POINT PROCESSES WITH ORDER STATISTIC PROPERTY, MIXTURES OF POISSON PROCESSES AND MIXTURES OF LINEAR DEATH PROCESSES

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ABSTRACT. In a recent paper Feigen (1979) considers the problem of characterizing point processes with order statistic (O.S.) property, called O.S.-point processes. However the class claimed there to characterize these processes fails to include an important class of point processes appropriately called "the mixtures of linear death processes". Besides pointing this out, the paper (1) studies some elementary properties of these latter processes, (2) shows that the class of O.S.-point processes consists only of either these mixtures or of the mixtures of Poisson processes save a time-scale transformation, (3) removes the customary assumption of finiteness of the first moment of these processes, (4) characterizes completely a multivariate analog of the O.S.-point processes, (5) considers briefly the state-dependent O.S.-point processes, among others.

KEY WORDS: POINT PROCESSES; ORDER STATISTIC PROPERTY; MIXTURES OF POISSON PROCESSES; MIXTURES OF LINEAR DEATH PROCESSES; NON-IDENTIFIABILITY PROBLEMS; MULTIVARIATE O.S.-POINT PROCESSES.

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1. INTRODUCTION. The author was just about ready to write this paper after completely characterizing point processes with the so called order statistic (O.S.) property studied previously by Crump (1975) among others, when a paper due to Feigin (1979) containing some of these results appeared. As it turns out in (Feigin (1979)) the class claimed to characterize these point processes fails to include an important class of point processes, which could properly be called as mixtures of linear death processes. In the next section we consider this latter class of processes and some of their elementary properties before returning in Section 3 to the question of characterizing point processes with O.S.-property henceforth called O.S.-point processes. The results of Crump (1975) as well as those of Feigin (1979) are restricted in that they assume the finiteness of the first moment of the process for all $t > 0$. This restriction is removed in Section 3.1. In Section 4 we characterize a multivariate analog of O.S.-point processes, while in Section 5 we briefly touch upon the processes with state-dependent O.S.-property. The paper ends with a few useful remarks, where it also gives a necessary and sufficient condition for a Markov point process to have an O.S.-property.

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2.0. MIXTURES OF LINEAR DEATH PROCESSES.

Suppose a process is initiated with a random number Z of particles at time $t = 0$. Each particle independent of others undergoes a death process with a common length of life distribution governed by a continuous distribution function (d.f.) $F(\cdot)$, with $F(0) = 0$ and $F(t) > 0$, $\forall\, t > 0$. The process $D(t)$, $t \geq 0$, denoting the number of particle-deaths occurring during $(0, t]$ will be called henceforth a mixture of linear death processes with $D(0) = 0$. Constructively we have for $t \geq 0$,

$$(1)\quad D(t) = \begin{cases} 0 & \text{when } Z = 0, \\ \sup\{j : T^{(j)} \leq t,\ j = 1,2,\ldots,k\}, & \text{when } Z = k \geq 1, \end{cases}$$

where for $k \geq 1$, when $Z = k$, $(T^{(1)},\ldots,T^{(k)})$ is an order statistic based on $T_1, T_2, \ldots, T_k$, which are positive mutually independent random variables (r.v.), denoting the lengths of lives of the particles with common d.f. F. Since F is assumed continuous with $F(0) = 0$, almost for every sample path, the process jumps only in unit steps. Also by definition it is separable nondecreasing and has right continuous sample paths. Again it is easily seen that

$$(2)\qquad G(s;t) \equiv E(s^{D(t)}) = g(1-(1-s)F(t)),$$

where $g(\cdot)$ is the probability generating function (p.g.f.) of Z, defined by

$$(3)\qquad g(s) = E(s^Z), \quad |s| \leq 1.$$

We now have the following theorem.

THEOREM 1. The process $\{D(t), t \geq 0\}$, a mixture of death processes with $D(0) = 0$, a.s., is Markovian with O.S.-property.

PROOF. Using the constructive definition of the process $D(t)$, it can be easily seen that for $0 = t_0 < t_1 <\ldots< t_k < t$ and $0 = n_0 \leq n_1 \leq n_2 \leq\ldots\leq n_k \leq n$, we have

$$(4)\quad P(D(t) = n,\ D(t_i) = n_i \quad i = 1,2,\ldots,k)$$

$$= \sum_{m=n}^{\infty} P(D(t) = n,\ D(t_i) = n_i,\quad i = 1,2,\ldots,k;\ Z = m)$$

$$= \sum_{m=n}^{\infty} P(Z=m)P(D(t) = n,\ D(t_i) = n_i,\quad i = 1,2,\ldots,k \mid Z = m)$$

$$= \left\{\prod_{i=1}^{k} \frac{[F(t_i) - F(t_{i-1})]^{n_i-n_{i-1}}}{(n_i - n_{i-1})!}\right\} \frac{[F(t) - F(t_k)]^{n-n_k}}{(n - n_k)!} \cdot g^{(n)}(1-F(t)),$$

where $g^{(m)}(\cdot)$ is the mth derivative of the p.g.f. $g(\cdot)$. A similar expression can be obtained for $P(D(t_i) = n_i,\ i = 1,2,\ldots,k)$ which along with (4) can be shown to yield

$$(5)\quad P(D(t) = n \mid D(t_i) = n_i,\ i = 1,2,\ldots,k)$$

$$= \frac{[F(t) - F(t_k)]^{n-n_k}}{(n - n_k)!} \quad \frac{g^{(n)}(1 - F(t))}{g^{(n_k)}(1 - F(t_k))},$$

provided $P(Z \geq n_k)$ is positive. Similarly one obtains an expression for $P(D(t) = n \mid D(t_k) = n_k)$ and finds it coinciding with (5), establishing thereby the Markovian property of the process. Again for $0 < t_1 < t_2 <\ldots< t_n < t$, and $n \geq 1$, a similar approach yields

(6) $P(t_i < \nu_i \leq t_i + dt_i, \quad i = 1,2,\ldots,n, \ D(t) = n)$

$$= g^{(n)}(1 - F(t)) \prod_{i=1}^{n} F(dt_i),$$

where $\nu_1 < \nu_2 < \ldots < \nu_n$ are the n points of jumps of the process. However since

(7) $P(D(t) = n) = \frac{1}{n!}[F(t)]^n \cdot g^{(n)}(1 - F(t)),$

we have using (6) and (7)

(8) $P(t_i < \nu_i \leq t_i + dt_i, \ i = 1,2,\ldots,n | D(t) = n) = n! \prod_{i=1}^{n} [\frac{F(dt_i)}{F(t)}],$

which corresponds to the distribution of an order statistic from distribution with d.f. given by $F(\tau)/F(t)$, for $0 \leq \tau \leq t$, for the fixed t. This establishes the O.S.-property of our process. □

2.1. A REMARK ABOUT NONIDENTIFIABILITY.

Based on the fact that a mixture of Poisson processes is also a Markov process (see McFadden (1965)), Cane (1977) observed in connection with probability models on accident proneness (see Bates and Neyman (1952 a,b)) a complete probabilistic nondistinguishability between two models that are otherwise conceptually different. One of these models is that of a mixture of Poisson processes where the Poisson parameter (symbolizing a measure of accident proneness of an individual) varies randomly over the population of individuals, while otherwise it remains constant throughout the individual's life time. The other process is a time nonhomogeneous Markov point process where the rate of a future accident depends in an appropriate manner on time as well as on the number of previous accidents (see Cane

(1977) and also Puri (1979) for related details). A somewhat analogous nondistinguishability is observed between the following two processes. The first one is the mixture of linear death process $\{D(t),\ t \geq 0\}$ as defined above, where the initial number Z of live particles is assumed random according to a p.g.f. $g(\cdot)$ given by (3) with particles' lifetimes being otherwise mutually independent and also of Z, with a common d.f. $F(\cdot)$. The second process $\{\tilde{D}(t),\ t \geq 0\}$ is a Markov (birth) point process with the instantaneous rate $\rho_n(t)$ for the (n+1)th event at time t, given by

$$(9) \qquad \rho_n(t) = F'(t) \cdot \frac{g^{(n+1)}(1 - F(t))}{g^{(n)}(1 - F(t))}, \quad n = 0,1,2,\ldots,$$

provided $P(Z \geq n)$ is positive, where $F'(\cdot)$, the probability density corresponding to $F(\cdot)$, is assumed to exist for $t > 0$. The validity of the above claim can be easily established following Cane (1977) by checking that the mixture of death processes in question is itself a Markov 'birth' process with the corresponding instantaneous (birth) rates given by (9).

2.2. SOME CLOSURE PROPERTIES OF CLASS OF MIXTURES OF DEATH PROCESS.

As in Section 2.0., a mixture of linear death processes is defined by $g(\cdot)$, the p.g.f. of Z and the life time d.f. $F(\cdot)$. Let $\mathscr{L}$ be the class of such mixtures with varying (g,F), where for each element of $\mathscr{L}$, we always assume F to be continuous with $F(0) = 0$ and $F(t) > 0,\ \forall\ t > 0$. Let $\mathscr{L}(F)$ be the subclass with F fixed but only g varying. In the following we briefly state some of the closure properties of these classes without proofs as they all follow rather easily.

a) If for any element $\{D(t),\ t \geq 0\}$ of $\mathscr{L}$ all the lengths of lives T_i's are multiplied by a positive constant c, while g remaining unchanged, the new process $D_c(t)$ still belongs to $\mathscr{L}$ and satisfies

(10) $$D_c(t) = D(t/c), \text{ a.s.}$$

If instead all the lengths T_i's are added a positive constant c, the new process $D^c(t)$ defined still as the number of deaths occurring in $(0,t]$, strictly speaking does not belong to $\mathscr{G}$. The new process of course is given a.s. by

(11) $$D^c(t) = \begin{cases} 0, & \text{for } t \leq c \\ D(t-c), & \text{for } t > c. \end{cases}$$

b) Let $\{D_n(t)\}$ be a sequence of mixtures belonging to $\mathscr{G}$ with (g_n, F_n) as the corresponding defining sequence of p.g.f.'s and d.f.'s. Then the sequence $\{D_n(t)\}$ of processes converges weakly to an element of $\mathscr{G}$ provided g_n converges to a proper p.g.f. g and F_n converges weakly to a proper continuous d.f. F with $F(0) = 0$ and $F(t) > 0$, $\forall\, t > 0$.

c) The class $\mathscr{G}(F)$ is closed under superimposition of a finite number of its elements. For instance if two elements of $\mathscr{G}(F)$ corresponding to (g_1, F) and (g_2, F) are superimposed, the resultant process belongs to $\mathscr{G}(F)$ with $g = g_1 g_2$.

d) The class $\mathscr{G}(F)$ is closed under mixing which has to be with respect to the distribution of Z or equivalently with respect to g. However the class $\mathscr{G}$ is not closed under mixing which is with respect to both g and F.

3.0. CHARACTERIZATION OF O.S.-POINT PROCESSES.

Let $\{M(t), t \geq 0\}$ with $P(M(t) < \infty) = 1$, $\forall\, t \geq 0$, be a separable point process with right continuous paths having unit steps at times $\mu_1, \mu_2, \ldots$. Unlike in Feigin (1979), instead of considering the process $\{M(t) - M(0), t \geq 0\}$

we assume without loss of generality that $M(0) = 0$. Other than this, we shall follow in this section the notation of Feigin (1979). The O.S.-property is now defined as follows:

Given that $M(t) = k$, the successive jump times $\{\mu_1, \mu_2, \ldots, \mu_k\}$ are distributed as the order statistic of k independent identically distributed (I.I.D.) r.v.'s with d.f. $F_t(\cdot)$ supported on $[0,t]$. We mention at the outset that since the process $M(t)$ takes jumps only in unit steps, the d.f. $F_t(\cdot)$ has to be continuous for every $t \geq 0$. Following Feigin (1979), we shall say that a point process $M(t)$ has property P if it has O.S.-property with

(12) $$F_t(x) = \frac{x}{t}, \; 0 \leq x \leq t, \quad t > 0.$$

We state the following two theorems due to Feigin (1979).

THEOREM 2. The point process M has property P if and only if $\exists$ a homogeneous Poisson process N with unit rate such that $N(0) = 0$ and

(13) $$M(\cdot) = N(W\,\cdot), \text{ a.s.},$$

where W is a nonnegative r.v. independent of N.

THEOREM 3. Let $EM(t) \equiv m(t)$ be finite $\forall\, t \geq 0$. Then if M is a point process with O.S.-property, there exists a homogeneous Poisson process N with unit rate and an independent nonnegative r.v. W, both defined on the same probability space $(\Omega, \mathcal{F}, P)$ as is M, such that

(14) $$M(\cdot) = N(Wm(\cdot)), \text{ a.s.}$$

Unfortunately theorem 3 is not quite correct, since not all O.S.-point processes have the representation (14) even under the condition that $m(t)$ be finite $\forall\, t \geq 0$, an assumption which we shall drop later in Section 3.1.

Subject to (14), for every $t > 0$, the distribution of $M(t)$ should be a mixture of Poisson distributions. However mixtures of linear death processes studied in Section 2 do enjoy O.S.-property and yet all such mixtures do not yield the distribution of $M(t)$ as mixture of Poisson distributions, unless of course the p.g.f. $g(\cdot)$ of r.v. Z is itself a mixture of poisson distributions (see also Puri and Goldie (1979) for such mixtures). A close scrutiny of Feigin's proof of theorem 3 reveals that he first introduces a process $R(\cdot)$ defined by

$$R(m(t)) = M(t), \tag{15}$$

and then applies theorem 2 to the process $R(\cdot)$ for establishing theorem 3. However one must remember that theorem 2 holds only when the process in question has property P for <u>all</u> $t > 0$. Thus for the case with $\lim_{t\to\infty} m(t) = \infty$, the process $R(t)$ is defined for all $t \geq 0$, through (15) and since it then enjoys property P, $\forall\, t > 0$, theorem 2 applies and the result of theorem 3 as stated is correct. If however $\lim_{t\to\infty} m(t) = \gamma < \infty$, the process $R(t)$ is defined through (15) only for $0 \leq t \leq \gamma$, so that $R(\cdot)$ enjoys property P (i.e. O.S.-property subject to (12)) only for $0 < t \leq \gamma$. Consequently theorem 2 does not apply in this case. What we need for this is the following analog of theorem 2 applicable to the case where property P holds only for $0 < t \leq t^* < \infty$.

THEOREM 4. <u>The point process</u> M <u>has property</u> P <u>for</u> $0 < t \leq t^*$ <u>for some fixed</u> $t^* < \infty$ <u>with</u> $P(M(t^*) < \infty) = 1$, <u>if and only if</u> $\exists$ <u>on the same probability space as that of</u> M, <u>a nonnegative integer valued r.v.</u> Z, <u>and for</u> $Z = k$, $k \geq 1$, <u>an</u>

order statistic $T^{(1)} < T^{(2)} <\ldots< T^{(k)}$, independent of Z but with its distribution identical to the one based on a sample of size k from d.f. $F_{t^*}(x)$ given by (12) such that they define a mixture D(t) of death processes as in (1) for $0 < t \leq t^*$, with

$$M(t) = D(t), \text{ a.s., } 0 \leq t \leq t^*. \tag{16}$$

PROOF. In view of theorem 1, if (16) holds, clearly M has property P for $0 < t \leq t^*$. On the other hand given that M has property P for $0 < t \leq t^*$, the proof follows in a rather straight forward manner by defining as in (1) a process $D(\cdot)$ by taking $Z = M(t^*)$ and $T^{(i)} = \mu_i$, $i = 1,2,\ldots,k$ whenever $M(t^*) = k \geq 1$, where $0 < \mu_1 < \mu_2 <\ldots< \mu_k < t^*$ are the times of unit steps for the process M(t) during $(0,t^*)$ and they form an order statistic based on k I.I.D. r.v.'s with d.f. $F_{t^*}(\cdot)$ given by (12). □

We now have a revised version of theorem 3.

THEOREM 5. Under the conditions of theorem 3, if $\lim_{t\to\infty} m(t) = \infty$, the assertion of the theorem 3 holds as stated. If on the other hand $\lim_{t\to\infty} m(t) = \gamma$ is finite, then on the same probability space as that of M, ∃ a nonnegative integer valued r.v. Z and for $Z = k \geq 1$, an order statistic $\tilde{T}^{(1)} <\ldots< \tilde{T}^{(k)}$ independent of Z, but with distribution identical to the one based on a sample of size k from d.f.

$$\tilde{F}(x) = \begin{cases} m(x)/\gamma, & 0 \leq x < \infty, \\ 0 & \text{elsewhere,} \end{cases} \tag{17}$$

such that they define a mixture $\tilde{D}(t)$ of death processes as in (1), with

$$(18) \qquad M(t) = \tilde{D}(t), \text{ a.s., } \forall\, 0 \le t < \infty.$$

PROOF. For the case with $\lim_{t\to\infty} m(t) = \infty$, the proof due to Feigin is valid. On the other hand for the case with $\lim_{t\to\infty} m(t) = \gamma < \infty$, following Feigin (1979), we consider a process $R(\cdot)$ defined by

$$(19) \qquad R(m(t)) = M(t),\ R(0) = M(0) = 0,\ R(\gamma) = M(\infty),$$

so that the process $R(t)$ is defined only for $0 \le t \le \gamma$. The argument of Feigin now goes through in showing that the process R has the property P for $0 < t \le \gamma$, so that by theorem 4, $\exists$ a nonnegative integer valued r.v. Z and for $Z = k$, $k \ge 1$, an order statistic $T^{(1)} < \ldots < T^{(k)}$, independent of Z with its distribution same as that of an order statistic based on a sample of size k from d.f. $F_\gamma(x)$ given by (12) such that they define a mixture $D(t)$ of death processes for $0 < t \le \gamma$, with

$$(20) \qquad R(t) = D(t), \text{ a.s., } 0 < t \le \gamma.$$

Now using (19), we have

$$(21) \qquad M(t) = R(m(t)) = D(m(t)), \text{ a.s., } 0 < t \le \gamma.$$

However note that since $m(t)$ is continuous, process $D(m(t))$ corresponds a.s. to the process $\tilde{D}(t)$ defined in the theorem with

$$(22) \qquad Z = R(\gamma) = D(\gamma) = M(\infty), \text{ a.s.,}$$

and

$$(23) \qquad \tilde{T}^{(i)} = m^{-1}(T^{(i)}),\ i = 1,2,\ldots k,$$

so that (23) forms an order statistic from d.f. (17), where $m^{-1}(t) = \inf\{s : m(s) > t\}$, for $0 < t \le \gamma$. □

The above theorem assumes the finiteness of $EM(t) = m(t)$, $\forall\, t > 0$. In the next section we attempt to remove this restriction.

3.1. REMOVING THE CONDITION OF FINITENESS OF EM(t).

The results of Crump (1975) are all subject to the assumption that $0 < m(t) \equiv EM(t) < \infty$, $\forall\, t > 0$. Same is true in the case of Feigin (1979) if we agree to exclude the trivial case where $m(t) \equiv 0$ (hence $M(t) \equiv 0$, a.s.) for some initial finite interval of the time axis. These results in particular include theorem 3 and 5 of the previous section and also that in this case one must have

$$F_t(x) = \frac{m(x)}{m(t)},\quad 0 \le x \le t < \infty,\ t > 0. \tag{24}$$

We now abandon the assumption of finiteness of $M(t)$, and instead make the following weaker assumption (A):

(A) $\exists\, t_0 > 0,\ \ni P(M(t_0) = 0) < 1,\ F_{t_0}(x) > 0, \forall\, 0 < x \le t_0$, and

$F_t(t_0) > 0, \forall\, t \ge t_0$.

With this we have the following theorem.

THEOREM 6. Let for an O.S.-point process $M(t)$ the assumption (A) hold and

$$q(t) \equiv \begin{cases} [F_t(t_0)]^{-1}, & \text{for } t \ge t_0, \\ F_{t_0}(t), & \text{for } t \le t_0. \end{cases} \tag{25}$$

Then

(i) $F_t(x) = q(x)/q(t),\quad 0 \le x \le t,$

(ii) for $t \ge t_0$, $F_t(t_0)$ is a nonincreasing continuous function of t so that $q(t)$ is positive, continuous, and nondecreasing in t with $q(0) = 0$.

(iii) If $0 < m(t) < \infty$, $\forall\, t > 0$, then

$$\frac{m(x)}{m(t)} = \frac{q(x)}{q(t)}, \;\forall\; 0 < x \leq t < \infty.$$

PROOF. Using (25) we may write (i) equivalently as

$$(26) \qquad F_t(x) = \begin{cases} F_{t_0}(x)/F_{t_0}(t), & 0 < x \leq t \leq t_0, \\ F_{t_0}(x)\; F_t(t_0), & 0 < x \leq t_0 \leq t \\ F_t(t_0)/F_x(t_0), & 0 < t_0 \leq x \leq t. \end{cases}$$

Other cases being analogous, we shall prove (26) only for the case with $0 < x \leq t \leq t_0$. The basic tool we use is the following identity which must hold due to the O.S.-property of the process $M(t)$.

$$(27) \qquad H(s;x) \equiv H(1-(1-s)F_t(x);t), \;\forall\; 0 < x < t; |s| \leq 1.$$

Here

$$(28) \qquad H(s;t) \equiv E(s^{M(t)}); |s| \leq 1.$$

In particular we have from (27)

$$(29) \qquad H(s;x) \equiv H(1-(1-s)F_{t_0}(x);t_0);\; 0 < x \leq t_0; |s| \leq 1.$$

Using this on the right side of (27) for $0 < t \leq t_0$, we have for $0 < x \leq t \leq t_0$,

$$\begin{aligned} (30) \qquad H(s;x) &= H(u;t)\big|_{u\,=\,1-(1-s)F_t(x)} \\ &= H(1-(1-u)F_{t_0}(t);t_0)\big|_{u\,=\,1-(1-s)F_t(x)} \\ &= H(1-(1-s)F_{t_0}(t)F_t(x);\; t_0). \end{aligned}$$

Note that since $P(M(t_0) = 0) < 1$, $H(s,t_0)$ is strictly increasing in s. Consequently on equating (29) and (30) it follows that we must have $F_t(x) = F_{t_0}(x)/F_{t_0}(t)$. Again for $t \geq t_0$, that $F_t(t_0)$ is nonincreasing in t follows from (26) for $0 < t_0 \leq x < t$ and the fact that $F_t(x) \leq 1$. Also its continuity and hence that of q(t) as well as its monstonicity now follows from the continuity in x of the d.f. $F_t(x)$. Finally part (iii) follows from (i) and (24). □

<u>REMARK 1</u>. Under (A) it is easily seen that for $t > t_0$, $F_t(t_0) = P(\mu_1 \leq t_0 | M(t) = 1)$, and for $t \leq t_0$, $F_{t_0}(t) = P(\mu_1 \leq t | M(t_0) = 1)$. Using these it is rather straightforward to show that the old assumption $0 < m(t) < \infty$, $\forall\, t > 0$, along with (24) implies (A).

We now have a generalized version of therorem 5, which is given below without proof since the same proof as that of theorem 5 works after replacing m(·) by the function q(·) of (25), which like m(·) is also continuous and nondecreasing.

<u>THEOREM 7</u>. <u>For an O.S.-point process</u> M(t) <u>satisfying the assumption</u> (A), <u>the assertions of theorem 5 hold with the function</u> m(·) <u>replaced by the function</u> q(·) <u>given in</u> (25).

4. MULTIVARIATE O.S.-POINT PROCESSES.

A simple multivariate O.S.-point process $\underset{\sim}{M}(t) = (M_1(t),\ldots,M_n(t))$ would be the one with the following O.S.-property:

Let $\underset{\sim}{M}(0) = 0$. Given $M_i(t) = k_i$, $i = 1,2,\ldots,n$, the successive times $\{\mu_{i1}, \mu_{i2},\ldots,\mu_{ik_i}\}$ of unit jumps for the ith component process $M_i(t)$ are distributed as the order statistic based on k_i I.I.D. r.v.'s with continuous d.f. $F_{it}(x)$ supported on $[0,t]$, the various sets of order statistics for $i = 1, 2, \ldots, n$, being mutually independent.

A multivariate separable point process with sample paths continuous from the right, having jumps only of unit steps for one of the component processes at a time, and having the above O.S.-property will be called a simple multivariate O.S.-point process. We introduce the following assumption, the analog of (A).

(A*). For every $i = 1,2,\ldots,n$, $\exists\ t_i > 0$, $\ni P(M_i(t_i) = 0) < 1$, $F_{it_i}(x) > 0$, $\forall\ 0 < x \le t_i$, and $F_t(t_i) > 0$, $\forall\ t > t_i$.

With this we have the following theorem.

THEOREM 8. Let $\underset{\sim}{M}(t)$ be a simple n- multivariate O.S.-point process satisfying condition (A*). Then we have

(i) for $i = 1,2,\ldots,n$, $F_{it}(x) = q_i(x)/q_i(t)$, $0 \le x \le t < \infty$, where as in theorem 6

$$q_i(t) = \begin{cases} [F_{it}(t_i)]^{-1}, & \text{for } t \ge t_i \\ [F_{it_i}(t), & \text{for } t \le t_i. \end{cases} \tag{31}$$

(ii) The process $\underset{\sim}{M}(t)$ is Markovian.

(iii) Let without loss of generality $\lim_{t\to\infty} q_i(t) = \gamma_i$, with $\gamma_i < \infty$, for $i = 1,2,\ldots,r$ and $\gamma_i = \infty$ for $i = r+1,\ldots,n$, with $0 \le r \le n$. Then $\exists$ a joint probability distribution of nonnegative integer valued r.v.'s $Z_1, Z_2,\ldots,Z_r$ and nonnegative r.v.'s $W_{r+1}, W_{r+2},\ldots,W_n$ with

(32) $$P(Z_i = k_i,\ i = 1,2,\ldots,r;\ W_i \le w_i,\ i = r+1,\ldots,n) = \rho(k_1,\ldots,k_r;\ w_{r+1},\ldots,w_n),$$

__for__ $w_i \ge 0,\ i = r+1,\ldots,n$, __and__ $k_i = 0,1,2,\ldots$, __such that__

(33) $$H(s_1,\ldots,s_n;t) \equiv E[\prod_{i=1}^{n} s_i^{M_i(t)}]$$

$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_r=0}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{r} [1-(1-s_i)\frac{q_i(t)}{\gamma_i}]^{k_i} \cdot$$

$$\cdot \exp[-\sum_{i=r+1}^{n} (1-s_i)q_i(t)w_i]\ \rho(k_1,\ldots,k_r;\ dw_{r+1},\ldots,dw_n),$$

$|s_i| \le 1,\ i = 1,2,\ldots,n.$

__PROOF.__ Proof of (i) is similar to that of theorem 6, and (ii) follows from a simple argument used by Crump (1975). To prove (iii) we use the following basic identity for the p.g.f. H, which follows from the O.S.-property.

(34) $$H(1-u_1,\ldots,1-u_n;x) \equiv H(1-u_1\frac{q_1(x)}{q_1(t)},\ldots,1-u_n\frac{q_n(x)}{q_n(t)};t),$$

for $0 < x \le t < \infty$, $|u_i| \le 1,\ i = 1,2,\ldots,n$. Let us fix $0 < x < \infty$, and define

(35) $$h(u_1,\ldots,u_n) \equiv H(1-u_1,\ldots,1-u_n;x).$$

Then through (34) h is well defined for $0 \le u_i \le 1,\ i = 1,2,\ldots,n$, taking u_i's real. However using right side of (34), by taking t large enough we can extend the definition of h, for $0 \le u_i < \theta_i,\ i = 1,2,\ldots,r$ and for

$0 \leq u_i < \infty$, $i = r+1,\ldots,n$, where $\theta_i = \gamma_i/q_i(x)$, $i = 1,2,\ldots,r$. Using (34) and (35), it is easy to see that

$$(36) \qquad (-1)^{k_1+\ldots+k_n} \frac{\partial^{k_1+\ldots+k_n} h(u_1,\ldots,u_n)}{\partial u_1^{k_1}\ldots\partial u_n^{k_n}} \geq 0,$$

for all combinations of $k_i \geq 0$, $i = 1,2,\ldots,n$ and $\forall\ 0 < u_i < \theta_i$, $i = 1,\ldots,r$; $0 < u_i < \infty$, $i = r+1,\ldots,n$. We now change the variables $(u_1,\ldots,u_n)$ to $(v_1,\ldots,v_n)$ with

$$(37) \qquad u_i = \theta_i(1-e^{-v_i}),\ i = 1,2,\ldots,r;\ v_i = u_i,\ i = r+1,\ldots,n,$$

and let

$$(38) \qquad h^*(v_1,\ldots,v_n) \equiv h(\theta_1(1-e^{-v_1}),\ldots,\ \theta_r(1-e^{-v_r}),\ v_{r+1},\ldots,v_n),$$

which is defined now for $\forall\ 0 \leq v_i \leq \infty$, $i = 1,2,\ldots,n$. Using (36) it can be seen that the function h^* is completely monotone in the n-dimensional open octant $0 < v_i < \infty$, $i = 1,2,\ldots n$, so that using a well known theorem (see Bochner (1960), page 87) $\exists$ a unique n-variate measure $\sigma_x(y_1,\ldots,y_n)$ (possibly dependent on x) over n-dimensional closed octant $y_i \geq 0$, $i = 1,2,\ldots,n$, such that for $0 \leq v_i < \infty$, $i = 1,2,\ldots,n$, we have

$$(39) \qquad h^*(v_1,\ldots,v_n) = \int_0^\infty\ldots\int_0^\infty \exp[-\sum_{i=1}^n v_i y_i]\ \sigma_x(dy_1,\ldots,dy_n).$$

From this and the preceding relations, it easily follows that

$$(40) \qquad H(s_1,\ldots,s_n;x) = \int_0^\infty\ldots\int_0^\infty \left(\prod_{i=1}^r [1-(1-s_i)\frac{q_i(x)}{\gamma_i}]^{y_i}\right)$$

$$\exp[-\sum_{i=r+1}^n (1-s_i)y_i]\ \sigma_x(dy_1,\ldots,dy_n),$$

valid $\forall\ x > 0$. For removing the dependence of the measure σ on x and in order to bring (40) into the form (33), we proceed as follows. Let $0 < x \leq 1$. Using (34) with $t = 1$ and (40), we have

$$(41)\qquad H(s_1,\ldots,s_n;x) = H(1-(1-s_1)\frac{q_1(x)}{q_1(1)},\ldots,1-(1-s_n)\frac{q_n(x)}{q_n(1)};1)$$

$$= \int_0^\infty\ldots\int_0^\infty (\prod_{i=1}^{r} [1-(1-s_i)\frac{q_i(x)}{\gamma_i}]^{y_i}).$$

$$\cdot \exp(-\sum_{i=r+1}^{n} (1-s_i)\frac{q_i(x)}{q_i(1)}\cdot y_i)\ \sigma_1(dy_1,\ldots,dy_n)$$

$$= \int_0^\infty\ldots\int_0^\infty (\prod_{i=1}^{r} [1-(1-s_i)\frac{q_i(x)}{\gamma_i}]^{w_i})$$

$$\cdot \exp(-\sum_{i=r+1}^{n} (1-s_i)q_i(x)w_i)\ \rho(dw_1,\ldots dw_n),$$

where at the end we changed the variables from $(y_1,\ldots,y_n)$ to $(w_1,\ldots,w_n)$, with $w_i = y_i$, $i = 1,2,\ldots,r$ and $w_i = y_i/q_i(1)$, $i = r+1,\ldots,n$, and ρ is the new measure for $\underset{\sim}{w}$.

Similarly let $1 \leq x < t < \infty$. Using (34) with $x = 1$, and taking $u_i = (1-s_i)\ q_i(t)/q_i(1)$, $i = 1,2,\ldots,n$, we have

$$(42)\qquad H(s_1,\ldots,s_n;t) = H(1-(1-s_1)\frac{q_1(t)}{q_1(1)},\ldots,1-(1-s_n)\frac{q_n(t)}{q_n(1)};1),$$

valid for $s_i \in [1-\frac{q_i(1)}{q_i(t)},1]$, $i = 1,2,\ldots,n$. Again using (40) in (42), we have for $t \geq 1$,

$$(43)\qquad H(s_1,\ldots,s_n;t) = \int_0^\infty \ldots \int_0^\infty \left(\prod_{i=1}^{r} \left[1-(1-s_i)\frac{q_i(t)}{\gamma_i}\right]^{y_i} \right) \cdot$$

$$\exp\left(-\sum_{i=r+1}^{n} (1-s_i)\frac{q_i(t)}{q_i(1)} \cdot y_i\right) \sigma_1(dy_1,\ldots,dy_n),$$

for $s_i \in [1-\frac{q_i(1)}{q_i(t)},1]$, $i = 1,2,\ldots,n$. We now extend through analytic continuation the validity of (43) for $|s_i| \leq 1$, $i = 1,2,\ldots,n$. Finally (43) can be equivalently written as in (41) yielding together with (41),

$$(44)\qquad H(s_1,\ldots,s_n;t) = \int_0^\infty \ldots \int_0^\infty \left(\prod_{i=1}^{r} \left[1-(1-s_i)\frac{q_i(t)}{\gamma_i}\right]^{y_i} \right).$$

$$\cdot \exp\left(-\sum_{i=r+1}^{n} (1-s_i)q_i(t)y_i\right) \rho(dy_1,\ldots,dy_n),$$

valid for $|s_i| \leq 1$, $i = 1,2,\ldots,n$ and $t \geq 0$. Note that ρ has to be a probability measure since $H(1,\ldots,1;t) \equiv 1$. Also putting $s_i = 1$, for $i = r+1,\ldots,n$ and then letting $t \to \infty$ on both sides of (44), since the left side limit is a p.g.f., we note that the measure ρ must concentrate its mass on nonnegative integer values for y_i, $i = 1,2,\ldots,r$. This makes (44) and (33) equivalent, thereby completing the proof. □

REMARK 2. The characterization of the distribution of $\underset{\sim}{M}(t)$ for $t \geq 0$, through (33) characterizes all the simple multivariate O.S.-point processes, since the distribution of $M(t)$ together with the O.S.-property allows us to obtain the distribution of any arbitrary finite dimensional vector $(\underset{\sim}{M}(t_1),\ldots,\underset{\sim}{M}(t_\ell))$, for $0 < t_1 <\ldots< t_\ell < \infty$. Moreover the O.S.-property and the form (33) admit the following probabilistically equivalent constructive interpretation for these processes. Observe first the random vector $(Z_1,\ldots Z_r;\ W_{r+1},\ldots,W_n)$ according to the probability measure ρ. Given this vector construct n independent point processes $M_i(t)$, $i = 1,2,\ldots,n$, where the first r processes are linear death processes using (Z_i,F_{it}), $i = 1,2,\ldots r$, and the remaining are Poisson processes with $E(M_i(t)|W_i = w_i) = w_i q_i(t)$, $i = r+1,\ldots,n$. With this, it is evident that $\lim_{t\to\infty} M_i(t) = Z_i$, a.s., for $i = 1,2,\ldots,r$, and $\lim_{t\to\infty} M_i(t) = \infty$, a.s., for $i = r + 1,\ldots,n$.

REMARK 3. Since $\{\underset{\sim}{M}(t);\ t \geq 0\}$ is a Markov process, with jumps of only unit steps for only one of the components $M_i(t)$, $i = 1,2,\ldots n$, at a time, the possible transitions are only of the type

$$(45)\qquad (\ell_1,\ldots,\ell_{j-1},\ \ell_j,\ \ell_{j+1},\ldots,\ell_n) \to (\ell_1,\ldots,\ell_{j-1},\ \ell_j+1,\ \ell_{j+1},\ldots,\ell_n),$$

for $j = 1,2,\ldots,n$. Now using the construction mentioned in Remark 2 and following Cane (1977), it can be easily shown that analogous to (9) the rates corresponding to the transitions (45) are given by

$$(46)\qquad \lambda^{(j)}_{\ell_1,\ldots,\ell_n}(t) = \frac{H^{(\ell_1,\ldots,\ell_{j-1},\ell_j+1,\ldots,\ell_n)}(0,\ldots,0;t)}{H^{(\ell_1,\ell_2,\ldots,\ell_j,\ldots,\ell_n)}(0,\ldots,0;t)} \cdot \frac{q_j'(t)}{q_j(t)},$$

for $j = 1,2,\ldots,n$, provided in the denominator $H^{(\ell_1,\ldots,\ell_n)}(0,\ldots,0;t)$ is positive, where $q_i(t)$'s are assumed to be continuously differentiable, H is given by (33) and

$$(47)\qquad H^{(\ell_1,\ldots,\ell_n)}(0,\ldots,0;t) = \left.\frac{\partial^{\ell_1+\ldots+\ell_n} H(s_1,\ldots,s_n;t)}{\partial s_1^{\ell_1}\ldots\partial s_n^{\ell_n}}\right|_{s_1=\ldots=s_n=0}.$$

In view of the previous remark we notice again a nondistinguishability between the two models, one based on a mixture of multivariate processes and the other on a single multivariate Markov process, an observation similar to the one made by Cane (1977) and also discussed earlier in Section 2.1.

REMARK 4. For the case with $r = n$, where $\lim_{t\to\infty} q_i(t) = \gamma_i < \infty$, $i = 1,2,\ldots,n$, it is possible to define directly an n-variate mixture of linear death processes $\underset{\sim}{D}(t) = (D_1(t),\ldots,D_n(t))$ on the same sample space as that of $\underset{\sim}{M}(t)$, with $\underset{\sim}{M}(t) \equiv \underset{\sim}{D}(t)$, a.s., by taking

$$(48) \qquad \underset{\sim}{Z} \equiv (Z_1,\ldots,Z_n) = \underset{\sim}{M}(\infty) \equiv (M_1(\infty),\ldots,M_n(\infty))$$

and

$$(49) \qquad T_i^{(j)} = \mu_{ij},\ i = 1,2,\ldots n;\ j = 1,2,\ldots,$$

where for each $i = 1,2,\ldots n$, $(T_i^{(j)}, j = 1,2,\ldots)$ is the order statistic needed to define $D_i(t)$ as in (1), and μ_{ij}'s are as defined in theorem 8. However for the case with $r < n$, where $\lim_{t\to\infty} q_i(t) = \infty$, for $i = r+1,\ldots,n$, the proof of Feigin (1979) does not go through, as it involves a transformation of time scale using the function $q(\cdot)$, which in the present case are more than one and are possibly different. Thus in the case with $r < n$, it was not possible to obtain slightly stronger results similar to Feigin (1979).

5. STATE-DEPENDENT O.S.-POINT PROCESSES.

As a generalization to the ordinary O.S.-point (univariate) processes studied in Section 3, we may consider processes with the state-dependent

O.S.-property. Here the process M(t) is defined exactly as in Section 3.0., except that now for each given M(t) = k, the continuous d.f. $F_{kt}^{(\cdot)}$ for the order statistic depends not only on t but also on the state k. These processes in general are highly nonMarkovian but are quite interesting. While their detailed investigation will be reported elsewhere, we close with the following theorem given here without proof.

THEOREM 9. Let M(t) with M(0) = 0, a.s., be a univariate state-dependent O.S.-point process satisfying the condition

(A_1) $\exists\ t_0 \ni P(M(t_0)=0) < 1,\ F_{1t_0}(x) > 0,\ \forall\ 0 < x \leq t_0$ and

$$F_{1t}(t_0) > 0,\ \forall\ t \geq t_0.$$

Let

$$q_1(t) = \begin{cases} [F_{1t}(t_0)]^{-1}, & \text{for } t \geq t_0 \\ F_{1t_0}(t), & \text{for } t \leq t_0. \end{cases} \tag{50}$$

Then $F_{1t}(x) = q_1(x)/q_1(t)$, for $0 \leq x \leq t$. Also the process is Markovian if and only if it is an ordinary O.S.-point process with

$$F_{kt}(x) \equiv F_{1t}(x),\ 0 \leq x \leq t < \infty,\ k \geq 1. \tag{51}$$

6. A FEW CONCLUDING REMARKS.

We make a few remarks for the univariate O.S.-point processes below, although some of these may also apply to the multivariate case also.

(a) We have shown that univariate O.S.-point processes are either mixtures of Poisson processes, save a time-scale transformation, with

$$H(s;t) = E(s^{M(t)}) = \int_0^\infty \exp[-(1-s)q(t)w]\ \psi(dw), \tag{52}$$

where $\lim_{t\to\infty} q(t) = \infty$, and ψ is the d.f. of the r.v. W of theorem 3, or they are mixtures of linear death processes with

$$H(s;t) = E(s^{M(t)}) = g(1-(1-s)\frac{q(t)}{\gamma}), \tag{53}$$

for some p.g.f. $g(\cdot)$ and a nondecreasing function $q(\cdot)$ with $q(0) = 0$, $\lim_{t\to\infty} q(t) = \gamma$ and $0 < \gamma < \infty$. The reader may refer to Puri and Goldie (1979) concerning the infinite-divisibility properties of mixture of Poisson processes for varying values of t. In particular the mixture (52) is infinitely divisible for all $t > 0$, if and only if ψ is an infinitely divisible law. Similarly it can be shown that (53) is infinitely divisible $\forall\, t > 0$, if and only if p.g.f. $g(\cdot)$ corresponds to an infinitely divisible law.

(b) Crump has given a condition (see (2.12) of Crump (1975)) which is necessary for a point process to have an O.S.-property. An analogous condition, which is both necessary and sufficient, is given without proof in the following theorem, as it can be easily established.

THEOREM 10. Let a point process M(t) be a nonhomogeneous (Markov) birth process constructively defined with birth rates $\lambda_i(t)$, $i = 0,1,2,\ldots,m$, which for $0 \le i < m$, are all strictly positive, continuous and integrable over $(0,t)$ for all $t > 0$, with $\lambda_m(t) \equiv 0$. Here m may be infinite or finite; in the latter case it is an absorption state for the process. Then in order that the process M(t) be an O.S.-point process, it is necessary and sufficient that for all $t > 0$, the λ's satisfy the identities (54) given below for some positive constants $L(i)$, $i = 0,1,2,\ldots,m$, with $L(0) = 1$, and a function $h(t)$ which is strictly positive, continuous and intergrable over $(0,t)$, $\forall\, t > 0$.

(54) $$\rho_i(t) \equiv h(t)\, L(i+1)/L(i), \quad i = 0,1,\ldots,m-1.$$

Here for $i = 0,1,2,\ldots$

(55) $$\rho_i(t) = \lambda_i(t) \exp[\Lambda_{i+1}(t) - \Lambda_i(t)]$$

and

(56) $$\Lambda_i(t) = \int_0^t \lambda_i(u)\, du.$$

Furthermore, if $H(t) = \int_0^t h(u)\, du$, *we also have*

(57) $$EM(t) = H(t) = \frac{L(i)}{L(i+1)} \int_0^t \lambda_i(u) \exp[\Lambda_{i+1}(u) - \Lambda_i(u)]\, du,$$

for $i = 0,1,2,\ldots,m-1$, *and*

(58) $$P(M(t) = k) = L(n)\, \frac{[H(t)]^k}{k!} \cdot \exp[-\Lambda_k(t)], \quad k = 0,1,\ldots,m.$$

(c) For an O.S.-point process, which is a mixture of Poisson processes with the corresponding p.g.f. given by (52), assuming that the function $q(t)$ is continuously differentiable and using (46) for $n = 1$, we get the corresponding birth rates as

(59) $$\lambda_k(t) = q'(t) \cdot \frac{\int_0^\infty w^{k+1} \exp[-q(t)w]\, \psi(dw)}{\int_0^\infty w^k \exp[-q(t)w]\, \psi(dw)},$$

for $k = 0,1,2,\ldots$, where ψ is a probability measure. If for some $0 \le j < \ell < \infty$, $\lambda_j(t) \equiv \lambda_\ell(t)$, it turns out that we must have $\lambda_k(t) \equiv \lambda_0(t)$, $\forall\, k \ge 0$. This follows by showing, using the lemma given below, that in this case the measure ψ must be degenerate.

LEMMA. *Let for an arbitrary nonnegative random variable* Y,

(60) $$E(Y)\, E(Y^k) = E(Y^{k+1}),$$

for some $k \ge 1$, *where the moments involved are assumed finite*, *then* $P(Y = c) = 1$, *for some* $c \ge 0$.

PROOF. The lemma is trivial if $P(Y=0) = 1$. Suppose therefore $P(Y=0) < 1$, so that $E[Y^x] > 0$, for all $x \geq 0$. It can be now easily seen that $\beta(x) \equiv \ln E(Y^x)$ is a convex function of x and that the condition (60), which is equivalent to $\beta(1) + \beta(k) = \beta(k+1)$, implies that $\beta(x) \equiv \alpha x$, for some constant α. From this the lemma follows with $c = \exp(\alpha)$.

(d) Consider again a mixture of Poisson processes corresponding to (52), from which we have

$$(61) \qquad P_0(t) \equiv P(M(t)=0) = \int_0^\infty \exp[-q(t)w]\ \psi(dw).$$

This relation may be useful in characterizing the mixing probability measure ψ. For instance, let $q^{-1}(\tau) = \inf\{t : q(t) > \tau\}$. Since $q(\cdot)$ is continuous, we have the Laplace Stieltjes transform of ψ given by

$$(62) \qquad \psi^*(\theta) \equiv \int_0^\infty \exp(-\theta w)\ \psi(dw) = P_0(q^{-1}(\theta)),\ \text{for } \theta \geq 0.$$

Similarly for a mixture of linear death processes corresponding to (53), we have

$$(63) \qquad P_0(t) = H(0;t) = g(1- \frac{q(t)}{\gamma}),$$

which yields

$$(64) \qquad g(s) = P_0(q^{-1}(\gamma(1-s))),\ 0 \leq s \leq 1.$$

(e) CLOSURE PROPERTIES. As in Section (2.2), we can note some of the closure properties of O.S.-point processes. For instance analogous to (d) of Section 2.2, it can be easily shown that if we have $\{M_i(t),\ t \geq 0\}$, $i = 1,2,\ldots,k$, all independent O.S.-point processes with $q_i(t)$, $i = 1,2,\ldots,k$, as the corresponding functions entering either in (52) or (53) as the case may be, then all mixtures of these k processes have O.S.-property if and only if

(65) $\quad q_i(t) \equiv c_i\, q(t),\ i = 1,2,\ldots,k,$

for some constants c_i and a function $q(t)$. This means either all the k processes have to be of the type (52) or all of the type (53); in either case the corresponding $q_i(t)$'s must satisfy (65). Similarly superimposition of the above k processes subject to (65), yields again a processes with O.S.-property (see also (c) of Section 2.2).

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