \le \phi^0(r)/r \le 1$ (which if not satisfied makes $\underset{\sim}{\delta}^0$ trivially inadmissible), it is easy to see that conditions (i), (ii), and (iii) of Theorem 3 are satisfied. (Note, from (2.3), that $w(r)$ must be continuously differentiable if ϕ is to be admissible.)

Clearly,

$$\phi^{0'}(r) = \frac{1}{r^2+c} - \frac{2r^2}{(r^2+c)^2} - \frac{w'(r)}{r^3} + \frac{3\,w(r)}{r^4}$$

and

$$\phi^{*'}(r) = \begin{cases} \phi^{0'}(r) & \text{if } r < K_1 \\ \dfrac{1}{K_1}\left[\phi^*(2K_1) - \phi^0(K_4)\right] & \text{if } K_1 < r < 2K_1 \\ \dfrac{1}{r^2+5-p} - \dfrac{2r^2}{(r^2+5-p)^2} - \dfrac{w'(r)}{r^3}\left(1 - \dfrac{\varepsilon}{r^2+c} + \dfrac{\beta}{r^2(r^2+c)}\right) & \\ + w(r)\left(\dfrac{3}{r^4} - \dfrac{\varepsilon(5r^2+3c)}{r^4(r^2+c)^2} + \dfrac{\beta(7r^2+5c)}{r^6(r^2+c)^2}\right) & \text{if } r > 2K_1 . \end{cases}$$

Thus, for $r \ge 2K_1$,

$$\Delta^*(r) = -2[\phi^0(r)-\phi^*(r)]+[r-(p-1)r^{-1}][\phi^0(r)^2-\phi^*(r)^2]+2[\phi^*(r)\phi^{*\prime}(r)-\phi^0(r)\phi^{0\prime}(r)]$$

$$= -2\left\{\frac{r}{r^2+c} - \frac{r}{r^2+5-p} + \frac{w(r)}{r^3}\left(\frac{-\varepsilon}{r^2+c} + \frac{\beta}{r^2(r^2+c)}\right)\right\}$$

$$+ [r-(p-1)r^{-1}]\left\{\frac{r}{r^2+c} - \frac{r}{r^2+5-p} + \frac{w(r)}{r^3}\left(\frac{-\varepsilon}{r^2+c} + \frac{\beta}{r^2(r^2+c)}\right)\right\}$$

$$\times\left\{\frac{r}{r^2+c} + \frac{r}{r^2+5-p} - \frac{w(r)}{r^3}\left(2 - \frac{\varepsilon}{r^2+c} + \frac{\beta}{r^2(r^2+c)}\right)\right\}$$

$$+ 2\left[\frac{r}{r^2+5-p} - \frac{w(r)}{r^3}\left(1 - \frac{\varepsilon}{r^2+c} + \frac{\beta}{r^2(r^2+c)}\right)\right]\left[\frac{1}{r^2+5-p} - \frac{2r^2}{(r^2+5-p)^2}\right.$$

$$- \frac{w'(r)}{r^3}\left(1-\frac{\varepsilon}{r^2+c}+\frac{\beta}{r^2(r^2+c)}\right) + w(r)\left(\frac{3}{r^4} - \frac{\varepsilon(5r^2+3c)}{r^4(r^2+c)^2} + \frac{\beta(7r^2+5c)}{r^6(r^2+c)^2}\right)\Bigg]$$

$$\left.-\left[\frac{r}{r^2+c} - \frac{w(r)}{r^3}\right]\left[\frac{1}{r^2+c} - \frac{2r^2}{(r^2+c)^2} - \frac{w'(r)}{r^3} + \frac{3w(r)}{r^4}\right]\right\} .$$

Ignoring terms which are $o(r^{-5})$ (recall $w = o(1)$), this becomes

$$(4.2)\quad \Delta^*(r) = \frac{-r\,\varepsilon}{(r^2+c)(r^2+5-p)}\left\{-2 + \left(r - \frac{p-1}{r}\right)\frac{[2r^3 + (5-p+c)r]}{(r^2+c)(r^2+5-p)}\right\} - \frac{8\varepsilon}{r^5}$$

$$+ \frac{2w'(r)}{r^3}\left\{\frac{-r\,\varepsilon}{(r^2+c)(r^2+5-p)} + \frac{r}{(r^2+5-p)}\left(\frac{\varepsilon}{r^2+c} - \frac{\beta}{r^2(r^2+c)}\right)\right.$$

$$\left. + \frac{w(r)}{r^3}\left[\left(1 - \frac{\varepsilon}{r^2+c} + \frac{\beta}{r^2(r^2+c)}\right)^2 - 1\right]\right\} + o(r^{-5})$$

$$= \frac{-r\,\varepsilon}{(r^2+c)(r^2+5-p)}\left\{-2 + 2 + \frac{(5-p+c)}{r^2} + 2\left(1 - \frac{c}{r^2}\right)\left(1 - \frac{5-p}{r^2}\right) - \frac{2(p-1)}{r^2}\right\}$$

$$- \frac{8\varepsilon}{r^5} + \frac{2w'(r)}{r^3}\left\{-\frac{\beta}{r^5} + o(r^{-5})\right\} + o(r^{-5})$$

$$= \frac{\varepsilon^2}{r^5} + \frac{w'(r)}{r^8}\left(\frac{\varepsilon^2}{4} + o(1)\right) + o(r^{-5}) .$$

Solving for w in (4.1) gives

$$w(r) = \frac{r^4}{r^2+c} - r^3\,\phi^0(r)\ ,$$

so that

$$w'(r) = \frac{4r^3}{r^2+c} - \frac{2r^5}{(r^2+c)^2} - 3r^2\,\phi^0(r) - r^3\,\phi^{0'}(r)\ . \tag{4.3}$$

Defining

$$h(r) = \int_0^\infty \gamma^{-1}\sinh(\gamma r)d\,\mu(\gamma)\ ,$$

it follows from (2.3) that, if ϕ^0 is admissible, then $\phi^0(r) = h(r)/h'(r)$. This implies that

$$\phi^{0'}(r) = 1 - h(r)\,h''(r)/[h'(r)]^2\ . \tag{4.4}$$

Observing that h and h'' are positive, it follows from (4.3) and (4.4) that

$$w'(r) \geq -3r^2\,\phi^0(r) - r^3\ .$$

From (4.1), it is clear that if K_1 is chosen large enough, then $\phi^0(r) \leq 2/r$ for $r \geq 2K_1$. Thus

$$w'(r) \geq -6r - r^3$$

for $r \geq 2K_1$. From this and (4.2), it can be concluded that, for large enough K_1 and $r \geq 2K_1$,

$$(4.5) \qquad \Delta^*(r) \geq \frac{\varepsilon^2}{r^5} - \frac{\varepsilon^2/4}{r^5} + o(r^{-5}) \geq \frac{\varepsilon^2}{2\,r^5} \quad .$$

Thus condition (iv) of Theorem 3 is satisfied.

To verify condition (v) of Theorem 3, note that for large enough K_1 and $r \geq K_1/2$,

$$\phi^0(r)^{-1} = \left(\frac{r}{r^2+c} - \frac{w(r)}{r^3}\right)^{-1} = r + \frac{c}{r} + \frac{h(r)}{r} ,$$

where $|h(r)| \leq \varepsilon/2$. Hence, for $r \geq K_1/2$,

$$\phi^0(r)^{-1} \geq r + \frac{c - \varepsilon/2}{r} ,$$

so that

$$\psi(r) = \int_{K_1/2}^{r} \phi^0(v)^{-1} dv \geq \frac{1}{2}\left(r^2 - \frac{K_1^2}{4}\right) + (c - \frac{\varepsilon}{2}) \log\left(\frac{r}{K_1/2}\right) \quad .$$

It follows that, for $r \geq K_1/2$,

$$\exp\{\psi(r)\} \geq K_2 \exp\{\tfrac{1}{2} r^2\} r^{c-\varepsilon/2} ,$$

where the constant K_2 depends on K_1 and is positive. Together with (4.5), this implies that

$$\int_{2K_1}^{\infty} r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\} \Delta^*(r) \exp\{\psi(r)\} dr \geq \frac{1}{2} K_2 \varepsilon^2 \int_{2K_1}^{\infty} r^{(p-1-5+c-\varepsilon/2)} dr$$

$$= \frac{1}{2} K_2 \varepsilon^2 \int_{2K_1}^{\infty} r^{-1+\varepsilon/2} dr = \infty \quad .$$

It is easy to check that

$$\int_{K_1}^{2K_1} r^{(p-1)} \exp\{-\tfrac{1}{2} r^2\} \, \Delta^*(r) \exp\{\psi(r)\} dr$$

is finite, so that condition (v) of the theorem is clearly satisfied. Hence $\underset{\sim}{\delta}^0$ is inadmissible.||

The above theorem suggests that the dividing line between admissibility and inadmissibility for an estimator, in terms of the large r behavior of $\phi(r)$, is

$$\phi(r) = r/(r^2 + 5-p) \; . \tag{4.6}$$

An estimator with a flatter ϕ should be inadmissible, while an estimator with a sharper ϕ (and which satisfies (2.3)) should be admissible. (It is shown in Stein and Zaman (1980), for example, that $\phi(r) = r/(r^2+1)$ corresponds to an admissible estimator when $p = 4$.) It is interesting that an estimator which behaves as in (4.6) for large r is "asymptotically efficient", as discussed in Zaman (1977).

A natural class of (generalized) prior distributions, for the control problem, is the class of priors of the form $\pi(d\underset{\sim}{\theta}) = |\underset{\sim}{\theta}|^{c-1} d\underset{\sim}{\theta}$. This includes the uniform prior $(c = 1)$, which has been extensively studied by Zellner (1971) and others. For $c \neq 1$, the generalized Bayes rule is hard to explicitly calculate for this prior. The following theorem shows that the generalized Bayes rule for this prior is of the form (4.1).

Theorem 5. Consider the (generalized) prior density $\pi(d\underset{\sim}{\theta}) = |\underset{\sim}{\theta}|^{c-1} d\underset{\sim}{\theta}$, where $c > 1 - p$ (so that the generalized Bayes rule $\underset{\sim}{\delta}^{\pi}$, with respect to π, exists). Then $\underset{\sim}{\delta}^{\pi}$ is of the form (8.1), and, hence, is inadmissible if $c > 5 - p$.

Proof: In Zaman (1980), it is shown that

$$\underset{\sim}{\delta}^{\pi}(|\underset{\sim}{x}|) = \phi^{\pi}(|\underset{\sim}{x}|)|\underset{\sim}{x}|^{-1}\underset{\sim}{x} ,$$

where

$$(4.7) \qquad \phi^{\pi}(|\underset{\sim}{x}|) = |\underset{\sim}{x}| \Big/ \left(1 + 2\,\frac{F''(|\underset{\sim}{x}|^2)}{F'(|\underset{\sim}{x}|^2)}\,|\underset{\sim}{x}|^2\right) ,$$

where

$$(4.8) \qquad F(|\underset{\sim}{x}|^2) = \int_{-\infty}^{\infty} \exp\{\theta_1 |\underset{\sim}{x}|\}\,\tilde{\pi}(d\theta_1)$$

$$= S_p (2\pi)^{-p/2} \exp\{\tfrac{1}{2}|\underset{\sim}{x}|^2\} \int_{\Theta} \exp\{-\tfrac{1}{2}|\underset{\sim}{x}-\underset{\sim}{\theta}|^2\}\pi(d\underset{\sim}{\theta}) .$$

(See (2.5) for the definition of $\tilde{\pi}$.) Clearly,

$$F'(v) = \frac{d}{dv}\int \exp\{\theta_1\, v^{1/2}\}\tilde{\pi}(d\theta_1)$$

$$= \frac{1}{2}\, v^{-1/2} \int \theta_1 \exp\{\theta_1\, v^{1/2}\}\tilde{\pi}(d\theta_1) ,$$

and

$$F''(v) = -\frac{1}{4} v^{-3/2} \int \theta_1 \exp\{\theta_1 v^{1/2}\}\tilde{\pi}(d\theta_1) + \frac{1}{4} v^{-1} \int \theta_1^2 \exp\{\theta_1 v^{1/2}\}\tilde{\pi}(d\theta_1) .$$

Hence

$$(4.9) \qquad \frac{F''(v)}{F'(v)} = -\frac{1}{2} v^{-1} + \frac{1}{2} v^{-1/2} \frac{\int \theta_1^2 \exp\{\theta_1 v^{1/2}\} \tilde{\pi}(d\theta_1)}{\int \theta_1 \exp\{\theta_1 v^{1/2}\} \tilde{\pi}(d\theta_1)} .$$

As in (4.8),

$$I_1 \equiv S_p^{-1} (2\pi)^{p/2} \int \theta_1 \exp\{\theta_1 |\underset{\sim}{x}|\} \tilde{\pi}(d\theta_1)$$

$$= \exp\{\tfrac{1}{2} |\underset{\sim}{x}|^2\} \int_{\Theta} \theta_1 \exp\{-\tfrac{1}{2}(\theta_1 - |\underset{\sim}{x}|)^2\} \exp\{-\tfrac{1}{2}|\underset{\sim}{\theta}^*|^2\}|\underset{\sim}{\theta}|^{c-1} d\underset{\sim}{\theta} ,$$

where $\underset{\sim}{\theta}^* = (\theta_2, \ldots, \theta_p)^t$. Making the change of variables $y = (\theta_1 - |\underset{\sim}{x}|)$, and then replacing the dummy variable y by θ_1 , results in the expression

$$(4.10) \quad I_1 = \exp\{\tfrac{1}{2}|\underset{\sim}{x}|^2\} \int (\theta_1 + |\underset{\sim}{x}|) \exp\{-\tfrac{1}{2}|\underset{\sim}{\theta}|^2\}(|\underset{\sim}{\theta}|^2 + 2\theta_1|\underset{\sim}{x}| + |\underset{\sim}{x}|^2)^{(c-1)/2} d\underset{\sim}{\theta} .$$

By a standard Chebyshev argument, it can be shown that, as $|\underset{\sim}{x}| \to \infty$,

$$(4.11) \quad I_2 = \int_{|\underset{\sim}{\theta}| > |\underset{\sim}{x}|^{1/4}} (\theta_1 + |\underset{\sim}{x}|) \exp\{-\tfrac{1}{2}|\underset{\sim}{\theta}|^2\}(|\underset{\sim}{\theta}|^2 + 2\theta_1|\underset{\sim}{x}| + |\underset{\sim}{x}|^2)^{(c-1)/2} d\underset{\sim}{\theta}$$

$$= o(|\underset{\sim}{x}|^{c-2}) .$$

For $|\underset{\sim}{\theta}| \leq |\underset{\sim}{x}|^{1/4}$, it is clear that, as $|\underset{\sim}{x}| \to \infty$,

$$(4.12)\quad (|\tilde{\theta}|^2 + 2\theta_1|\tilde{x}| + |\tilde{x}|^2)^{\alpha/2} = |\tilde{x}|^{\alpha}\left(1 + \frac{2\theta_1}{|\tilde{x}|} + \frac{|\tilde{\theta}|^2}{|\tilde{x}|^2}\right)^{\alpha/2}$$

$$= |\tilde{x}|^{\alpha}\left\{1 + \frac{\alpha\theta_1}{|\tilde{x}|} + \frac{\alpha|\tilde{\theta}|^2}{2|\tilde{x}|^2} + \frac{1}{2}(\frac{\alpha}{2})(\frac{\alpha}{2} - 1)\left(\frac{2\theta_1}{|\tilde{x}|}\right)^2 + o(|\tilde{x}|^{-2})\right\}$$

$$= |\tilde{x}|^{\alpha}\left\{1 + \frac{\alpha\theta_1}{|\tilde{x}|} + \frac{\alpha}{2|\tilde{x}|^2}\,[|\tilde{\theta}|^2 + (\alpha-2)\theta_1^2] + o(|\tilde{x}|^{-2})\right\} ,$$

the $o(|\tilde{x}|^{-2})$ term being uniform in $|\tilde{\theta}| \leq |\tilde{x}|^{1/4}$. Hence

$$I_3 \equiv \int_{|\tilde{\theta}|\leq|\tilde{x}|^{1/4}} (\theta_1+|\tilde{x}|)\,\exp\{-\tfrac{1}{2}|\tilde{\theta}|^2\}(|\tilde{\theta}|^2+2\theta_1|\tilde{x}|+|\tilde{x}|^2)^{(c-1)/2}\,d\tilde{\theta}$$

$$= |\tilde{x}|^{c-1} \int_{|\tilde{\theta}|\leq|\tilde{x}|^{1/4}} (\theta_1 + |\tilde{x}|)\,\exp\{-\tfrac{1}{2}|\tilde{\theta}|^2\}\left\{1 + \frac{(c-1)\theta_1}{|\tilde{x}|} + \frac{(c-1)}{2|\tilde{x}|^2}\,[|\tilde{\theta}|^2 + (c-3)\theta_1^2]\right\}d\tilde{\theta} + o(|\tilde{x}|^{c-2}) .$$

A second Chebyshev argument allows one to replace the region of integration in the last integral above by Θ . Calculating the indicated moments (involving θ_1 and $|\tilde{\theta}|$) of a normal distribution, it follows that

$$(4.13)\quad I_3 = |\tilde{x}|^{c-1}\,(2\pi)^{p/2}\left\{|\tilde{x}| + \frac{(c-1)}{2|\tilde{x}|}[p + (c-3)] + \frac{(c-1)}{|\tilde{x}|}\right\} + o(|\tilde{x}|^{c-2})$$

$$= |\tilde{x}|^{c}\,(2\pi)^{p/2}\left\{1 + \frac{(c-1)}{2|\tilde{x}|^2}\,(p + c - 1)\right\} + o(|\tilde{x}|^{c-2}) .$$

It can be concluded from (4.10), (4.11), and (4.13) that

$$I_1 = (2\pi)^{p/2} \; |\underset{\sim}{x}|^c \; \exp\{\tfrac{1}{2}|\underset{\sim}{x}|^2\}\left\{1 + \frac{(c-1)}{2|\underset{\sim}{x}|^2}(p+c-1) + o(|\underset{\sim}{x}|^{-2})\right\} .$$

A similar argument, using (4.12) and Chebyshev arguments, shows that

$$I_4 \equiv S_p^{-1} \;\; (2\pi)^{p/2} \int \theta_1^2 \exp\{\theta_1|\underset{\sim}{x}|\} \; \tilde{\pi}(d\theta_1)$$

$$= \exp\{\tfrac{1}{2}|\underset{\sim}{x}|^2\} \int_\Theta (\theta_1 + |\underset{\sim}{x}|)^2 \exp\{-\tfrac{1}{2}|\underset{\sim}{\theta}|^2\}(|\underset{\sim}{\theta}|^2 + 2\theta_1|\underset{\sim}{x}| + |\underset{\sim}{x}|^2)^{(c-1)/2} d\underset{\sim}{\theta}$$

$$= \exp\{\tfrac{1}{2}|\underset{\sim}{x}|^2\}|\underset{\sim}{x}|^{c-1}\left[\int \exp\{-\tfrac{1}{2}|\underset{\sim}{\theta}|^2\}(\theta_1^2 + 2\theta_1|\underset{\sim}{x}| + |\underset{\sim}{x}|^2)\right.$$

$$\left. \times \left\{1 + \frac{(c-1)\theta_1}{|\underset{\sim}{x}|} + \frac{(c-1)}{2|\underset{\sim}{x}|^2}[|\underset{\sim}{\theta}|^2 + (c-3)\theta_1^2]\right\} d\underset{\sim}{\theta} + o(1)\right]$$

$$= (2\pi)^{p/2} \exp\{\tfrac{1}{2}|\underset{\sim}{x}|^2\}|\underset{\sim}{x}|^{c+1} \; \{1 + |\underset{\sim}{x}|^{-2}[(\tfrac{c-1}{2})(p+c+1) + 1] + o(|\underset{\sim}{x}|^{-2})\} .$$

Hence

$$\frac{\int \theta_1^2 \exp\{\theta_1|\underset{\sim}{x}|\} \; \tilde{\pi}(d\theta_1)}{\int \theta_1 \exp\{\theta_1|\underset{\sim}{x}|\} \; \tilde{\pi}(d\theta_1)} = \frac{I_4}{I_1}$$

$$= \frac{|\underset{\sim}{x}|\{1 + |\underset{\sim}{x}|^{-2}[(\frac{c-1}{2})(p+c+1) + 1] + o(|\underset{\sim}{x}|^{-2})\}}{\{1 + |\underset{\sim}{x}|^{-2}[(\frac{c-1}{2})(p+c-1)] + o(|\underset{\sim}{x}|^{-2})\}}$$

$$= |\underset{\sim}{x}|\{1 + |\underset{\sim}{x}|^{-2}[\frac{(c-1)}{2}(p+c+1) + 1 - \frac{(c-1)}{2}(p+c-1)] + o(|\underset{\sim}{x}|^{-2})\}$$

$$= |\underset{\sim}{x}|[1 + |\underset{\sim}{x}|^{-2}(c) + o(|\underset{\sim}{x}|^{-2})] .$$

Together with (4.7) and (4.9) this gives

$$\phi^{\pi}(|\underset{\sim}{x}|) = |\underset{\sim}{x}|/(1+2\{-\tfrac{1}{2}|\underset{\sim}{x}|^{-2}+\tfrac{1}{2}|\underset{\sim}{x}|^{-1}|\underset{\sim}{x}|[1+|\underset{\sim}{x}|^{-2}c + o(|\underset{\sim}{x}|^{-2})]\}|\underset{\sim}{x}|^{2})$$

$$= |\underset{\sim}{x}|/(|\underset{\sim}{x}|^{2} + c + o(1))$$

$$= \frac{|\underset{\sim}{x}|}{|\underset{\sim}{x}|^{2} + c} + \frac{o(1)}{|\underset{\sim}{x}|^{3}} ,$$

which is the desired result. ||

It appears that the dividing line between admissibility and inadmissibility, in terms of the tail of the (generalized) prior, is $|\underset{\sim}{\theta}|^{4-p}$. A flatter tailed prior should be inadmissible, and a sharper tailed prior should be admissible. The generalized Bayes estimator with respect to the generalized prior $\pi(d\underset{\sim}{\theta}) = |\underset{\sim}{\theta}|^{(4-p)}d\underset{\sim}{\theta}$, thus, seems like it might be reasonable for the control problem. This estimator is discussed in Zaman (1977).

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