THE SOLUTION OF THE FUNCTIONAL EQUATION OF D'ALEMBERT'S TYPE FOR COMMUTATIVE GROUPS*

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ABSTRACT

A functional equation of the form $\phi_1(x+y) + \phi_2(x-y) = \sum_1^n \alpha_i(x)\beta_i(y)$, where functions $\phi_1, \phi_2, \alpha_i, \beta_i$, $0 = 1,\ldots,n$ are defined on a commutative group, is solved. We also obtain conditions for the solutions of this equation to be matrix elements of a finite dimensional representation of the group.

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1. INTRODUCTION

Consider the functional equation

$$\phi_1(x+y) + \phi_1(x-y) = \alpha_1(x)\beta_1(y) + \ldots + \alpha_n(x)\beta_n(y), \tag{1.1}$$

where $\phi_1, \phi_2, \alpha_i, \beta_i$, $i = 1,\ldots,n$ are functions given on a commutative group $\mathcal{G}$, taking values in a field $\mathcal{F}$ of characteristic zero.

Clearly, if $f(x) = \phi_1(x)-\phi_2(x)$, $g(x) = \phi_1(x) + \phi_2(x)$, then

$$f(x+y)-f(x-y) = \sum_{i=1}^{n} [\alpha_i(x) + \alpha_i(-x)]\beta_i(y) = \sum_{j=1}^{m} h_j(x)k_j(y) \tag{1.2}$$

and

$$g(x+y) + g(x-y) = \sum_{i=1}^{n} [\alpha_i(x) + \alpha_i(-x)]\beta_i(y) = \sum_{i=1}^{p} u_i(x)v_i(y), \tag{1.3}$$

where the functions h_j, k_j, $j = 1,\ldots,m$ and u_i, v_i, $i = 1,\ldots,p$ are linearly independent. Therefore it suffices to consider the case when $\phi_1 = \phi_2$ or $\phi_1 = -\phi_2$ in (1.1). Note that linear independence of h_j and u_i implies $k_j(-y) = -k_j(y)$, $j = 1,\ldots,m$ and $v_i(-y) = v_i(y)$ $i = 1,\ldots,p$.

The equation (1.1) can be viewed as a generalization of D'Alembert's (cosine) functional equation

$$\phi(x+y) + \phi(x-y) = 2\phi(x)\phi(y), \tag{1.4}$$

which has been much studied (cf [1 p. 176], [2], [4], [5], [6], [9]). It also arises in statistical applications (see [10]).

In Section 3 we obtain the general form of the solutions f and g of equations (1.2) and (1.3). These solutions are expressed as linear combinations of matrix elements of inequivalent finite dimensional representations of the group $\mathcal{G}$ and also of terms involving homomorphisms of $\mathcal{G}$ into a vector space $\mathcal{F}^n$ over the field $\mathcal{F}$ and homomorhisms of $\mathcal{G}$ into additive matrix group over $\mathcal{F}$. While the former terms are well known in the theory of functional equations, the latter terms seem to be new. Section 2 contains some preliminary results about polynomials on Abelian groups. The discussion of the main result is given in Section 4, where sufficient conditions for a solution to be a matrix element of a finite dimensional representation are derived.

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2. POLYNOMIALS OVER COMMUTATIVE GROUPS

Let $\mathcal{L}$ be a finite dimensional vector space over the field $\mathcal{F}$. (In this paper $\mathcal{L}$ will be the vector space $\mathcal{F}_n$ of all n×n matrices over the field $\mathcal{F}$, or the vector space $\mathcal{F}^n$ of dimension n on $\mathcal{F}$). If ψ is an $\mathcal{L}$-valued function defined on the Abelian group $\mathcal{G}$, then $L(x)$, $x \in \mathcal{G}$ is the translation operator, $L(x)\psi(\cdot) = \psi(\cdot+x)$. Thus L is a regular representation of $\mathcal{G}$ which acts in the linear space spanned by the translates of the function ψ. The function ψ is called a polynomial if for some n $(L(x)-I)^{n+1}\psi(y) \equiv 0$ for all $x,y \in \mathcal{G}$. The smallest number n for which this identity holds is called the degree of the polynomial.

Thus a polynomial of degree one satisfies the identity

$$\psi(x+y) + \psi(x-y) = 2\psi(x).$$

If $2\mathcal{G} = \mathcal{G}$ this condition implies that $\psi(x) = \chi(x) + c$, where $c \in \mathcal{L}$, $\chi \in \text{Hom}\,(\mathcal{G}, \mathcal{L})$, i.e. $\chi(x+y) = \chi(x) + \chi(y)$ for all $x,y \in \mathcal{G}$.

A polynomial φ is said to be homogeneous, if

$$(L(x)-I)^n\psi(\cdot) = n!\psi(x).$$

The following elementary results [6] will be used in Section 3.

1°. If φ is a homogeneous polynomial of degree n, then for all integer j $\varphi(jx) = j^n\varphi(x)$, $x \in \mathcal{G}$.

2°. If ψ is a polynomial of degree n, then $\varphi(x) = (L(x)-I)^n\psi(y)$ does not depend on y and is an homogeneous polynomial of degree n in x.

3°. If ψ is a polynomial of degree n, then $(L(x_1)-I)\ldots(L(x_j)-I)\psi(x)$ is a polynomial in x of degree n-j.

4°. If ψ is a polynomial of degree n, then

$$\psi(x) = \varphi_n(x) + \ldots + \varphi_0(x),$$

where $\varphi_j(x)$ is a homogeneous polynomial of degree j, $j = 0,1,\ldots,n$. One has

$$\varphi_n(x) = \frac{1}{n!}(L(x)-I)^n\psi(\cdot)$$

and for $j = n-1,\ldots,0$,

$$\varphi_j(x) = \frac{1}{j!}(L(x)-I)^n(f(\cdot)-\psi_n(\cdot)-\ldots-\psi_{j+1}(\cdot)).$$

5°. If φ is a homogeneous polynomial of degree n, then $\varphi(x) = \chi(x,\ldots,x)$ where $\chi(x_1,\ldots,x_n)$ is a symmetric function of $x_1,\ldots,x_n$ and for fixed $x_2,\ldots,x_n$, $\chi(\cdot,x_2,\ldots,x_n) \in \mathrm{Hom}\,(\mathcal{G}, \mathcal{L})$.

If ψ is a polynomial of even degree and $2\mathcal{G} = \mathcal{G}$ then in all formulas above $L(x)-I$ can be replaced by $L(x/2)-L(-x/2)$.

If $k \in \mathfrak{J}^n$ and $k \in \mathfrak{J}^{*n}$ where $\mathfrak{J}^{*n}$ is the dual space, then $\langle h,k\rangle$ will always denote the value of the linear functional k on the element h. With this convention equation (1.2), for instance, can be rewritten

$$f(x+y)-f(x-y) = \langle h(x),k(y)\rangle, \tag{2.1}$$

where $h(x) \in \mathfrak{F}^m$, and $k(y) \in \mathfrak{F}^{*m}$. Also A^t will denote the transpose of a linear transformation A.

3. THE MAIN RESULT

A structure theorem for the solutions of the functional equations (1.2) and (1.3) is obtained in this Section.

Theorem 1. Assume that $\mathcal{G}$ is a commutative group such that $2\mathcal{G} = \mathcal{G}$. A function f taking values in an algebraically closed field $\mathfrak{F}$ of characteristic zero is a solution of the equation (1.2) with linearly independent functions h_j, k_j, $j = 1,\ldots,m$ if, and only if, there exist nonnegative integers $m_1,\ldots,m_R, m_1 + \ldots + m_R = m$ such that

$$f(x) = \langle S(x)f_1, \varphi(x)\rangle + \langle T(x)Q_1\varphi(x), \varphi(x)\rangle$$

$$+ \sum_{r=2}^{R} [\langle F_r(x)f_r, \ell_r\rangle + \langle F_r(-x)d_r, \ell_r\rangle] + c. \qquad (3.1)$$

Here $\varphi \in \mathrm{Hom}(\mathcal{G}, \mathfrak{F}^{*m_1})$, $S(x) = \sum_{k=0}^{m_1-1} H^k(x,x)/(2k+1)!$, $T(x) = \sum_{k=0}^{m_1-1} H^k(x,x)/(2k+2)!$, where for each $y \in \mathcal{G}$ $H(\cdot,y) \in \mathrm{Hom}(\mathcal{G}, \mathfrak{F}_{m_1})$, $H(x,y) = H(y,x)$, $H^{m_1}(x,y) = 0$ if $m_1 \geq 1$, $H^2(x,y) = H(x,x)\,H(y,y)$, $H^t(x,x)\varphi(y) = H^t(x,y)\varphi(x)$ for all x,y; F_r, $r = 2,\ldots,R$ are pairwise inequivalent matrix representations of the group $\mathcal{G}$ of degree m_r, all eigenvalues of F_r are equal and different from one; Q_r are invertible linear operators from $\mathfrak{F}^{*m_r}$ to $\mathfrak{F}^{m_r}$, $H(x,x)Q_1 = Q_1H^t(x,x)$, $F_r(x)Q_r = Q_rF_r^t(x)$, $r = 2,\ldots,R$; $f_1 \in \mathfrak{F}^{m_1}$ $\ell_r \in \mathfrak{F}^{*m_r}$, $f_r, d_r \in \mathfrak{F}^{m_r}$, $r = 2,\ldots,R$, $f_r + d_r = 2Q_r\ell_r$, $r = 2,\ldots,R$, $c \in \mathfrak{F}$. Also the vectors $C(x)f_1 + S(x)Q_1\varphi(x)$, $x \in \mathcal{G}$ span $\mathfrak{F}^{m_1}$ and the vectors $S^t(x)\varphi(x)$ span

$\mathfrak{F}^{*m_1}$, $C(x) = \sum_{k=0}^{m_1-1} H^k(x,x)/(2k)!$; the spaces $\mathfrak{F}^{*m_r}$ and $\mathfrak{F}^{m_r}$, $r = 2,\ldots,R$ are spanned by the vectors $[F_r^t(x)-F_r^t(-x)]\ell_r$, $x\in G$ and by the vectors $F_r(x)f_r-F_r(-x)d_r$ $x\in G$, correspondingly. The representation (3.1) is unique up to equivalence for matrices $H(x,x)$ and $F_r(x)$, r $2,\ldots,R$.

We do not prove the next Theorem 2 since its proof is analogous to that of Theorem 1.

Theorem 2. Under assumptions of Theorem 1 a function g is a solution of the equation (1.3) with linearly independent functions u_i, v_1 $i = 1,\ldots,p$ if, and only if, there exist nonnegative integers $p_1,\ldots,p_R$, $p_1 + \ldots + p_R = p$, such that

$$g(x) = \langle C(x)Q_1g_1, a_1\rangle + \langle S(x)\psi(x), a_1\rangle$$

$$+ \sum_{r=2}^{R} [\langle F_r(x)g_r, a_r\rangle + \langle F_r(-x)b_r, a_r\rangle] + c.$$

Here $C(x)$, $S(x)$, $F_r(x)$, Q_r and Q_r have the same meaning as in Theorem 1 with m_r replaced by p_r, $\psi\in \mathrm{Hom}(G, \mathfrak{F}^{p_1})$ b_r, $g_r\in\mathfrak{F}^{m_r}$, $g_r-b_r = 2Q_ra_r$, $a_r\in\mathfrak{F}^{*m_r}$, $r = 2,\ldots,R$, $c\in\mathfrak{F}$. The vectors $C^t(x)a_1$, $x\in G$ span $\mathfrak{F}^{*p_1}$, and the vectors $C(x)a_1 + S(x)\psi(x)$, $x\in G$, span $\mathfrak{F}^{p_1}$; the spaces $\mathfrak{F}^{p_r}$ and $\mathfrak{F}^{*p_r}$ $r = 2,\ldots,R$ are spanned by the vectors $F_r(x)g_r + F_r(-x)b_r$ and by the vectors, $[F_r^t(x)+F_r^t(-x)]a_r$ correspondingly. The matrix functions $H(x,x)$ and $F_r(x)$ $r = 2,\ldots,R$ are defined uniquely up to equivalence.

Proof of the Theorem 1. The functional equation (1.2) rewritten in the from (2.1) implies that for all x,y,z

$$f(x+y+z)-f(x-y-z) = \langle h(x),\ k(y+z)\rangle$$

and

$$f(x+y-z)-f(x-y+z) = \langle h(x),\ k(y-z)\rangle.$$

Combining these formulas one obtains

$$\langle h(x), k(y+z)-k(y-z)\rangle = f(x+y+z)-f(x+y-z) + f(x-y+z)-f(x-y-z)$$
$$= \langle h(x+y) + h(x-y), k(z)\rangle.$$

Since the functions $k_j(z)$, $j = 1,\ldots,m$ are linearly independent there exist $z_j \in G$ such that the vectors $k(z_j) \in \mathfrak{F}^{*m}$ $j = 1,\ldots,m$ are linearly independent. If the linear operator $A(y)$ in $\mathfrak{F}^{*m}$ is defined by the formula

$$k(y+z_j)-k(y-z_j) = 2A(y)k(z_j) \qquad j = 1,\ldots,m ,$$

then for all x,y

$$h(x+y) + h(x-y) = 2A^t(y)h(x). \tag{3.2}$$

We also deduce

$$\langle h(x), k(y+z)-k(y-z)\rangle = \langle h(x), 2A(y)k(z)\rangle,$$

what because of linear independence of $h_j(x)$, $j = 1,\ldots,m$ implies that

$$k(y+z)-k(y-z) = 2A(y)k(z). \tag{3.3}$$

Since $k(-z) = -k(z)$ it is clear that $A(-y) = A(y)$ for all y. Also

$$2A(x)A(y)k(z) = A(x)[k(y+z)-k(y-z)]$$
$$= [k(x+y+z)-k(x-y-z)-k(x+y-z) + k(x-y+z)]/2$$
$$= [A(x+y) + A(x-y)]k(z).$$

Thus the matrices $A(x)$ satisfy D'Alembert's functional equation

$$A(x+y) + A(x-y) = 2A(x)A(y). \tag{3.4}$$

An immediate consequence of (3.4) is that al matrices $A(x)$ commute. It is known (see [11 p. 16]) that the whole space $\mathfrak{F}^{*m}$ can be represented as a direct sum of invariant subspaces W_r, with respect to all $A(x)$, for $r = 1,\ldots,R$. The irreducible $A(x)|W_r$ are equivalent, while for $r \neq s$ the irreducible parts of $A(x)|W_r$ and $A(x)|W_s$ are not equivalent. Since the field $\mathfrak{F}$ is algebraically closed Shur's lemma shows that all irreducible parts of $A(x)|W_r$, $r = 1,\ldots,R$, are one-dimensional operators. Thus all matrices $A(t)$ have the form $A(x) = T^{-1}B(x)T$, where $B(x)$ is a quasi-diagonal matrix with blocks $B_1(x),\ldots,B_R(x)$ on the principal diagonal, and $B_r(x)$ is a lower triangular matrix of dimension $m_r = \dim W_r$, $r = 1,\ldots,R$ with the same diagonal elements $b^{(r)}(x)$, $b^{(r)}(x) \neq b^{(s)}(x)$, $r \neq s$. Clearly $m = m_1 + \ldots + m_R$ and all matrices $B_r(x)$, $r = 1,\ldots,R$ commute.

Returning to (3.2) and (3.3) we see that if $Tk(y) = \ell(y)$, $T^t w(x) = h(x)$, then

$$\ell(y+z)-\ell(y-z) = 2B(y)\ell(z)$$

and

$$w(x+y) + w(x-y) = 2B^t(y)w(x).$$

Moreover

$$\langle h(x),k(y)\rangle = \langle w(x),\ell(y)\rangle.$$

Let $\ell(y) = \ell_1(y) \oplus \ldots \oplus \ell_R(y)$ with $\ell_r \in \mathfrak{F}^{*m_r}$, and $w(x) = w_1(x) \oplus \ldots \oplus w_r(x)$ with $w_r \in \mathfrak{F}^{m_r}$, $r = 1,\ldots,R$, be partitions of $\ell(y)$ and $w(x)$ into direct sums corresponding to that of the matrix $B(x)$. Then

$$\langle w(x), \ell(y)\rangle = \sum_{r=1}^{k} \langle w_r(x), \ell_r(y)\rangle \quad .$$

Also

$$\ell_r(x+y) + \ell_r(x-y) = 2B_r(y)\ell_r(x), \tag{3.5}$$

and

$$w_r(x+y) + w_r(x-y) = 2B_r^t(y)w_r(x). \tag{3.6}$$

Note that if $f_1(x) = [f(x) + f(-x)]/2$, then

$$f_1(x+y)-f_1(x-y) = \langle d(x), \ell(y)\rangle,$$

where $d(x) = [w(x)-w(-x)]/2$. It follows that

$$\langle d(x), \ell(y)\rangle = \langle d(y), \ell(x)\rangle.$$

Therefore there exists an invertible linear operator Q from $\mathfrak{F}^{*m}$ to $\mathfrak{F}^m$ such that $Q^t = Q$ and for all $x \in \mathcal{G}$

$$d(x) = Q\ell(x).$$

It is easy to see that $QB(x) = B^t(x)Q$, and because of Shur's lemma $Q = Q_1 \oplus \ldots \oplus Q_R$ where Q_r is of dimension m_r, and $Q_r B_r(x) = B_r^t(x) Q_r$, $r = 1,\ldots,R$. Also,

$$[w_r(x)-w_r(-x)]/2 = Q_r \ell_r(x) \quad r = 1,\ldots,R.$$

It follows from (3.4)

$$B_r(x+y) + B_r(x-y) = 2B_r(x)B_r(y) \quad r = 1,\ldots,R \tag{3.7}$$

so that in particular

$$b^{(r)}(x+y) + b^{(r)}(x-y) = 2b^{(r)}(x)b^{(r)}(y).$$

All solutions of this D'Alembert's functional equation are known to be of the form (cf. [5])

$$b^{(r)}(x) = [\chi_r(x) + \chi_r(-x)]/2,$$

where χ_r is a multiplicative homomorphism of $\mathcal{G}$ into $\mathcal{F}$:

$$\chi_r(x+y) = \chi_r(x)\chi_r(y).$$

If χ_r is not identically one there exists $x_0 \in \mathcal{G}$ such that $\chi_r(2x_0) \neq 1$ and the matrix $B_r^2(x_0)-I = [B_r(2x_0)-I]/2$ is nonsingular. Moreover one can find a nonsingular lower triangular matrix G_r such that $G_r^2 = B_r^2(x_0)-I$. Indeed

$$B_r^2(x_0)-I = [(\chi_r(x_0)-\chi_r(-x_0))/2]^2[I + P_r]$$

where P_r is a nilpotent matrix, $P_r^{m_r} = 0$.

Thus one can put

$$G_r = [(\chi_r(x_0)-\chi_r(-x_0))/2][I + P_r/2 + \sum_{i=2}^{m_r-1} \frac{(-1)^{i+1}(2i-1)!!}{2^i \cdot i!} P_r^i] .$$

Clearly G_r commutes with all matrices $B_r(x)$ and $Q_r G_r = G_r^t Q_r$.

Now let

$$G_r(x) = G_r^{-1}[B_r(x)(G_r - B_r(x_0)) + B_r(x+x_0)]$$

$$= B_r(x) - G_r^{-1}[B_r(x)B_r(x_0) - B_r(x+x_0)].$$

It is easy to check (cf. [5]) that

$$G_r(x+y) = G_r(x)G_r(y),$$

and

$$Q_r G_r(x) = G_r^t(x) Q_r.$$

Evidently $G_r(x)$ and $G_s(x)$ are inequivalent for $r \neq s$ and

$$G_r(x) + G_r(-x) = 2B_r(x) - G_r^{-1}[2B_r(x)B_r(x_0) - B_r(x+x_0) - B_r(-x+x_0)] = 2B_r(x)$$

It is also clear that $G_r(x)$ is a lower triangular matrix with all diagonal elements (and hence eigenvalues) equal to $\chi_r(x)$.

It follows from (3.5)

$$\ell_r(x+y) + \ell_r(y-x) = 2B_r(x)\ell_r(y),$$

so that

$$\ell_r(x-y) = B_r(y)\ell_r(x) - B_r(x)\ell_r(y).$$

Using again (3.5) we see that

$$2B_r(x)[\ell_r(x+y) + \ell_r(x-y)] = 2B_r(y)\ell_r(2x)$$

$$= 2B_r(y)[B_r(x-y)\ell_r(x+y) + B_r(x+y)\ell_r(x-y)].$$

Now one deduces from (3.7)

$$B_r(y)B_r(x-y) = [B_r(x) + B_r(x-2y)]/2,$$

and

$$B_r(y)B_r(x+y) = [B_r(x) + B_r(x+2y)]/2.$$

Thus

$$[B_r(x) - B_r(x-2y)]\ell_r(x+y) = -[B_r(x) - B_r(x+2y)]\ell_r(x-y).$$

It is easy to check that

$$B_r(x) - B_r(x-2y) = [G_r(y) - G_r(-y)][G_r(x-y) - G_r(-x+y)]/2,$$

and

$$-B_r(x) + B_r(x+2y) = [G_r(y)-G_r(-y)][G_r(x+y)-G_r(-x-y)]/2.$$

Let $K_r = \{x: \chi_r(2x) = 1\}$. If $x \notin K_r$ the matrix $G_r(x)-G_r(-x)$ is nonsingular. Thus if $y \notin K_r$ $[G_r(x+y)-G_r(-x-y)]\ell_r(x-y) = [G_r(x-y)-G_r(-x+y)]\ell_r(x+y)$. It follows that the relations $x+y \notin K_r$ and $x-y \notin K_r$ imply

$$[G_r(x+y)-G_r(-x-y)]^{-1}\ell_r(x+y) = [G_r(x-y)-G_r(-x+y)]^{-1}\ell_r(x-y).$$

In other words for $z \notin K_r$

$$\ell_r(z) = [G_r(z)-G_r(-z)]\ell_r \tag{3.8}$$

with some vector ℓ_r if z has the form $z = x+y$ with $y \notin K_r$ and $x-y \notin K_r$ or $z = x+2y$, x, $y \notin K_r$. We prove now that every element $z \notin K_r$ has this form.

If there exists $x_0 \in K_r$ such that $\chi_r(x_0) \neq 1$ we put $z = (z+x_0)-x_0$. Clearly $z + x_0 \notin K_r$ and $x_0/2 \notin K_r$. If for all $x \in K_r$ one has $\chi_r(x) = 1$, then we show that $z = x+y$ with $x,y \notin K_r$. Indeed in this case it suffices to take $x = y = z/2$.

Thus (3.8) holds for all $z \notin K_r$. We prove now that (3.8) is valid for all $z \in \mathcal{G}$. Let $z \in K_r$, $x \notin K_r$, then $x + z \notin K_r$ and $x-z \notin K_r$. Therefore

$$\ell_r(z+x) + \ell_r(z-x) = [G_r(x+z)-G_r(-x-z) + G_r(-z-x)-G_r(x-z)]\ell_r$$

$$= 2B_r(x)[G_r(z)-G_r(-z)]\ell_r.$$

From this relation and (3.5) it follows that (3.8) holds if there exists $x \notin K_r$ such that the matrix $B_r(x)$ is nonsingular. The latter condition is met if $2x \notin K_r$. If $2x \in K_r$ for all $x \in \mathcal{G}$, then because of the condition $2\mathcal{G} = \mathcal{G}$ it follows $x \in K_r$ for all x. Thus $\chi_r(x) = 1$ for all x contrary to our assumption. Thus (3.8) is true for all $z \in \mathcal{G}$.

From the relation (3.6) it follows

$$w_r(y) + w_r(-y) = 2B_r^t(y)w_r(0) = [G_r^t(y) + G_r^t(-y)]w_r(0),$$

and

$$w_r(x+y)-w_r(-x-y) + w_r(x-y)-w_r(y-x) = 2B_r^t(y)[w_r(x)-w_r(-x)].$$

Thus if $d_r(x) = [w_r(x)-w_r(-x)]/2$, then $d_r(-x) = -d_r(x)$ and

$$d_r(x+y) + d_r(x-y) = d_r(x+y)-d_r(y-x) = 2B_r^t(y)d_r(x).$$

The latter equation is of the form (3.5) so that the result just obtained shows that for some vector $\tilde{d}_r$

$$d_r(x) = [G_r^t(x)-G_r^t(-x)]\tilde{d}_r.$$

We also know that $d_r(x) = Q_r\ell_r(x)$, i.e. $\tilde{d}_r = Q_r\ell_r$.

Thus

$$w_r(x) = [w_r(x) + w_r(-x)]/2 + d_r(x)$$

$$= \frac{1}{2}[G_r^t(x) + G_r^t(-x)]w_r(0) + [G_r^t(x)-G_r^t(-x)]\tilde{d}_r.$$

Therefore

$$\langle w_r(x),\ell_r(y)\rangle = \langle G_r^t(x)f_r-G_r^t(-x)d_r,[G_r(y)-G_r(-y)]\ell_r\rangle$$

with some vectors $f_r,d_r \in \mathfrak{F}^{m_r}$ and $\ell_r \in \mathfrak{F}^{*m_r}$, $f_r + d_r = 2Q_r\ell_r$.

Now we have to consider the more difficult situation when some χ_r, say χ_1, is identically equal to one. In this case $B_1(x) = I + N(x)$, where $N^q(x) = 0$, $q = m_1$.

Thus

$$\ell_1(x+y) + \ell_1(x-y)-2\ell_1(x) = 2N(y)\ell_1(x) \tag{3.9}$$

and

$$N(x+y) + N(x-y)-2N(x) = 2N(y) + 2N(x)N(y).$$

The latter identity can be rewritten

$$[L(y/2)-L(-y/2)]^2N(x) = 2N(y)[I + N(x)].$$

Easy induction shows that for $k = 1,2,\ldots$

$$[L(y/2)-L(-y/2)]^{2k}N(x) = 2^k N^k(y)[I + N(x)].$$

Thus in particular

$$[L(y/2)-L(-y/2)]^{2q-2}N(x) = 2^{q-1}N^{q-1}(y),$$

which implies

$$[I-L(y)]^{2q-1}N(x) = 0,$$

i.e. $N(x)$ is a polynomial of degree $2q-2$. Because of the result mentioned in Section 2

$$N(x) = N_{2q-2}(x) + \ldots + N_2(x),$$

where for $k = 1,\ldots,q-1$

$$[L(y/2)-L(-y/2)]^{2k}N_{2k}(x) = (2k)!N_{2k}(y),$$

i.e. $N_{2k}(x)$ is a homogeneous polynomial of degree $2k$, $N_{2k}(nx) = n^{2k}N_{2k}(x)$. These polynomials are defined by the formulas

$$N_{2q-2}(x) = \frac{1}{(2q-2)!}[L(x/2)-L(-x/2)]^{2q-2}N(\cdot),$$

and for $k = q-2,\ldots,1$

$$N_{2k}(x) = \frac{1}{(2k)!}[L(x/2)-L(-x/2)]^{2k}[N(\cdot)-N_{2q-2}(\cdot)-\ldots-N_{2k+2}(\cdot)] .$$

We prove at first that

$$N_{2k}(x) = \sum_{j=k}^{q-1} d_{jk}[2N(x)]^j/(2j)! ,$$

where the coefficients d_{jk} can be found in the following way. If D is the lower triangular matrix formed by d_{jk} $k \leq j$, then $D = P^{-1}$ here the elements p_{jk} of P have the form

$$p_{jk} = \frac{1}{(2k)!}\sum_{i=0}^{2k}\binom{2k}{i}(-1)^i(i-k)^{2j}.$$

(Clearly $p_{jk} = 0$ if $k > j$).

Indeed

$$N_{2q-2}(x) = \frac{1}{(2q-2)!}[L(x/2)-L(-x/2)]^{2q-2}N(\cdot) = \frac{[2N(x)]^{q-1}}{(2q-2)!} ,$$

so that $d_{q-1\ q-1} = 1$.

Also,

$$[L(x/2)-L(-x/2)]^{2k}N_{2j}(0) = \sum_{i=0}^{2k} \binom{2k}{i}(-1)^i N_{2j}((k-i)x)$$

$$= \sum_{i=0}^{2k} \binom{2k}{i}(-1)^i (i-k)^{2j} N_{2j}(x)$$

$$= (2k)!\,p_{jk}N_{2j}(x).$$

Thus

$$N_{2k}(x) = \frac{1}{(2k)!}[L(x/2)-L(-x/2)]^{2k}[N(0)-N_{2q-2}(0)-\ldots-N_{2k+2}(0)]$$

$$= [2N(x)]^k/(2k)! - \sum_{j=k+1}^{q-1} p_{jk}N_{2j}(x) \tag{3.10}$$

and it follows by induction that

$$d_{jk} = -\sum_{i=k+1}^{j} d_{ji}\ p_{ik} \quad , \quad j > k$$

$$d_{jj} = 1 \ .$$

But these identities mean $D = -D(P-I) + I$ or $DP = I$.

We prove now that

$$[L(y/2)-L(-y/2)]^2 N_{2k}(x) = 2[N_{2k}(y) + \sum_{j<k} N_{2j}(y)N_{2(k-j)}(x)] \ .$$

Indeed

$$[L(y/2)-L(-y/2)]^2[2N(x)]^k \ell_1(z)$$

$$= [L(y/2)-L(-y/2)]^2 \sum_{i=0}^{2k} \binom{2k}{i}(-1)^i \ell_1(z+(i-k)x)$$

$$= \sum_{i=0}^{2k} \binom{2k}{i}(-1)^i 2N((i-k)y)\ell_1(z+(i-k)x)$$

$$= \sum_{i=0}^{k-1} \binom{2k}{i}(-1)^i 2N((i-k)y)[\ell_1(z+(i-k)x)+\ell_1(z-(i-k)x)]$$

$$= \sum_{i=0}^{k-1} \binom{2k}{i}(-1)^i 4N((i-k)y)N((i-k)x)\ell_1(z)$$

$$+ \sum_{i=0}^{k-1} \binom{2k}{i}(-1)^i b_N((i-k)y)\ell_1(z)$$

$$= \sum_{i=0}^{2k} \binom{2k}{i}(-1)^i 2N((i-k)y)N((i-k)x)\ell_1(z)$$

$$+ \sum_{i=0}^{2k} \binom{2k}{i}(-1)^i 2N((i-k)y)\ell_1(z).$$

Therefore

$$[L(y/2)-L(-y/2)]^2[2N(x)]^k = 2(2k)! \sum_{j,i} p_{j+ik} N_{2j}(y)N_{2i}(x)+2(2k)!\sum_j p_{jk}N_{2j}(y)$$

and

$$[L(y/2)-L(-y/2)]^2 N_{2k}(x) = \sum_{j=k}^{q-1} d_{jk}[L(y/2)-L(-y/2)]^2 \frac{[2N(x)]^j}{(2j)!}$$

$$= 2\sum_{j=k}^{q-1} d_{jk} \sum_{n,i} p_{n+ij}N_{2n}(y)N_{2i}(x)+2\sum_{j=k}^{q-1} d_{jk}\sum_i p_{ij}N_{2i}(y)$$

$$= 2[\sum_{i<k} N_{2i}(y)N_{2(k-i)}(y) + N_{2k}(y)].$$

Using (3.10) repeatedly we can now establish the following formula

$$[L(y/2)-L(-y/2)]^{2k}N_{2k}(\cdot) = 2^k \sum_{j_1+\ldots+j_k=k} N_{2j_1}(y)\ldots N_{2j_k}(y)$$

$$= [2N_2(y)]^k, \qquad (3.11)$$

which gives the basic result:

$$N_{2k}(y) = \frac{1}{(2k)!}[L(y/2)-L(-y/2)]^{2k}N_{2k}(\cdot) = \frac{[2N_2(y)]^k}{(2k)!}.$$

Note that there exists a function $M(x,y)$ on $\mathcal{G}\times\mathcal{G}$ with values in $\mathcal{F}_q$ such that

(i) $2N_2(x) = M(x,x)$,

(ii) $M(x_1+x_2,y) = M(x_1,y) + M(x_2,y)$,

(iii) $M(x,y) = M(y,x)$,

(iv) $Q_1M(x,x) = M^t(x,x)Q_1$,

(v) $M^q(x,x) = 0$,

(vi) $M^2(x,y) = M(x,x)M(y,y)$.

The last formula follows from (3.11) for $k = 2$. Now we return to the equation (3.9) which can be rewritten in the following form

$$[L(y/2)-L(-y/2)]\,\ell_1(x) = 2N(y)\ell_1(x).$$

It is easy to check that for $k = 1,2,\ldots$

$$[L(y/2)-L(-y/2)]^{2k}\ell_1(x) = [2N(y)]^k\ell_1(x).$$

Thus

$$[L(y/2)-L(-y/2)]^{2q}\ell_1(x) = 0,$$

and $\ell_1(x)$ is a polynomial of degree $2q-1$.

Analogously to previous considerations

$$\ell_1(x) = \varphi_{2q-1}(x) + \ldots + \varphi_1(x),$$

where $\varphi_{2k+1}(x)$ is an homogeneous polynomial of degree $2k+1$, $\varphi_{2k+1}(nx) = n^{2k+1}\varphi_{2k+1}(x)$.

Note that if $2x = 2y$ then $\varphi_{2k+1}(x) = \varphi_{2k+1}(y)$, $k = 0,1,\ldots,q-1$. Thus the function $\ell(x) = 2\ell_1(x/2)$ is defined.

Similarly to (3.10) we prove

$$\varphi_{2k+1}(x) = \sum_{j=k}^{q-1} c_{jk}[(2j+1)!]^{-1}[2N(x)]^j \ell(x), \tag{3.12}$$

where the lower triangular matrix C formed by the coefficients c_{jk}, $k \leq j$ has the form $C = V^{-1}$. Here V is the matrix with elements

$$v_{jk} = \frac{2}{(2k+1)!} \sum_{i=0}^{2k} \binom{2k}{i}(-1)^i (i-k+1/2)^{2j+1} .$$

Clearly $v_{jk} = 0$ if $k > j$.

Also

$$[L(y/2)-L(-y/2)]^2[2N(x)]^j \ell(x)$$

$$= 2[L(y/2)-L(-y/2)]^2 \sum_{i=0}^{2j} \binom{2j}{i}(-1)^i \ell_1((i-j)x+x/2)$$

$$= 2 \sum_{i=0}^{2j} \binom{2j}{i}(-1)^i 2N((i-j+1/2)y)\ell_1((i-j+1/2)x)$$

$$= 4 \sum_{i=0}^{2j} \binom{2j}{i}(-1)^i \sum_{n,k} N_{2n}(y)\varphi_{2k+1}(x)(i-j+1/2)^{2n+2k+1},$$

so that

$$[L(y/2)-L(-y/2)]^2 \varphi_{2k+1}(x) = 2 \sum_{j \geq k} c_{jk} v_{n+i\,j} \sum_{n,i} N_{2n}(y)\varphi_{2i+1}(x)$$

$$= 2 \sum N_{2(k-i)}(y)\varphi_{2i+1}(x) .$$

Using this identity repeatedly one obtains

$$[L(y/2)-L(-y/2)]^{2k} \varphi_{2k+1}(x) = 2^k \sum_{i_1+\ldots+i_k+i_{k+1}=k} N_{2i_1}(y)\ldots N_{2i_k}(y)\varphi_{2i_{k+1}+1}(x)$$

$$= [2N_2(y)]^k \varphi_1(x) ,$$

and

$$[L(y/2)-L(-y/2)]^{2k+1}\varphi_{2k+1}(x) = [2N_2(y)]^k\varphi_1(y). \tag{3.13}$$

Therefore

$$\varphi_{2k+1}(x) = \frac{[2N_2(x)]^k}{(2k+1)!}\varphi_1(x), \qquad k = 0,1,\ldots,q-1,$$

and

$$\ell_1(x) = \sum_{k=0}^{q-1} \frac{[2N_2(x)]^k}{(2k+1)!}\varphi_1(x).$$

The relation (3.13) for k = 1 implies that

$$M(x,y)\varphi_1(y) = M(y,y)\varphi_1(x).$$

Now we are able to give the formula for the function $w_1(x)$. Since $d_1(x) = [w_1(x)-w_1(-x)]$ satisfies equation (3.9) with N replaced by N^t,

$$d_1(x) = \sum_{k=0}^{q-1} \frac{[M^t(x,x)]^k}{(2k+1)!}\psi_1(x) .$$

Here $\psi_1 \in \mathrm{Hom}\,(\mathcal{G}, \mathcal{F}^q)$. Since $d_1(x) = Q_1\ell_1(x)$ we deduce

$$\psi_1(x) = Q_1\varphi_1(x).$$

Also

$$w_1(x) + w_1(-x) = 2(I + N^t(x))w_1(0),$$

so that

$$w_1(x) = 2\sum_{k=0}^{q-1} \frac{[M^t(x,x)]^k}{(2k)!} f_1 + 2\sum_{k=0}^{q-1} \frac{[M^t(x,x)]^k}{(2k+1)!} Q_1\varphi(x).$$

The desired formula for the function f is obtained from the identity

$$f(x)-f(0) = \langle h(x/2),k(x/2)\rangle$$

$$= \langle \sum_{k=0}^{m_1-1} \frac{[M^t(x/2,x/2)]^k}{(2k)!} f_1 , \sum_{i=0}^{m_1-1} \frac{M^i(x/2,x/2)}{(2i+1)!} \varphi(x)\rangle$$

$$+ \frac{1}{2} \langle \sum_{k=0}^{m_1-1} \frac{[M^t(x/2,x/2)]^k}{(2k+1)!} Q_1\varphi(x), \sum_{i=0}^{m_1-1} \frac{M^i(x/2,x/2)}{(2i+1)!} \varphi(x)\rangle$$

$$+ \sum_{r=2}^{R} \langle G_r^t(x/2)f_r - G_r^t(-x/2)d_r, [G_r(x/2)-G_r(-x/2)]\ell_r\rangle$$

$$= \langle \sum_{k=0}^{m_1-1} \frac{[M^t(x,x)]^k}{(2k+1)!} f_1 ; \varphi(x)\rangle + \langle \sum_{k=0}^{m_1-1} \frac{[M^t(x,x)]^k}{(2k+2)!} Q_1\varphi(x) ; \varphi(x)\rangle$$

$$+ \sum_{r=2}^{R} [\langle f_r, G_r(x)\ell_r\rangle + \langle d_r, G_r(-x)\ell_r\rangle + \langle f_r - d_r, \ell_r\rangle] .$$

The formula (3.1) follows with $H(x,x) = M^t(x,x)$ and $F_r(x) = G_r^t(x)$.

We prove now that every function f of the form (3.1) satisfies the equation (1.3). Note that for $k \geq 1$

$$H^k(x+y,x+y)-H^k(x-y,x-y)$$

$$= 2 \sum_{i:k-i \text{ odd}} \binom{k}{i}[H(x,x)+H(y,y)]^i[2H(x,y)]^{k-i}$$

$$= 2 \sum_{i:k-i \text{ odd}, j\leq i} \binom{k}{i}\binom{i}{j}[H(x,x)]^{j+(k-i-1)/2}[H(y,y)]^{i-j+(k-i-1)/2}2^{k-i}H(x,y)$$

$$= 2 \sum_{i+1\leq k} \binom{2k}{2i+1}[H(x,x)]^i[H(y,y)]^{k-1-i}H(x,y) . \qquad (3.14)$$

The last identity follows from the formula

$$\sum_{\substack{i:k-i \text{ odd} \\ 2j+k-i=2p-1}} \binom{k}{i}\binom{i}{j}2^{k-i} = \binom{2k}{2p-1} ,$$

which is easily obtained by comparison of coefficients of $a^{2p}b^{2k-2p}$ in expansions $(a^2+b^2+2ab)^k-(a^2+b^2-2ab)^k$ and $(a+b)^{2k}-(a-b)^{2k}$.

Also,

$$H^k(x+y,x+y) + H^k(x-y,x-y) = 2 \sum_{i \le k} \binom{2k}{2i} H^i(x,x)H^{k-i}(y,y).$$

Using this formula, (3.14) and properties of the function H one obtains

$$f(x+y)-f(x-y) = 2 \left\langle \sum_{k=0}^{m_1-1} \frac{H^k(x,x)}{(2k)!} f_1 + \sum_{k=0}^{m_1-1} \frac{H^k(x,x)}{(2k+1)!} (x), \sum_{i=0}^{m_1-1} \frac{[H^t(y,y)]^i}{(2i+1)!} \varphi(y) \right\rangle$$

$$+ \sum_{r=2}^{R} \langle F_r(x)f_r - F_r(-x)d_r, [F_r^t(y)-F_r^t(-y)]\ell_r \rangle . \qquad (3.15)$$

Thus (1.2) holds and the statements of the Theorem 1 about vectors $S^t(x)\varphi(x)$, $C(x)f_1 + S(x)Q_1\varphi(x)$, $F_r(x)f_r - F_r(-x)d_r$ and $[F_r^t(x)-F_r^t(-x)]\ell_r$, $x \in \mathcal{G}$, $r = 2,\ldots,R$ follow from the assumed linear independence of functions h_j, k_j, $j = 1,\ldots,m$. The uniqueness up to equivalence of the matrices in formula (3.1) is a corollary of the uniqueness of the decomposition of the space $\mathfrak{F}^m$ into direct sum of subspaces invariant with respect to commuting matrices A(x) from (3.4).

Remark 1. If $\mathcal{G}$ is a topological group and f (or g) is assumed to be a continuous function, then the condition $2\mathcal{G} = \mathcal{G}$ of the Theorem 1 (or 2) can be replaced by the following one: the subgroup $2\mathcal{G}$ is dense in $\mathcal{G}$. Incidentally, this condition means that the dual group does not have elements of order two.

Remark 2. Theorems 1 and 2 are true if $\mathfrak{F}$ is not algebraically closed field. In this case all homomorphisms from $\mathcal{G}$ into corresponding vector spaces over $\mathfrak{F}$ should be replaced by homomorphisms from $\mathcal{G}$ into vector spaces over a finite extension of the field $\mathfrak{F}$. Of course if $\mathfrak{F}$ is the field of reals, this extension coincides with the field of complex numbers.

For instance, any solution of the classical D'Alembert's equation (1.4) has the form $[\chi(x)+\chi(-x)]/2$ where χ is a multiplicative homomorphism into a simple extension of the initial field $\mathfrak{F}$.

Remark 3. The general form of a solution of (1.1) easily follows from Theorems 1 and 2.

4. DISCUSSION

If f is a solution of the fuctional equation (1.2) let V = V(f) denote the vector space spanned by the translates of f. Then (1.2) means that

$$[L(y)-L(-y)]f = \sum_{j=1}^{m} u_i\ (y)v_j,$$

where f denotes the cyclic vector of the representation L (which corresponds to the function f as an element of V) and $v_1,\ldots,v_m$ are some vectors from V. This fact implies that the linear subspace V_- of V spanned by the vectors $[L(x)-L(-x)]f$, $x \in \mathcal{G}$ is of dimension m.

Under this interpretation the operator A(x) introduced in the proof of Theorem 1 is just the restriction of $[L(x)+L(-x)]$ onto V. The functional equation (3.4) is an immediate corollary of this fact.

Thus every solution f of (1.2) has the form

$$f(x) = \langle L(x)f, \Delta \rangle . \qquad (4.1)$$

Here L is a cyclic representation of the group $\mathcal{G}$ in the space V with a cyclic vector f, and the space V_- spanned by the vectors $[L(x)-L(-x)]f$, $x \in \mathcal{G}$ has dimension m. The element Δ of the dual space V* is a cyclic vector for the conjugate representation L*, $L^*(\cdot) = L^t(-x)$. (Indeed we define Δ in the following way: $\langle h, \Delta \rangle = h(0)$ for all h from V. Then $\langle h, L^*(x)\Delta \rangle = h(-x)$ and the vectors $L^*(x)$ $x \in \mathcal{G}$ must span the whole space V*.) Clearly the representation L under these conditions is defined uniquely up to equivalence. A natural question is if the representation L is finite dimensional. Bounds for the dimension of L in terms of m are also of interest. The same question can be formulated for the functional equation (1.3).

It was proven in [8] that for both equations the space V is finite dimensional if $\mathcal{G}$ is a compact group. In the noncompact Abelian case the situation for the equations (1.2) and (1.3) is different. Here is an example of a solution to (1.3) with infinite dimensional representation L.

Let $\mathcal{G}$ be an infinite dimensional Hilbert space, $g(x) = ||x||^2$. Then

$$g(x+y) + g(x-y) = 2(||x||^2 + ||y||^2) ,$$

so that (1.3) holds, and the dimension of the subspace V_+ spanned by the vectors $[L(x)+L(-x)]g$, $x \in \mathcal{G}$ is two.

However

$$g(x+y)-g(x-y) = 4 < x,y > ,$$

and the space V_- is infinite dimensional one. Therefore $V = V(g)$ is an infinite dimensional space as well.

Note that in this example the homomorphism ψ of Theorem 2 is zero. Also note that if g is an odd function, $g(-x) = -g(x)$, and g satisfies (1.3), then

$$g(x+y)-g(x-y) = g(x+y) + g(y-x) = < u(y), v(x) >$$

so that g also satisfies (1.2). Thus both spaces V_+ and V_- have dimension p, and the dimension of V does not exceed 2p. Of course the same remark refers to equation (1.2).

Now let f be a solution of (1.2). Then f has the form (3.1), and

$$\begin{aligned} f(x+y)+f(x-y) = {} & 2<C(y)f_1, S^t(x)\varphi(x)> \\ & + 2<H(x,y)T(x)Q_1\varphi(x), T^t(y)\varphi(y)> + 2<T(x)Q_1\varphi(x), \varphi(y)> \\ & + 2<T(y)Q_1\varphi(y), \varphi(y)> + \sum_{r=2}^{R} <F_r(x)f_r + F_r(-x)d_r, [F_r^t(x)+F_r^t(-x)]\varphi_r> . \end{aligned} \tag{4.2}$$

The proof of (4.2) is analogous to that of the identity (3.15).

Note that the second term in (4.1) has the form

$$<H(x,y)T(x)Q_1\varphi(x), T^t(y)\varphi(y)> = <H(x,x)T(x)Q_1\varphi(y), T^t(y)\varphi(y)>$$

$$= \sum_{i,j=1}^{m_1} \alpha_{ij}(x)\varphi_i(y)n_j(y),$$

where $\alpha_{ij}(x)$ are elements of the matrix $H(x,x)T(x)Q_1$ and $\varphi_i(y)$ and $n_j(y)$ are coordinates of the functions $\varphi(y)$ and $T^t(y)\varphi(y)$.

Therefore the dimension of the space V_+ does not exceed

$$m_1 + m_1^2 + 2 + \sum_{r=2}^{R} m_r = m_1^2 + m + 2.$$

Thus

$$\dim V(f) \leq m_1^2 + 2m + 2 \leq m^2 + 2m + 2,$$

and the next result follows.

Theorem 3. Every solution f of the equation (1.2) has the form (4.1) with a finite dimensional representation L, $\dim L \leq m^2 + m + 2$. The representation L is defined uniquely up to equivalence.

Theorem 4. Every solution g of the equation (1.3) has the form (4.1) with a finite dimensional representation L under one of the two following conditions:

(i) $g(-x) = -g(x), \quad x \in \mathcal{G}$,

(ii) $\dim \mathrm{Hom}(\mathcal{G}, \mathfrak{F}^n) = \rho_n < \infty$ for $n = 1,2,\ldots$

Under condition (i) $\dim L \leq 2p$; under condition (ii) $\dim L \leq p(\rho_\rho+2)$.

The proof of Theorem 4 under condition (ii) follows from the following formula valid for any solution of (1.3)

$$g(x+y)-g(x-y) = 2<H(x,y)S(x)Q_1a_1, S^t(y)a_1>+2<S(y)\psi(y), C^t(x)a_1>$$

$$+ \sum_{r=2}^{R} <[F_r(x)g_r-F_r(-x)b_r, [F_r^t(y)-F_r^t(-y)]a_r> .$$

This identity implies, that the dimension of the subspace V_- is less or equal to $p_1\rho_{p_1} + p_1 + \sum_{r=2}^{R} p_r = p + p_1\rho_{p_1}$.

Therefore

$$\dim V(g) \leq p + p_1(\rho_{p_1}+1) \leq p(\rho_p+2),$$

and Theorem 2 follows.

Assume now that $\mathcal{G}$ is a topological group and continuous solutions of equations (1.2) and (1.3) are considered. The $\varphi(x) = 0$ for all x belonging to a compact subgroup of $\mathcal{G}$. Therefore the first term in the formula (3.1) vanishes if $\mathcal{G}$ is a compact group.

If the group $\mathcal{G}$ does not contain nontrivial compact groups, then any matrix homomorphism F(x) has the form $F(x) = \exp\{H(x)\}$ where $H \in \text{Hom}(\mathcal{G}, \mathfrak{I}_n)$. (cf. [3 p. 393] for one dimensional result.) In this case, the power series for, say, $[F(x)+F(-x)]/2$ bears some resemblance to the function C(x) and explains the structure of the latter.

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