Γ-MINIMAX AND MINIMAX DECISION RULES FOR COMPARISON OF TREATMENTS WITH A CONTROL*

by

Shanti S. Gupta and Woo-Chul Kim
Purdue University

Department of Statistics
Division of Mathematical Sciences
Mimeograph Series #79-10

August 1979

*This research was supported by the Office of Naval Research contract N00014-75-C-0455 at Purdue University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Γ-MINIMAX AND MINIMAX DECISION RULES

FOR COMPARISON OF TREATMENTS WITH A CONTROL*

by

Shanti S. Gupta and Woo-Chul Kim
Purdue University

1. Introduction.

In many fields of research one is faced with the problem of comparing k experimental categories with reference to a 'standard' or a 'control'. Following the initial investigation by Paulson (1952), this problem has been studied in several different formulations by Dunnett (1955), Gupta and Sobel (1958) and Lehmann (1961) among others.

Let $\pi_1,\ldots,\pi_k$ denote the k experimental categories or 'treatment' populations and let π_0 denote the 'control' population, where the quality of each population π_i is characterized by a real-valued parameter θ_i $(i = 0,1,\ldots,k)$. Each treatment population π_i is said to be 'superior', 'equivalent' or 'inferior' to the control population π_0 if $\theta_i-\theta_0 \geq \Delta$, $-\Delta < \theta_i - \theta_0 < \Delta$, $\theta_i-\theta_0 \leq -\Delta$, respectively, where Δ is a given positive constant. We consider a problem in which the treatment populations are to be classified as one of the above three cases based on the observations from the populations. Bhattacharyya (1956, 1958) studied this problem for the normal populations with unknown means when the control population is assumed known. A similar problem has been considered by Seeger (1972). We apply the Γ-minimax principle to this problem.

*This research was supported by the Office of Naval Research contract N00014-75-C-0455 at Purdue University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Γ-minimax principle is known as one of the techniques for the use of incomplete prior information. Such an idea was first used by Robbins (1951) and independently by Hodges and Lehmann (1952) and Menges (1966). The name Γ-minimax was first used by Blum and Rosenblatt (1967). Randles and Hollander (1971) applied such a principle to a problem of selecting the treatments 'better' than the control. It has been applied to various problems, and recently to selection problems by Gupta and Huang (1975, 1977), Berger (1977) and Miescke (1979).

In Section 2, necessary notations, definitions, a loss function and the incomplete prior are introduced. A lemma is given to help find Γ-minimax rules. Section 3 treats the case of known control population, and a Γ-minimax rule and a minimax rule are derived. In Section 4, the case in which the control population parameter θ_0 is unknown is treated. Rules are derived which are Γ-minimax among rules for which the decision about the i-th population depends only on the observations from π_i and π_0. A minimax rule is also derived. A normal means problem and a normal variances problem are given as specific examples. Section 5 consists of comparisons of Γ-minimax rules with Bayes rules for independent normal priors for the normal means problem.

2. Formulation of the problem.

Let $X_0, X_1, \ldots, X_k$ be k+1 independent random variables representing the control population π_0 and the k treatment populations $\pi_1, \ldots, \pi_k$, respectively, with X_i having pdf $f_i(x-\theta_i)$ with respect to the Lebesgue measure on the real line R where $\theta_i \in \Theta = R$, $i = 0,1,\ldots,k$. The random variables $X_0, \ldots, X_k$ may be sufficient statistics or other statistics based on which we wish to make statistical decisions. We assume that each $f_i(\cdot)$ $(i = 0,1,\ldots,k)$ is symmetric about the origin and strongly unimodal, i.e., $f_i(\cdot)$ is log-concave on the real line. Hence $f_i(x-\theta_i)$ has the monotone likelihood ratio (MLR) property. Obviously, we do not need any observations from π_0 when θ_0 is assumed known; therefore, it will be understood that, in such a case, the random variable X_0 is deleted from our consideration.

The action space $\mathcal{A}$ can be written as $\mathcal{A} = \mathcal{A}_1 \times\ldots\times \mathcal{A}_k$ where $\mathcal{A}_i = \{1,2,3\}$ for $i = 1,\ldots,k$. The action $a = (a_1,\ldots,a_k) \in \mathcal{A}$ is to be interpreted in such a way that, for $i = 1,\ldots,k$, the treatment population π_i is classified as 'inferior', 'equivalent' and 'superior' to π_0 for $a_i = 1,2,3$, respectively. The loss $L(\underline{\theta},a)$ incurred by the action $a \in \mathcal{A}$ for $\underline{\theta} = (\theta_0,\ldots,\theta_k)$ is assumed to be of the following form.

$$L(\underline{\theta},a) = \sum_{i=1}^{k} L_i(\underline{\theta},a_i) \tag{2.1}$$

where $L_i(\underline{\theta},a_i)$ is defined as in the following table;

Table of loss $L_i(\underline{\theta},a_i)$

State of nature \ a_i	1	2	3
$\theta_i-\theta_0 \le -\Delta_2$	0	ℓ_1	$\ell_1+\ell_3$
$-\Delta_2<\theta_i-\theta_0 \le -\Delta_1$	0	0	ℓ_4
$\lvert\theta_i-\theta_0\rvert < \Delta_1$	ℓ_2	0	ℓ_2
$\Delta_1 \le \theta_i-\theta_0 < \Delta_2$	ℓ_4	0	0
$\theta_i-\theta_0 \ge \Delta_2$	$\ell_1+\ell_3$	ℓ_1	0

$(\ell_i \ge 0,\ i=1,\ldots,4)$

Here, $\Delta_1 = \Delta-\epsilon$, $\Delta_2 = \Delta+\epsilon$ for a given constant ϵ: $0 \le \epsilon < \Delta$ and it will be understood that the second row and the fourth row will disappear when $\epsilon = 0$. Bhattacharyya (1956) derived a minimax rule assuming the above loss function with $\ell_1 = \ell_2 = \ell_1+\ell_3 = 1$ and $\epsilon = 0$ when θ_0 is assumed known and $\theta_1,\ldots,\theta_k$ are the unknown means of normal distributions. However, the irregularity of such a loss function has been pointed out in the sense that the minimax risk does not tend to zero even if the sample sizes increase indefinitely, and the same problem has been studied afresh by Bhattacharyya (1958) assuming the above loss function with $\ell_1 = \ell_2 = \ell_4 = \ell_1+\ell_3 = 1$ and $\epsilon > 0$. Note that the above loss function with $\epsilon > 0$ assumes the indifference zones.

For given $\underline{x} = (x_0,x_1,\ldots,x_k)$ consider decision rules of the form

$$\delta(\underline{x}) = (\delta_1(\underline{x}),\ldots,\delta_k(\underline{x})) \tag{2.2}$$

where $\delta_i(\underline{x}) = (\delta_i(1|\underline{x}),\ \delta_i(2|\underline{x}),\ \delta_i(3|\underline{x}))$ and, for $j = 1,2,3$, $\delta_i(j|\underline{x})$ denotes the conditional probability of taking action j in the i-th component decision problem. Note that there is no loss of generality in considering decision rules of the form given in (2.3). The risk function of a rule δ for fixed $\underline{\theta}$ is then $R(\underline{\theta},\delta) = \sum_{i=1}^{k} R_i(\underline{\theta},\delta_i)$ where $R_i(\underline{\theta},\delta_i) = E_{\underline{\theta}}[L_i(\underline{\theta},\delta_i(\underline{X}))]$. For a prior distribution $\tau(\underline{\theta})$ of $\underline{\theta}$, the overall risk of a rule δ wrt τ is denoted by $r(\tau,\delta) = \sum_{i=1}^{k} r_i(\tau,\delta_i)$ where $r_i(\tau,\delta_i) = \int R_i(\underline{\theta},\delta_i)d\tau(\underline{\theta})$.

It is assumed that partial prior information is available to a decision maker such that, for each i, he can specify $\gamma_i = P[|\theta_i-\theta_0| \geq \Delta_2]$ and $\gamma_i' = P[|\theta_i-\theta_0| < \Delta_1]$ where $\gamma_i+\gamma_i' \leq 1$ for $i = 1,\ldots,k$. Let Γ denote the class of all such prior distributions, i.e.,

$$\Gamma = \{\tau(\underline{\theta}): \int_{|\theta_i-\theta_0|\geq\Delta_2} d\tau(\underline{\theta}) = \gamma_i,\ \int_{|\theta_i-\theta_0|<\Delta_1} d\tau(\underline{\theta}) = \gamma_i' \text{ for } i=1,\ldots,k\}. \tag{2.3}$$

Note that when $\epsilon = 0$, i.e., $\Delta_1 = \Delta_2$, $\gamma_i + \gamma_i' = 1$.

A rule δ^Γ is called a Γ-minimax rule if $\sup_{\tau\in\Gamma} r(\tau,\delta^\Gamma) = \inf_\delta \sup_{\tau\in\Gamma} r(\tau,\delta)$, and $\sup_{\tau\in\Gamma} r(\tau,\delta^\Gamma)$ is called the Γ-minimax value. The next result is useful to find the Γ-minimax rule.

Lemma 2.1. Suppose $\{\tau_n, n = 1,2,\ldots\}$ is a sequence of priors in Γ. If $\overline{\lim_n} \inf_\delta r(\tau_n,\delta) \geq c$ and if $\sup_{\tau\in\Gamma} r(\tau,\delta^\Gamma) \leq c$, then δ^Γ is a Γ-minimax rule and c is the Γ-minimax value.

Proof. The result follows from the following inequalities.

$$\begin{aligned}\sup_{\tau\in\Gamma} \inf_\delta r(\tau,\delta) &\geq \overline{\lim_n} \inf_\delta r(\tau_n,\delta)\\ &\geq c\\ &\geq \sup_{\tau\in\Gamma} r(\tau,\delta^\Gamma)\\ &\geq \inf_\delta \sup_{\tau\in\Gamma} r(\tau,\delta)\\ &\geq \sup_{\tau\in\Gamma} \inf_\delta r(\tau,\delta).\end{aligned}$$

3. Known control population

In this section θ_0 is assumed known and thus we may assue $\theta_0 = 0$ without loss of generality. Hence $\underline{x}$ and $\underline{\theta}$ in this section denote $(x_1,\ldots,x_k)$ and $(\theta_1,\ldots,\theta_k)$, respectively. Let us consider a rule $\delta(\underline{x})$ of the form in (2.2) where $\delta_i(j|\underline{x})$ $(j = 1,2,3)$ is given by

$$\begin{cases}\delta_i(1|\underline{x}) = I_{(-\infty,-d_i]}(x_i),\\ \delta_i(2|\underline{x}) = I_{(-d_i,d_i)}(x_i),\\ \delta_i(3|\underline{x}) = I_{[d_i,\infty)}(x_i),\end{cases} \tag{3.1}$$

for $0 < d_i \leq \infty$ and $i = 1,\ldots,k$.

Lemma 3.1. Suppose that a decision rule $\delta(x)$ is given by (2.2) and (3.1). Then, for $i = 1,\ldots,k$,

$$\sup_{\tau\in\Gamma} r_i(\tau,\delta_i) \le v_i$$

where
$$v_i = \int_{d_i}^{\infty} [\ell_3\gamma_i f_i(x+\Delta_2)+\ell_2\gamma_i'(f_i(x-\Delta_1)+f_i(x+\Delta_1))+\ell_4(1-\gamma_i-\gamma_i')f_i(x+\Delta_1)]dx + \int_{-\infty}^{d_i} \ell_1\gamma_i f_i(x-\Delta_2)dx.$$

Proof. It follows from the definition of $L_i(\underline{\theta},a_i)$ and the symmetry of $f_i(\cdot)$ that, for $|\theta_i| < \Delta_1$,

$$R_i'(\underline{\theta},\delta_i) = \ell_2[f_i(d_i-\theta_i)-f_i(-d_i-\theta_i)]$$

$$= \ell_2 f_i(\theta_i+d_i)\left[\frac{f_i(\theta_i-d_i)}{f_i(\theta_i+d_i)} -1\right],$$

where R_i' denotes the derivative of R_i wrt θ_i.

It follows from the MLR property of $f_i(x-\theta_i)$ that $R_i'(\underline{\theta},\delta_i)$ has at most one change of sign, from negative to positive if there is any sign change at all; therefore, $R_i(\underline{\theta},\delta_i)$ attains the supremum over $\theta_i \in (-\Delta_1,\Delta_1)$ at either $\theta_i = -\Delta_1$ or $\theta_i = \Delta_1$. Hence, for $|\theta_i| < \Delta_1$,

$$R_i(\underline{\theta},\delta_i) \le \ell_2 \int_{d_i}^{\infty} [f_i(x-\Delta_1)+f_i(x+\Delta_1)]dx.$$

It can be easily shown that

$$R_i(\underline{\theta},\delta_i) \begin{cases} \le \ell_1 \int_{-\infty}^{d_i} f_i(x-\Delta_2)dx+\ell_3 \int_{d_i}^{\infty} f_i(x+\Delta_2)dx & \text{for } |\theta_i| \ge \Delta_2, \\ \le \ell_4 \int_{d_i}^{\infty} f_i(x+\Delta_1)dx & \text{for } \Delta_1 \le |\theta_i| < \Delta_2. \end{cases}$$

Therefore, it follows from (2.3) that $\sup_{\tau\in\Gamma} r_i(\tau,\delta_i) \le v_i$ which completes the proof.

Now we derive a Γ-minimax rule for the case where θ_0 is known.

Theorem 3.1. Assume that independent random variables $X_1,\ldots,X_k$ have $f_1(x_1-\theta_1),\ldots,f_k(x_k-\theta_k)$, respectively, with $f_i(\cdot)$ being symmetric and strongly unimodal, and that the loss function is given by (2.1). Then the Γ-minimax rule δ^Γ is given by (2.2) and (3.1) where each $d_i = d_i^\Gamma$ in (3.1) is defined by $d_i^\Gamma = \max(c_i,0)$ with c_i being determined by

$$\ell_3\gamma_i f_i(x+\Delta_2)+\ell_2\gamma_i'(f_i(x-\Delta_1)+f_i(x+\Delta_1))+\ell_4(1-\gamma_i-\gamma_i')f_i(x+\Delta_1) \le, > \ell_1\gamma_i f_i(x-\Delta_2) \quad \text{as} \quad x \ge, < c_i. \tag{3.2}$$

Proof. The existence of a c_i satisfying (3.2) follows from the MLR property of $f_i(x-\theta_i)$. Therefore, the decision rule δ^Γ is well defined. First, we will consider the case when $\epsilon > 0$, i.e., $\Delta_2 > \Delta_1$. For $n > \Delta_1^{-1}$, let τ_n be a prior distribution in Γ under which $\theta_1,\ldots,\theta_k$ are independent, $P(\theta_i=\Delta_2) = P(\theta_i = -\Delta_2) = \gamma_i/2$, $P(\theta_i=\Delta_1) = P(\theta_i = -\Delta_1) = (1-\gamma_i-\gamma_i')/2$ and $P(\theta_i=\Delta_1-n^{-1}) = P(\theta_i = -\Delta_1+n^{-1}) = \gamma_i'/2$ for $i = 1,\ldots,k$. Then it can be easily verified that $\inf_\delta r(\tau_n,\delta) = \sum_{i=1}^k \inf_{\delta_i} r_i(\tau_n,\delta_i)$ and, for $i = 1,\ldots,k$,

$$\inf_{\delta_i} r_i(\tau_n,\delta_i) = \int_{-\infty}^{\infty} p_n(x)dx/2,$$

where $p_n(x) = \min\{p_n(1,x),\ p(2,x),\ p_n(1,-x)\}$ with $p(2,x) = \ell_1\gamma_i[f_i(x+\Delta_2) + f_i(x-\Delta_2)]$ and $p_n(1,x) = \ell_2\gamma_i'[f_i(x-\Delta_1+n^{-1})+f_i(x+\Delta_1-n^{-1})] + \ell_4(1-\gamma_i-\gamma_i')f_i(x-\Delta_1)+(\ell_1+\ell_3)\gamma_i f_i(x-\Delta_2)$. Since $f_i(\cdot)$ is strongly unimodal on the real line, $f_i(\cdot)$ is continuous and thus $p_n(x)$ converges, as $n \to \infty$, to $p(x) = \min\{p(1,x),\ p(2,x),\ p(1,-x)\}$ where $p(1,x) = \lim_{n\to\infty} p_n(1,x)$. Note that $p(1,x) \geq p(1,-x)$ if and only if $x \geq 0$. This follows from the fact that, for any $t > 0$ $f_i(x-t) \geq f_i(x+t)$ if and only if $x \geq 0$. Since $p_n(x)$ is bounded above by $p(2,x)$ which is integrable, it follows from the Lebesgue convergence theorem that

$$\begin{aligned}\lim_n \inf_{\delta_i} r_i(\tau_n,\delta_i) &= \int_{-\infty}^{\infty} p(x)dx/2 \\ &= \int_0^{\infty} \min\{p(2,x),\ p(1,-x)\}dx.\end{aligned} \tag{3.3}$$

Note that $\int_0^{\infty} \min\{p(2,x),\ p(1,-x)\}dx$ can be written as

$$\begin{aligned}&\int_0^{\infty} \min\{p(2,x),\ p(1,-x)\}dx \\ &= \int_0^{\infty} \min\{\ell_3\gamma_i f_i(x+\Delta_2)+\ell_4(1-\gamma_i-\gamma_i')f_i(x+\Delta_1)+\ell_2\gamma_i'[f_i(x+\Delta_1)+f_i(x-\Delta_1)], \\ &\qquad \ell_i\gamma_i f_i(x-\Delta_2)\}dx + \int_{-\infty}^{0} \ell_1\gamma_i f_i(x-\Delta_2)dx \\ &= \int_{d_i}^{\infty} [\ell_3\gamma_i f_i(x+\Delta_2)+\ell_4(1-\gamma_i-\gamma_i')f_i(x+\Delta_1)+\ell_2\gamma_i'(f_i(x-\Delta_1)+f_i(x+\Delta_1))]dx \\ &\quad + \int_{-\infty}^{d_i} \ell_1\gamma_i f_i(x-\Delta_2)dx,\end{aligned}$$

where $d_i = \max(0,c_i)$ with c_i defined as in (3.2).

It follows from Lemma 3.1 that $\lim_n \inf_{\delta_i} r_i(\tau_n,\delta_i) \geq \sup_{\tau\in\Gamma} r_i(\tau,\delta_i^{\Gamma})$. Therefore,

$$\begin{aligned}\lim_n \inf_{\delta} r(\tau_n,\delta) &= \lim_n \sum_{i=1}^{k} \inf_{\delta_i} r_i(\tau_n,\delta_i) \\ &\geq \sum_{i=1}^{k} \sup_{\tau\in\Gamma} r_i(\tau,\delta_i^{\Gamma}) \\ &\geq \sup_{\tau\in\Gamma} r_i(\tau,\delta^{\Gamma}).\end{aligned}$$

Hence Lemma 2.1 yields that δ^Γ is a Γ-minimax rule. This completes the proof of the case when $\epsilon > 0$. Note that $\Delta_1 = \Delta_2 = \Delta$ and $\gamma_i + \gamma_i' = 1$ for $i = 1,\ldots,k$ if $\epsilon = 0$. When $\epsilon = 0$, let us consider a sequence of prior distributions, $\{\tau_n, n > \Delta^{-1}\}$, in Γ under which $\theta_1,\ldots,\theta_k$ are independent, $P(\theta_i=\Delta) = P(\theta_i= -\Delta) = \gamma_i/2$ and $P(\theta_i=\Delta-n^{-1}) = P(\theta_i= -\Delta+n^{-1}) = \gamma_i'/2$ for $i = 1,\ldots,k$. Then we can prove in the exactly same manner as the above that δ^Γ is a Γ-minimax rule.

Now we discuss the derivation of the minimax rule for some special cases. A minimax rule can be derived from the arguments in the proof of Theorem 2.1. For this purpose, assume that $\ell_1 = \ell_2$, $\ell_4 \le 2\ell_1$ and $\ell_3 \le \ell_1$. We may assume that $\ell_1 = \ell_2 = 1$ without loss of generality. Let us consider a rule δ^* of the type given by (2.2) and (3.1) where each $d_i = d_i^*$ in (3.1) is determined so that, for $F_i(x) = \int_{-\infty}^{\infty} f_i(t)dt$,

$$F_i(d_i-\Delta_2)+\ell_3F_i(-d_i-\Delta_2) = F_i(-d_i-\Delta_1)+F_i(-d_i+\Delta_1). \tag{3.3}$$

Note that the existence of such a non-negative d_i^* follows from the strong unimodality and the symmetry of $f_i(\cdot)$. Let us define γ_i and $\gamma_i' = 1-\gamma_i$ for $i = 1,\ldots,k$ by

$$\gamma_i = [f_i(d_i^*-\Delta_1)+f_i(d_i^*+\Delta_1)]/[f_i(d_i^*-\Delta_2)-\ell_3f_i(d_i^*+\Delta_2)+f_i(d_i^*-\Delta_1)+f_i(d_i^*+\Delta_1)].$$

Since $\gamma_i \in [0,1]$, we can consider a family of prior distributions, Γ, given by (2.3). Then it follows from Theorem 2.1 that the corresponding Γ-minimax rule is of the same type as δ^* except that now $d_i^\Gamma = \max(c_i,0)$ where c_i is determined so that

$$H(c_i) = \gamma_i[\ell_3f_i(c_i+\Delta_2)-f_i(c_i-\Delta_2)]+\gamma_i'[f_i(c_i-\Delta_1)+f_i(c_i+\Delta_1)] = 0.$$

Since $H(d_i^*) = 0$ and $d_i^* \ge 0$, $d_i^\Gamma = d_i^*$, i.e., the rule δ^* is the Γ-minimax rule; therefore it follows from the arguments in the proof of Theorem 2.1 that

$$\liminf_{n}\ \inf_{\delta} r(\tau_n,\delta) \geq \sum_{i=1}^{k} \gamma_i[F_i(d_i^*-\Delta_2)+\ell_3 F_i(-d_i^*-\Delta_2)]+\gamma_i'[F_i(-d_i^*-\Lambda_1)+F_i(-d_i^*+\Lambda_1)]$$

$$= \sum_{i=1}^{k} [F_i(-d_i^*-\Delta_1)+F_i(-d_i^*+\Delta_1)]$$

$$= \sum_{i=1}^{k} \sup_{\underline{\theta}} R_i(\underline{\theta},\delta^*)$$

$$\geq \sup_{\underline{\theta}} R(\underline{\theta},\delta^*)$$

Therefore, we have the next result which includes the results in Bhattacharyya (1956, 1958) as special cases.

Corollary 3.1. Under the assumptions in Theorem 3.1, if $\ell_1 = \ell_2 = 1$, $\ell_3 \leq 1$ and $\ell_4 \leq 2$, then a rule δ^M of the type given by (2.2) and (3.1) with $d_i = d_i^M$ in (3.1) being determined by (3.3) is minimax.

4. Unknown control population.

In this section we will consider the case when θ_0 is unknown and will derive a Γ-minimax decision rule δ^Γ in the class $\mathcal{D}_0$ of decision rules for which $\delta_i(\underline{x})$ in (2.2) depends only on x_0 and x_i for $i = 1,\ldots,k$. Let us consider rules $\delta(\underline{x})$ in $\mathcal{D}_0$ where $\delta_i(j|\underline{x})$ $(j = 1,2,3)$ are given by

$$\begin{cases} \delta_i(1|\underline{x}) = I_{(-\infty,-d_i]}(x_i-x_0), \\ \delta_i(2|\underline{x}) = I_{(-d_i,d_i)}(x_i-x_0), \\ \delta_i(3|\underline{x}) = I_{[d_i,\infty)}(x_i-x_0), \end{cases} \tag{4.1}$$

for $0 \leq d_i \leq \infty$ and $i = 1,\ldots,k$.

Note that the pdf of $Y_i = X_i - X_0$ is given by

$$g_i(y-(\theta_i-\theta_0)) = \int_{-\infty}^{\infty} f_i(t+y-\theta_i)f_0(t-\theta_0)dt, \tag{4.2}$$

and that $g_i(\cdot)$ is strongly unimodal by the result of Ibragimov (1956) and symmetric about the origin. Therefore, the next follows from this fact and Lemma 3.1.

Lemma 4.1. Suppose that a rule $\delta(\underline{x})$ in $\mathcal{D}_0$ is given by (2.2) and (4.1). Then, for $i = 1,\ldots,k$,

$$\sup_{\tau\in\Gamma} r_i(\tau,\delta_i) \leq \omega_i,$$

where, for $i = 1,\ldots,k$,

$$\omega_i = \int_{d_i}^{\infty} [\ell_3\gamma_i g_i(y+\Delta_2)+\ell_2\gamma_i'(g_i(y-\Delta_1)+g_i(y+\Delta_1))+\ell_4(1-\gamma_i-\gamma_i')g_i(y+\Delta_1)]dy$$
$$+ \int_{-\infty}^{d_i} \ell_1\gamma_i g_i(y-\Delta_2)dy.$$

We now proceed as in Theorem 3.1 by considering the following sequence $\{\tau_n, n > \Delta^{-1}\}$ of prior distributions in Γ for the case when $\epsilon > 0$. Under τ_n,

(i) $\theta_1 - \theta_0,\ldots,\theta_k-\theta_0$ are independent,

(ii) $P[\theta_i-\theta_0=\Delta_2] = P[\theta_i-\theta_0 = -\Delta_2] = \gamma_i/2,$

$P[\theta_i-\theta_0 = \Delta_1] = P[\theta_i-\theta_0= -\Delta_1] = (1-\gamma_i-\gamma_i')/2,$

$P[\theta_i-\theta_0 = \Delta_1-n^{-1}] = P[\theta_i-\theta_0 = -\Delta_1+n^{-1}] = \gamma_i'/2$ and

(iii) θ_0 has uniform distribution over $[-n,n]$ and is independent of $\theta_1-\theta_0,\ldots,\theta_k-\theta_0$.

It can be easily shown that the overall risk of the Bayes rule is given by

$$\inf_{\delta\in\mathcal{D}_0} r(\tau_n,\delta) = \frac{1}{4n}\sum_{i=1}^{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(i,x,y)dxdy \tag{4.3}$$

where $p_n(i,x,y) = \min\{s_n(i,x,y),\ t_n(i,x,y),\ s_n(i,-x,-y)\}$ with

$$s_n(i,x,y) = \ell_2\gamma_i' \int_{-n}^{n} [f_i(x-u-\Delta_1+n^{-1})+f_i(x-u+\Delta_1-n^{-1})]f_0(y-u)du +$$

$$+ \ell_4(1-\gamma_i-\gamma_i') \int_{-n}^{n} f_i(x-u-\Delta_1)f_0(y-u)du +$$

$$+ (\ell_1+\ell_3)\gamma_i \int_{-n}^{n} f_i(x-u-\Delta_2)f_0(y-u)du \text{ and}$$

$$t_n(i,x,y) = \ell_1\gamma_i \int_{-n}^{n} [f_i(x-u+\Delta_2)+f_i(x-u-\Delta_2)]f_0(y-u)du.$$

From change of variables $x = nv-w$ and $y = nv+w$, it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(i,x,y)dxdy/4n &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(i,nv-w,nv+w)dvdw/2 \\ &\geq \int_{-\infty}^{\infty}[\int_{-1}^{1} p_n(i,nv-w,nv+w)dv]dw/2.\end{aligned} \tag{4.4}$$

Note that

$$s_n(i,nv-w,nv+w) = \ell_2\gamma_i \int_{n(v-1)}^{n(v+1)} [f_i(z-w-\Delta_1+n^{-1})+f_i(z-w+\Delta_1-n^{-1})]f_0(z+w)dz +$$

$$+ \ell_4(1-\gamma_i-\gamma_i')\int_{n(v-1)}^{n(v+1)} f_i(z-w-\Delta_1)f_0(z+w)dz +$$

$$+ (\ell_1+\ell_3)\gamma_i \int_{n(v-1)}^{n(v+1)} f_i(z-w-\Delta_2)f_0(z+w)dz \text{ and}$$

$$t_n(i,nv-w,\ nv+w) = \ell_1\gamma_i \int_{n(v-1)}^{n(v+1)} [f_i(z-w+\Delta_2)+f_i(z-w-\Delta_2)]f_0(z+w)dz.$$

Therefore, for any $(v,w) \in (-1,1)\times R$, $p_n(i,\ nv-w,nv+w)$ converges, as $n \to \infty$, to $p(i,w) = \min\{s(i,w),\ t(i,w),\ s(i,-w)\}$ where

$$s(i,w) = \ell_2\gamma_i \int_{-\infty}^{\infty}[f_i(z-w-\Delta_1)+f_i(z-w+\Delta_1)]f_0(z+w)dz +$$

$$+ \ell_4(1-\gamma_i-\gamma_i') \int_{-\infty}^{\infty} f_i(z-w-\Delta_1)f_0(z+w)dz +$$

$$+ (\ell_1+\ell_3)\gamma_i \int_{-\infty}^{\infty} f_i(z-w-\Delta_2)f_0(z+w)dz \text{ and}$$

$$t(i,w) = \ell_1\gamma_i \int_{-\infty}^{\infty}[f_i(z-w+\Delta_2)+f_i(z-w-\Delta_2)]f_0(z+w)dz.$$

It follows from (4.4) that, for $i = 1,\ldots,k$,

$$\underline{\lim}_n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_n(i,x,y)dxdy/4n \geq \int_{-\infty}^{\infty} p(i,w)dw$$

$$= \int_0^{\infty} h(i,y)dy,$$

where $h(i,y) = \min\{\ell_2\gamma_i'[g_i(y-\Delta_1)+g_i(y+\Delta_1)]+\ell_4(1-\gamma_i-\gamma_i')g_i(y+\Delta_1) +$

$$+ (\ell_1+\ell_3)\gamma_i g_i(y+\Delta_2),\ \ell_1\gamma_i[g_i(y+\Delta_2)+g_i(y-\Delta_2)]\}.$$

Then from (4.3), we have

$$\underline{\lim}_n \inf_{\delta\in\mathcal{D}_0} r(\tau_n,\delta) \geq \sum_{i=1}^{k} \int_0^{\infty} h(i,y)dy. \tag{4.5}$$

Note that $\int_0^{\infty} h(i,y)dy$ can be written as

$$\int_0^{\infty} h(i,y)dy = \int_0^{\infty}\min\{\ell_3\gamma_i g_i(y+\Delta_2)+\ell_4(1-\gamma_i-\gamma_i')g_i(y+\Delta_1)+\ell_2\gamma_i'(g_i(y+\Delta_1)+g_i(y-\Delta_1)),$$

$$\ell_1\gamma_i g_i(y-\Delta_2)\}dy + \int_{-\infty}^{0} \ell_1\gamma_i g_i(y-\Delta_2)dy$$

$$= \int_{d_i}^{\infty} [\ell_3\gamma_i g_i(y+\Delta_2)+\ell_2\gamma_i'(g_i(y-\Delta_1)+g_i(y+\Delta_1))+\ell_4(1-\gamma_i-\gamma_i')g_i(y+\Delta_1)]dy$$
$$+ \int_{-\infty}^{d_i} \ell_1\gamma_i g_i(y-\Delta_2)dy,$$

where $d_i = \max(c_i, 0)$ with c_i being determined so that

$$\ell_3\gamma_i g_i(y+\Delta_2)+\ell_2\gamma_i'(g_i(y-\Delta_1)+g_i(y+\Delta_1))+\ell_4(1-\gamma_i-\gamma_i')g_i(y+\Delta_1) \tag{4.6}$$
$$\leq, \geq \ell_1\gamma_i g_i(y-\Delta_2) \quad \text{as} \quad y \geq, \leq c_i.$$

Let δ^Γ be the rule given by (2.2) and (4.1) where $d_i = d_i^\Gamma$ in (4.1) is defined by $d_i = \max(c_i, 0)$ with c_i determined by (4.6). Then it follows from (4.5) and Lemma 4.1 that

$$\underline{\lim_n} \inf_{\delta \in \mathcal{D}_0} r(\tau_n, \delta) \geq \sum_{i=1}^{k} \sup_{\tau\in\Gamma} r_i(\tau, \delta_i^\Gamma)$$
$$\geq \sup_{\tau\in\Gamma} r(\tau, \delta^\Gamma).$$

Therefore, Lemma 2.1 yields the next result.

Theorem 4.1. Assume that independent random variables $X_0, \ldots, X_k$ have pdf's $f_0(x_0-\theta_0), \ldots, f_k(x_k-\theta_k)$, respectively, with $f_i(\cdot)$ being strongly unimodal and symmetric, and that the loss function is given by (2.1). Then the Γ-minimax rule δ^Γ in $\mathcal{D}_0$ is given by (2.2) and (4.1) where $d_i = d_i^\Gamma$ in (4.1) is defined by $d_i^\Gamma = \max(c_i, 0)$ with c_i being determined by (4.6), for all $i = 1, 2, \ldots, k$.

Remark 4.1. It can be easily shown that the symmetry of $f_i(\cdot)$ in Theorem 4.1 can be replaced by that of $g_i(\cdot)$. It should be noted that the symmetry of $g_i(\cdot)$ follows when $f_0(\cdot), \ldots, f_k(\cdot)$ are identical.

The next result follows in exactly the same manner as Corollary 3.1 was proved.

Corollary 4.1. Under the assumptions in Theorem 4.1, if $\ell_1 = \ell_2 = 1$, $\ell_3 \leq 1$ and $\ell_4 \leq 2$, then a minimax rule δ^M in $\mathcal{D}_0$ is given by (2.2) and (4.1) where $d_i = d_i^M$ in (4.1) is determined so that, for $G_i(x) = \int_{-\infty}^{x} g_i(t)dt$,

$$G_i(d_i-\Delta_2) + \ell_3 G_i(-d_i-\Delta_2) = G_i(-d_i-\Delta_1)+G_i(-d_i+\Delta_1).$$

Now we provide some examples to illustrate the application of the above results.

Example 4.1. Suppose π_i represents a normal population $N(\theta_i,\sigma_i^2)$ for $i = 0,\ldots,k$ with σ_i^2 $(i = 0,\ldots,k)$ known. We assume that a random sample of size n_i is taken from each of the k+1 populations $\pi_0,\ldots,\pi_k$. By sufficiency we can restrict our attention to the decision rules depending only on the sample means $X_0,\ldots,X_k$ where X_i has normal distribution with mean θ_i and variance $\eta_i^2 = \sigma_i^2/n_i$ for $i = 0,1,\ldots,k$.

(A) Γ-minimax rule: The Γ-minimax rule δ^Γ in $\mathcal{D}_0$ in Theorem 4.1 is determined by $d_i^\Gamma = (\eta_i^2 + \eta_0^2)^{\frac{1}{2}} \max(c_i,0)$ where c_i is defined so that

$$\begin{aligned}&\ell_3\gamma_i e^{-2(\lambda_i+\epsilon_i)x} + \ell_2\gamma_i'[e^{-2\epsilon_i(x-\lambda_i)}+e^{-2\lambda_i(x-\epsilon_i)}] + \\ &+ \ell_4(1-\gamma_i-\gamma_i')e^{-2\lambda_i(x-\epsilon_i)} - \ell_1\gamma_i \geq, \leq 0 \text{ as } x \leq, \geq c_i,\end{aligned} \tag{4.7}$$

where $\lambda_i = \Lambda(\eta_i^2+\eta_0^2)^{-\frac{1}{2}}$ and $\epsilon_i = \epsilon(\eta_i^2+\eta_0^2)^{-\frac{1}{2}}$.

(B) Minimax rule: Assume $\ell_1 = \ell_2 = 1$, $\ell_3 \leq 1$ and $\ell_4 \leq 2$. Then the minimax rule δ^M in $\mathcal{D}_0$ in Corollary 4.1 is determined by $d_i^\Gamma = (\eta_i^2+\eta_0^2)^{\frac{1}{2}}t_i$ where t_i is defined so that

$$\Phi(t_i-\lambda_i-\epsilon_i)+\ell_3\Phi(-t_i-\lambda_i-\epsilon_i) = \Phi(-t_i-\lambda_i+\epsilon_i)+\Phi(-t_i+\lambda_i-\epsilon_i) \qquad (4.8)$$

with λ_i and ϵ_i defined as in (A) and Φ denoting the cdf of the standard normal distribution.

Example 4.2. Assume that π_i represents a normal population $N(0,\sigma_i^2)$ for $i = 0,1,\ldots,k$ with σ_i^2 unknown, and that we have a random sample of size n taken from each population π_i. Consider a problem of partitioning the treatment populations in terms of variances with a loss structure analogous to that given by (2.1), i.e., a loss function obtained from the latter by substituting $\log \sigma_i^2$, $\log \Delta$ and $\log \epsilon$ for θ_i, Δ and ϵ, respectively. Thus Δ and ϵ are assumed such that $1 \leq \epsilon < \Delta$. By sufficiency we need to consider only the decision rules depending on $s_0^2,\ldots,s_k^2$ where s_i^2 denotes the sample variance corresponding to π_i. Since ns_i^2/σ_i^2 $(i = 0,1,\ldots,k)$ are independently distributed chi-square random variables with degrees of freedom n, it can be easily seen that the associated location parameter problem satisfies the assumptions in Theorem 4.1 except the symmetry which is not necessary in this problem because of Remark 4.1. Therefore, with obvious modifications we have the following results. Let $\mathcal{D}_0$ denote the class of decision rules $\delta = (\delta_1,\ldots,\delta_k)$ for which δ_i depends only on s_0^2 and s_i^2 and let x_i denote s_i^2/s_0^2 for $i = 1,\ldots,k$.

(A) Γ-minimax rule: A Γ-minimax rule δ^Γ in $\mathcal{D}_0$ is given by

$$\delta_i^\Gamma(1|x_i) = I_{(0,d_i^{-1}]}(x_i),\ \delta_i^\Gamma(2|x_i)=I_{(d_i^{-1},d_i)}(x_i) \text{ and } \delta_i^\Gamma(3|x_i)=I_{[d_i,\infty)}(x_i)$$

for $i = 1,\ldots,k$ where $d_i = \max(c_i,1)$ with c_i being determined so that

$$\ell_3\gamma_i\left(\frac{\Delta_2+y}{1+\Delta_2 y}\right)^n + \ell_2\gamma_i'\left[\left(\frac{\Delta_2+y}{\Delta_1+y}\right)^n + \left(\frac{\Delta_2+y}{1+\Delta_1 y}\right)^n\right]\left(\frac{\Delta_1}{\Delta_2}\right)^{n/2} +$$

$$+ \ell_4(1-\gamma_i-\gamma_i')\left(\frac{\Delta_2+y}{1+\Delta_1 y}\right)^n\left(\frac{\Delta_1}{\Delta_2}\right)^{n/2} \le, \ge \ell_1\gamma_i \text{ as } y \ge, \le c_i. \tag{4.9}$$

Here $\Delta_1 = \Delta\epsilon^{-1}$, $\Delta_2 = \Delta\epsilon$.

(B) Minimax rule: Assume $\ell_1 = \ell_2 = 1$, $\ell_3 \le 1$ and $\ell_4 \le 2$. The minimax rule δ^M in $\mathcal{D}_0$ is the same as δ^Γ in (A) except that $d_i = d$ is determined so that

$$G_n(d/\Delta_2)+\ell_3[1-G_n(d\Delta_2)] = G_n(\Delta_1/d)+1-G_n(d\Delta_1) \tag{4.10}$$

where G_n denotes the cdf of F-distribution with degrees of freedom n and n.

We note that if π_i represents $N(\mu_i,\sigma_i^2)$ with both μ_i and σ_i^2 unknown, then the above results still hold with n-1 replacing n.

5. Comparison of Γ-minimax rules with Bayes rules.

When we represent our a prior information about the parameters by prior distributions over the parameter space, one method for the use of such information is to find a rule which is Γ-minimax with respect to the class, Γ, of such prior distributions.

Another way is to select one such prior distribution and use the corresponding Bayes rule. Thus Bayes rules wrt prior distributions in Γ are natural competitors of a Γ-minimax rule.

In this section we consider k+1 normal populations $N(\theta_i,\sigma^2)$ with σ^2 known, and derive Bayes rules wrt normal priors and then compare them with the corresponding Γ-minimax rules from both points of view.

For this purpose, assume that $(\theta_0,\ldots,\theta_k)$ have prior distribution τ_0 under which $\theta_0,\ldots,\theta_k$ are independent and each θ_i has a normal distribution with mean μ_i and variance v_i^2. Let $x_0,\ldots,x_k$ denote the observed sample means based on samples of size n_i $(i = 0,1,\ldots,k)$. To simplify forthcoming formulas, let us introduce the following notations;

$$\sigma_i^2 = \sigma^2/n_i,\ b_i = [(\sigma_i^{-2}+v_i^{-2})^{-1} + (\sigma_0^{-2}+v_0^{-2})^{-1}]^{\frac{1}{2}},$$
$$m_i = (\sigma_i^{-2}x_i + v_i^{-2}\mu_i)(\sigma_i^{-2}+v_i^{-2})^{-1},\ y_i = (m_i-m_0)/b_i. \tag{5.1}$$

The following theorem describes the Bayes rule.

Theorem 5.1. Assume the loss function is given by (2.1). Then the Bayes rule δ^B wrt τ_0 is given by $\delta_i^B(1|\underline{y}) = I_{(-\infty,-d_i]}(y_i)$, $\delta_i^B(2|\underline{y}) = I_{(-d_i,d_i)}(y_i)$ and $\delta_i^B(3|\underline{y}) = I_{[d_i,\infty)}(y_i)$ for $i = 1,\ldots,k$ where $d_i = \max(c_i,0)$ with c_i being determined so that

$$\begin{aligned}&\ell_3\Phi(-\Delta_2 b_i^{-1}-y)+\ell_4[\Phi(-\Delta_1 b_i^{-1}-y)-\Phi(-\Delta_2 b_i^{-1}-y)]\\ &+\ell_2[\Phi(\Delta_1 b_i^{-1}-y)-\Phi(-\Delta_1 b_i^{-1}-y)]-\ell_1\Phi(-\Delta_2 b_i^{-1}+y)\\ &\geq, \leq 0 \quad \text{as} \quad y \leq, \geq c_i.\end{aligned}$$

Proof. It suffices to find the Bayes rule for each of the k component decision problems. This reduces to the comparison of posterior risks of three possible actions. We will do this for the first component decision problem without loss of generality. Let $p_1(y_1)$, $p_2(y_1)$ and $p_3(y_1)$ denote the posterior risks of the actions 1, 2 and 3, respectively, in the first component problem. Then it can be shown that

$$p_1(y) = (\ell_1+\ell_3)\Phi(-\Delta_2 b_1^{-1}+y) + \ell_4[\Phi(\Delta_2 b_1^{-1}-y)-\Phi(\Delta_1 b_1^{-1}-y)] + $$
$$+ \ell_2[\Phi(\Delta_1 b_1^{-1}-y)-\Phi(-\Delta_1 b_1^{-1}-y)],$$
$$p_2(y) = \ell_1[\Phi(-\Delta_2 b_1^{-1}-y)+\Phi(-\Delta_2 b_1^{-1}+y)] \text{ and}$$
$$p_3(y) = p_1(-y).$$

Note that $p_1(y) - p_3(y)$ can be written as $E_y H(Z)$ where Z has a normal distribution with mean y and variance 1 and $H(\cdot)$ is given by

$$H(z) = \begin{cases} \ell_1+\ell_3 & \text{if } z \geq \Delta_2 b_1^{-1} \\ \ell_4 & \text{if } \Delta_1 b_1^{-1} < z < \Delta_2 b_1^{-1} \\ 0 & \text{if } -\Delta_1 b_1^{-1} \leq z \leq \Delta_1 b_1^{-1} \\ -\ell_4 & \text{if } -\Delta_2 b_1^{-1} < z < -\Delta_1 b_1^{-1} \\ -(\ell_1+\ell_3) & \text{if } z \leq -\Delta_2 b_1^{-1}. \end{cases}$$

Since the density of the normal distribution $N(y,1)$ has the MLR property, it follows that $p_1(y) - p_3(y)$ has at most one sign change. Furthermore, it can be shown that $p_1(y)-p_3(y)$ is strictly increasing on $(-\Delta b_1^{-1}, \Delta b_1^{-1})$ and $p_1(0)-p_3(0) = 0$. Thus $p_1(y)-p_3(y) \geq, \leq 0$ as $y \geq, \leq 0$. Similarly, we can show that $p_3(y)-p_2(y) \geq, \leq 0$ as $y \leq, \geq c_1$ for some real number c_1 unless $p_3(y)-p_2(y) \leq 0$ for all y. Therefore the result follows.

Now we compare the Γ-minimax rule δ^Γ given in Example 4.1 and the Bayes rule δ^B given in Theorem 5.1 under the assumption that $\ell_1 = \ell_2 = \ell_4 = 1$, $\ell_3 = \ell$, $n_i = n$ and $v_i^2 = v^2$ for $i = 0,\ldots,k$. Note that we compare these rules under the relations $\gamma_i = \Phi[(-\Delta_2+\mu_i-\mu_0)(2v^2)^{-\frac{1}{2}}]+\Phi[(-\Delta_2-\mu_i+\mu_0)(2v^2)^{-\frac{1}{2}}]$ and $\gamma_i' = \Phi[(\Delta_1-\mu_i+\mu_0)(2v^2)^{-\frac{1}{2}}]-\Phi[(-\Delta_1-\mu_i+\mu_0)(2v^2)^{-\frac{1}{2}}]$ for $i = 1,\ldots,k$. Each of them is the best in its own merit. Therefore there are two ways of any meaningful comparison of these rules. One way is to examine the increase in the overall risk wrt τ_0 resulting from the use of δ^Γ. Another way is to compare them in terms of $\sup_{\tau\in\Gamma} r(\tau,\delta)$. When $n_i = n$ and $v_i^2 = v^2$ for $i = 0,1,\ldots,k$, the Bayes rule depends on $\underline{x}$ only through $x_1-x_0,\ldots,x_k-x_0$ and it can be shown that $\sup_{\tau\in\Gamma} r(\tau,\delta^B) = \sum_{i=1}^{k} \sup_{\tau\in\Gamma} r_i(\tau,\delta_i^B)$. Thus it suffices to compare these rules wrt classification of one population. We choose π_1 for this purpose without loss of generality.

Now we introduce the parameters used in the comparison as follows.

$$\beta_1 = \frac{nv^2}{\sigma^2}\ ,\ \beta_2 = \frac{\Delta}{\sqrt{2v^2}},\ \beta_3 = \frac{\epsilon}{\sqrt{2v^2}} \text{ and } \beta_4 = \frac{\mu_1-\mu_0}{\sqrt{2v^2}}\ .$$

It can be verified that the overall risk wrt τ_0 of these rules can be written as

$$\begin{aligned}
&\ell\Phi(-A-B-C)+\Phi(A-B-C)+\Phi(D-E)+\ell\Phi(-D-E) \\
&-\Phi_0(-A-B-C,\ -D-E;\ \rho)+(1-\ell)\Phi_0(-A-B-C,\ D-E;\rho) \\
&-\ \Phi_0(-A+B-C,\ D-E;\ \rho)\ +\ \Phi_0(A-B-C,\ -D-E;\ \rho) \\
&-\ \Phi_0(-A+B-C,\ -D-E;\ \rho)-\Phi_0(A-B-C,\ D-E;\ \rho) \\
&+\ \Phi_0(-A+B-C,\ D-E;\ \rho)-\Phi_0(A-B-C,\ -D-E;\ \rho) \\
&-\ \Phi_0(A+B-C,\ D-E;\ \rho)-(1-\ell)\Phi_0(A+B-C,\ -D-E;\ \rho)
\end{aligned}$$

where $\Phi_0(\cdot,\cdot;\ \rho)$ is the cdf of a bivariate normal distribution with zero means, unit variances and correlation coefficient ρ, and where $A = \beta_2$, $B = \beta_3$, $C = \beta_4$, $\rho = \beta_1^{\frac{1}{2}}(1+\beta_1)^{-\frac{1}{2}}$, $D = d_1\beta_1^{-\frac{1}{2}}$ for δ^B, $D = \max(c_1,0)(1+\beta_1)^{-\frac{1}{2}}$ for δ^Γ, $E = \rho^{-1}\beta_4$ for δ^B and $E = \rho\beta_4$ for δ^Γ with d_1 and c_1 being those in Theorem 5.1 and Example 4.1, respectively. Also $\sup_{\tau\in\Gamma} r_1(\tau,\delta_1)$ for both rules can be written as

$$\gamma_1[\Phi(R+|S|-T-U)-\Phi(-R+|S|-T+U)+\ell\Phi(-R+|S|-T-U)]$$

$$+\ \gamma_1'[\Phi(-R-S+T-U)+\Phi(-R+S-T+U)]^{V}[\Phi(-R+S+T-U)+\Phi(-R-S-T+U)]$$

$$+\ (1-\gamma_1')\Phi(-R+|S|-T+U)$$

where $x^{V}y = \max(x,y)$, $T = \beta_2\beta_1^{\frac{1}{2}}$, $U = \beta_3\beta_1^{\frac{1}{2}}$, $S = \beta_4\beta_1^{-\frac{1}{2}}$ for δ^B, $S = 0$ for δ^Γ, $R = \beta_1^{-\frac{1}{2}}(1+\beta_1)^{\frac{1}{2}}\, d_1$ for δ^B, $R = \max(c_1,0)$ for δ^Γ with d_1 and c_1 being those in Theorem 5.1 and Example 4.1, respectively. For selected values of β_i $(i = 1,\ldots,4)$, Table I and Table II give $r_1(\tau_0,\delta_1)$ and $\sup_{\tau\in\Gamma} r_1(\tau,\delta_1)$ for $\delta_1 = \delta_1^\Gamma$, δ_1^B for $\ell = 0$ and $\ell = 1$, respectively. It can be observed from these tables that, in many cases, the increase in the overall risk wrt τ_0 from the use of δ_1^Γ is only slight compared to that in $\sup_{\tau\in\Gamma} r_1(\tau,\delta_1)$ from the use of δ_1^B. In this sense, δ^Γ is more robust against other formulation than δ^B. Such properties of δ^Γ become more prominent as the difference between the prior means (β_4) increases and the prior variance (β_1) gets smaller. When we have the same prior means and the prior variance is large, both rules compare favorably with each other. In most cases, we can observe that δ^Γ compares favorably with δ^B in terms of the overall risk.

Table I

Overall risks and the values of $\sup_{\tau \in \Gamma} r(\tau,\delta)$ of δ^B and δ^Γ when $\ell = 0$.

β_4 \ β_1	δ^Γ (.25)	δ^B (.25)	δ^Γ (1.0)	δ^B (1.0)	δ^Γ (4.0)	δ^B (4.0)
		$\beta_2 = 0.5$		$\beta_3 = 0.125$		
1.0	.4961	.9696	.4097	.7784	.2816	.5404
	.3930	.2587	.2864	.2097	.2001	.1189
0.5	.5344	.8274	.4577	.5201	.3312	.4359
	.4520	.3957	.3321	.2963	.2121	.1615
0.0	.5164	.5527	.4568	.4626	.3413	.3492
	.4599	.4544	.3341	.3315	.2103	.1786
		$\beta_2 = 0.5$		$\beta_3 = 0.05$		
1.0	.5250	.9731	.4471	.8117	.3261	.6276
	.4140	.2853	.3103	.2395	.2413	.1493
0.5	.5768	.8460	.5088	.6703	.3999	.5306
	.4898	.4330	.3840	.3368	.2899	.2021
0.0	.5680	.5937	.5170	.5190	.4210	.4391
	.5027	.4978	.3828	.3764	.2856	.2232
		$\beta_2 = 0.8$		$\beta_3 = 0.2$		
1.0	.4770	.9598	.3793	.7074	.2574	.4408
	.3909	.2824	.2539	.1931	.1369	.0923
0.5	.3739	.7920	.3450	.5344	.2692	.3449
	.3594	.3163	.2424	.2237	.1152	.1083
0.0	.3172	.5226	.3066	.3794	.2546	.2667
	.3151	.3050	.2404	.2298	.1133	.1131
		$\beta_2 = 0.8$		$\beta_3 = 0.08$		
1.0	.5365	.9699	.4541	.7840	.3573	.5860
	.4380	.3309	.3190	.2411	.2407	.1353
0.5	.4357	.8308	.4278	.6286	.3960	.4890
	.4311	.3742	.3137	.2801	.1632	.1588
0.0	.3789	.5827	.3785	.4658	.3679	.3967
	.3789	.3658	.3509	.2887	.1939	.1659

The numbers on the first (second) row in each box are the values of $\sup_{\tau \in \Gamma} r(\tau,\delta)$ $(r(\tau_0,\delta))$.

Table II

Overall risks and the value of $\sup_{\tau \in \Gamma} r(\tau, \delta)$ of δ^B and δ^P when $\ell = 1$.

$\beta_4 \backslash \beta_1$	.25 δ^Γ	.25 δ^B	1.0 δ^Γ	1.0 δ^B	4.0 δ^Γ	4.0 δ^B
			$\beta_2 = 0.5$	$\beta_3 = 0.125$		
1.0	.6655	1.6167	.5435	1.0979	.3258	.5696
	.5218	.2929	.2947	.2195	.1674	.1194
0.5	.5746	1.2157	.5098	.7758	.3496	.4498
	.5280	.4325	.3361	.3082	.1941	.1620
0.0	.5299	.6991	.4865	.5298	.3529	.3560
	.5085	.4817	.3474	.3437	.1985	.1791
			$\beta_2 = 0.5$	$\beta_3 = 0.05$		
1.0	.7117	1.6582	.6084	1.1721	.4027	.6698
	.5688	.3264	.3280	.2523	.2059	.1500
0.5	.6242	1.2762	.5775	.8566	.4372	.5520
	.5872	.4778	.3828	.3522	.2445	.2030
0.0	.5817	.7627	.5545	.6032	.4452	.4502
	.5683	.5316	.4017	.3923	.2510	.2242
			$\beta_2 = 0.8$	$\beta_3 = 0.2$		
1.0	.5021	1.4114	.3982	.8181	.2580	.4413
	.4248	.2883	.2465	.1938	.1355	.0923
0.5	.3745	.9929	.3470	.5702	.2693	.3451
	.3642	.3195	.2448	.2243	.1151	.1083
0.0	.3172	.5774	.3070	.3900	.2546	.2667
	.3157	.3062	.2420	.2303	.1133	.1131
			$\beta_2 = 0.8$	$\beta_3 = 0.08$		
1.0	.5665	1.4765	.4861	.9276	.3617	.5872
	.4891	.3393	.2966	.2423	.2239	.1353
0.5	.4357	1.0748	.4289	.6799	.3960	.4894
	.4337	.3789	.3222	.2811	.1632	.1588
0.0	.3789	.6540	.3785	.4823	.3679	.3969
	.3789	.3675	.3520	.2895	.1939	.1659

The numbers on the first (second) row in each box are the values of $\sup_{\tau \in \Gamma} r(\tau, \delta)$ $(r(\tau_0, \delta))$.

References

Berger, R. (1977). Minimax, Admissible and Gamma-Minimax Multiple Decision Rules. Ph.D. Thesis. Dept. of Statistics, Purdue Univ., West Lafayette, IN, 47907.

Bhattacharyya, P. K. (1956). Comparison of the means of k normal populations. Calcutta Statist. Assoc. Bull. 7: 1-20.

Bhattacharyya, P. K. (1958). On the large sample behavior of a multiple decision procedure. Calcutta Statist. Assoc. Bull. 8: 43-47.

Blum, J. R. and Rosenblatt, J. (1967). On partial information in statistical inference. Ann. Math. Statist. 38: 1671-1678.

Dunnett, C. W. (1955). A multiple comparison procedure for comparing several treatments with a control. J. Amer. Statist. Assoc. 50: 1096-1121.

Gupta, S. S. and Huang, D. Y. (1975). On Γ-minimax classification procedures. Proceedings of the 40th session of the International Statistical Institute, Vol. 46, Book 3: 330-335.

Gupta. S. S. and Huang, D. Y. (1977). On some Γ-minimax subset selection and multiple comparison procedures. Statistical Decision Theory and Related Topics II (Ed. S. S. Gupta and D. S. Moore). Academic Press, New York: 139-155.

Gupta, S. S. and Sobel, M. (1958). On selecting a subset which contains all populations better than a standard. Ann. Math. Statist. 29: 235-244.

Hodges, J. L., Jr. and Lehmann, E. L. (1952). The use of previous experience in reaching statistical decisions. Ann. Math. Statist. 23: 396-407.

Ibragimov, J. A. (1956). On the composition of unimodal distributions. (Russian) Teoriya veroyatnostey 1: 283-288.

Lehmann, E. L. (1961). Some model I problems of selection. Ann. Math. Statist. 32: 990-1012.

Menges, G. (1966). On the "Bayesification" of the minimax principle. Unternehmesforschung. 10: 81-91.

Miescke, K. J. (1979). Γ-minimax selection procedures in simultaneous testing problems. Mimeo. Ser. No. 79-1, Dept. of Statist., Purdue Univ., West Lafayette, IN 47907.

Paulson, E. (1952). On the comparison of several experimental categories with a control. Ann. Math. Statist. 23: 239-246.

Randles, R. H. and Hollander, M. (1971). Γ-minimax selection procedures in treatment versus control problems. Ann. Math. Statist. 42: 330-341.

Robbins, H. (1951). Asymptotically subminimax solutions of compound statistical decision problems. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. University of California Press, Berkeley and Los Angeles: 131-148.

Seeger, P. (1972). A note on partitioning a set of normal populations by their locations with respect to two controls. Ann. Math. Statist. 43: 2019-2023.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER Mimeograph Series #79-10	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Γ-Minimax and Minimax Decision Rules for Comparison of Treatments with a Control		5. TYPE OF REPORT & PERIOD COVERED Technical 6. PERFORMING ORG. REPORT NUMBER Mimeo. Series #79-10
7. AUTHOR(s) Shanti S. Gupta and Woo-Chul Kim		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0455
9. PERFORMING ORGANIZATION NAME AND ADDRESS Purdue University Department of Statistics West Lafayette, IN 47907		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Washington, DC		12. REPORT DATE August 1979 13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Γ-minimax, Minimax Rules, Selection Rules, Subset Selection, Control, New Treatments

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper deals with the problem of partitioning treatments in comparison with a standard or control under a decision-theoretic formulation. It is assumed that π_i, the ith treatment population, is characterized by a random variable X_i having strongly unimodal and symmetric density $f_i(x-\theta_i)$, $-\infty < \theta_i < \infty$, $i = 0,1,\ldots,k$, where π_0 is the control population. Under the assumptions of a linear loss structure and incomplete prior information about θ_i, $i = 0,1,\ldots,k$,

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601 |

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) optimal selection rules are derived for classifying the treatments into superior, equivalent, or inferior groups. These Γ-minimax rules are compared with Bayes rules for the normal means problem. It is shown that the Γ-minimax rules compare quite favorably with the Bayes rules. Minimax rules are also derived for the same problem.