

ON THE LEAST FAVORABLE CONFIGURATIONS IN  
CERTAIN TWO-STAGE SELECTION PROCEDURES\*

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ON THE LEAST FAVORABLE CONFIGURATIONS IN  
CERTAIN TWO-STAGE SELECTION PROCEDURES \*

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SUMMARY

The problem of finding the least favorable configuration for selecting the "best" of  $k$  populations i.e. the one with the largest location parameter by use of six different two-stage selection procedures is considered. Each of the six procedures consists of a subset selection (screening) rule at the first stage followed by another rule based on (the first stage and) additional samples from the selected populations to decide finally which of the selected populations is the best. In the indifference-zone approach it is (or was) conjectured that the least favorable parameter configuration is of the slippage type. It is shown that this conjecture is true for four of these procedures. For a fifth procedure it is proved that at least a certain lower bound of the probability of a correct selection has this property which is analogous to the result of Tamhane and Bechhofer (1979) concerning the sixth procedure.

Some key words: Selection procedures; Two-stage procedures; Least favorable configurations; Indifference-zone approach.

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## 1. INTRODUCTION

Suppose we are given  $k$  normal populations  $\pi_1, \dots, \pi_k$  with different unknown means and a common (known or unknown) variance. If the experimenter's goal is to find that population having the largest mean by using suitably chosen samples, then a large variety of possible sampling plans and selection procedures can be found in the literature. In this paper we are dealing with the so-called two-stage procedures of the following type:

Stage 1: Take  $k$  independent samples  $(X_{i1}, \dots, X_{in})$  of size  $n$ ,  $i = 1, \dots, k$ , from  $\pi_1, \dots, \pi_k$  and select a non-empty subset of these populations according to a pre-specified rule  $S(\underline{X})$  where  $\underline{X} = (X_1, \dots, X_k)$  and  $X_i = X_{i1} + \dots + X_{in}$ ,  $i = 1, \dots, k$ . If the resulting subset consists of only one population, stop and decide that this is the population with the largest mean. Otherwise proceed to Stage 2.

Stage 2: Take additionally independent samples of size  $m$   $(Y_{i1}, \dots, Y_{im})$  from those populations  $\pi_i$  selected in Stage 1. Among the selected populations decide finally in favor of that population yielding the largest  $Y_i$  (or  $X_i + Y_i$ ), where  $Y_i = Y_{i1} + \dots + Y_{im}$ .

For convenience, let us represent the rules for Stage 1 in the form  $S: \mathbb{R}^k \rightarrow \{s \mid \emptyset \neq s \subseteq \{1, \dots, k\}\}$  where  $i \in s$  means that the  $i$ 'th population is included in the subset of selected populations. Moreover, let us represent the rules for Stage 2 in the form  $d = \{d_s \mid \emptyset \neq s \subseteq \{1, \dots, k\}\}$ , where for every  $s, d_s: \mathbb{R}^{2k} \rightarrow \{1, \dots, k\}$  and  $d_s(\underline{\xi}, \underline{\eta})$  depends only on those  $\xi_i$ 's and  $\eta_i$ 's with  $i \in s$ .

Let us now study in more detail the four possible two-stage procedures  $(S_\alpha, d_\beta)$ ,  $\alpha, \beta = 1, 2$  which we get after combining any two of the different

single-stage procedures given below  $[(S_3, d_1)$  and  $(S_3, d_2)$  will be discussed at the end of this section].

Stage 1: For  $i \in \{1, \dots, k\}$  let

$$i \in S_1(\underline{X}) \text{ iff } X_i \geq \max_{j=1, \dots, k} X_j - c; \quad c > 0 \text{ fixed,}$$

$$i \in S_2(\underline{X}) \text{ iff } X_i \text{ is one of the } t \text{ largest values of } X_1, \dots, X_k; \\ t \in \{2, \dots, k-1\} \text{ fixed,}$$

$$[i \in S_3(\underline{X}) \text{ iff } X_i \geq c_i; \quad c_1, \dots, c_k \in \mathbb{R} \text{ fixed.}]$$

Stage 2: For  $\emptyset \neq s \subseteq \{1, \dots, k\}$  and  $i \in s$  let

$$d_{1,s}(\underline{X}, \underline{Y}) = i \text{ iff } Y_i = \max_{j \in s} Y_j,$$

$$d_{2,s}(\underline{X}, \underline{Y}) = i \text{ iff } X_i + Y_i = \max_{j \in s} (X_j + Y_j).$$

A correct selection occurs whenever a procedure finally ends up with the population associated with the largest mean, which we may assume to be  $\pi_k$  without loss of generality, since we obviously are dealing with permutation invariant two stage procedures.

To implement such a procedure one usually wishes to guarantee that the probability of a correct selection is at least  $P^* > k^{-1}$  over a certain set of parameter configurations. Now if the means  $v_i$ , say,  $i = 1, \dots, k$ , are restricted to the condition  $v_1, \dots, v_{k-1} \leq v_k - \Delta$  with  $\Delta \geq 0$  fixed, then it seems intuitively clear that the infimum of the probability of a correct selection should occur at parameter configurations  $(v, v, \dots, v, v + \Delta)$  with  $v \in \mathbb{R}$ , which are called the least favorable configurations. Since there is no proof for these conjectures till now in the literature except for the

special case of  $k = 2$  populations for  $(S_1, d_2)$  (as we shall discuss below more explicitly) we have tried to fill this gap and solve the problems in a more general setup (without the assumption of normality). Briefly, we have been successful in proving the conjectures for procedures using  $S_2$  and  $S_3$  but not for those using  $S_1$  (cf. Remark below).

Discussion of different two-stage procedures:

$(S_1, d_\beta)$ :  $(S_1, d_2)$  has been studied by Alam (1970) and Tamhane and Bechhofer (1977, 1979). Alam has proved the conjecture for  $k = 2$  and his subsequent results are based on the assumption that the conjecture is true for all  $k$ . Tamhane and Bechhofer (1977, 1979) on the other hand used lower bounds for the probability of a correct selection which assume their infima at the desired parameter configurations.

$(S_1, d_1)$  has not been studied up to now. Surprisingly, it turns out that it is even difficult to prove the conjecture for this simpler procedure. Therefore, we propose a lower bound for the probability of a correct selection which appears to be quite good and which is minimal at the conjectured parameter configuration.

Remark: Recently Miescke and Sehr (1979) have shown that the conjecture holds true also for  $(S_1, d_1)$  and  $(S_2, d_2)$  in case of  $k = 3$ . Their proof is non-standard and uses geometrical arguments.

$(S_2, d_\beta)$ :  $(S_2, d_2)$  and  $(S_2, d_1)$  have been studied by Somerville (1971a) and (1974). In both papers he has claimed that the corresponding conjectures have been proved by Somerville (1954), Fairweather (1968) and Somerville (1971b).

Now the last paper Somerville (1971b) was shown to be in error by Carroll and Santner (1975), and, in fact, his method of proof does not even work for  $(S_2, d_1)$ . Moreover, the "loss function approach" of Somerville (1954) and Fairweather (1968) is not applicable in our problem since the corresponding function  $W$  used there turns out to be here an indicator function which clearly does not have a continuous second derivative. Therefore, the conjectures for  $(S_2, d_1)$  and  $(S_2, d_2)$  remained totally unproved up to now.

$(S_3, d_\beta)$ : These types of procedures may be used when the  $k$  populations are compared with a predetermined standard value  $v_0$ , say, for the means. They proceed in the same manner as the procedures discussed above, with the only difference that at Stage 1  $S_3(\underline{X})$  now may be empty, in which case we stop and decide that no population is better than the standard. The probability of this event is now desired to be at least  $\beta^*$ , say, if  $v_1, \dots, v_k \leq v_0$  and the probability of a correct selection is then studied over all parameter configurations with  $v_1, \dots, v_{k-1} \leq v_0^{-\Delta}$  and  $v_0 < v_k$ . We will show in this paper that the infimum occurs at the point  $(v_0^{-\Delta}, \dots, v_0^{-\Delta}, v_0)$ .

Remark: Let us finally mention that  $S_1, S_2, S_3$  and  $d$  are well-established one-stage multiple decision procedures, studied and used in a variety of papers which can not all be mentioned here. To give a few references, Gupta (1956, 1965) proposed and studied  $S_1$ , Bechhofer (1954)  $S_2$ , Dunnett (1955), Gupta and Sobel (1958) and Lehmann (1961) studied  $S_3$  and Bahadur and Goodman (1952), Lehmann (1966) and Miescke (1979) investigated  $d$ .

## 2. GENERAL CONSIDERATIONS CONCERNING ALL PROCEDURES

Let  $X_i, Y_i, i = 1, \dots, k$  be independent random variables where  $X_i$  and  $Y_i$  have distribution functions  $F(\xi - \theta_i), \xi \in \mathbb{R}$ , and  $G(\eta - \mu_i), \eta \in \mathbb{R}, i = 1, \dots, k$ .  $F$  and  $G$  are assumed to be known continuous functions and the  $\theta_i$ 's and  $\mu_i$ 's represent unknown location parameters. Since we restrict ourselves to two-stage procedures which are invariant under permutations of the  $k$  populations, we may assume, without loss of generality, that we have  $\theta_1, \dots, \theta_{k-1} \leq \theta_k$  and  $\mu_1, \dots, \mu_{k-1} \leq \mu_k$ .

Now let  $S$  be any subset selection rule for Stage 1. Then using  $d_1$  or  $d_2$  in Stage 2 the probabilities of correct selections are as follows:

$$P_k(S, d_1) = \sum_{\tilde{s} \subseteq \{1, \dots, k-1\}} P\{S(\underline{X}) = s\} \int_{\mathbb{R}} \prod_{i \in \tilde{s}} G(\eta + \mu_k - \mu_i) dG(\eta) \quad (2.1)$$

$$P_k(S, d_2) = \sum_{\tilde{s} \subseteq \{1, \dots, k-1\}} P\{S(\underline{X}) = s; \quad X_i + Y_i < X_k + Y_k, i \in \tilde{s}\} \quad (2.2)$$

with the understanding that here and in the sequel  $s = \tilde{s} \cup \{k\}$  if both  $s$  and  $\tilde{s}$  appear simultaneously. The product appearing in (2.1) is defined to be equal to one if  $\tilde{s}$  is empty.

In the sequel let  $|A|$  denote the size of any finite set  $A$ . Now we state our main result:

Theorem: For every  $\delta, \Delta \geq 0, \beta \in \{1, 2\}$  and  $\theta_0, \mu_0 \in \mathbb{R}$  the following holds:

- (i) Subject to  $\theta_1, \dots, \theta_{k-1} \leq \theta_k - \delta$  and  $\mu_1, \dots, \mu_{k-1} \leq \mu_k - \Delta$   $P_k(S_2, d_\beta)$  assumes its minimal value at every parameter configuration  $(\theta, \dots, \theta, \theta + \delta)$  and  $(\mu, \dots, \mu, \mu + \Delta)$  with  $\theta, \mu \in \mathbb{R}$ .
- (ii) Subject to the additional restrictions  $\theta_k \geq \theta_0$  and  $\mu_k \geq \mu_0$   $P_k(S_3, d_\beta)$  assumes its minimal value at every parameter configuration

$(0, \dots, 0, 0+\delta)$  and  $(\mu, \dots, \mu, \mu+\Delta)$  with  $\mu \geq \mu_0^{-\Delta}$  and  $\theta \geq \theta_0^{-\delta}$ .

Proof (first part): From expression (2.1) it is clear that  $P_k(S, d_1)$  for every  $S$  is non-increasing in  $\mu_1, \dots, \mu_{k-1}$  and non-decreasing in  $\mu_k$ . The same is seen to hold true for  $P_k(S, d_2)$  since for every  $\underline{\xi} \in \mathbb{R}^k$  and  $\tilde{s} \subseteq \{1, \dots, k-1\}$   $P\{S(\underline{X}) = s; X_i + Y_i < X_k + Y_k, i \in \tilde{s} | \underline{X} = \underline{\xi}\}$  also has this property.

This accomplishes the first step towards a solution of our problem. We can assume from now on that  $(\mu_1, \dots, \mu_k) = (\mu, \dots, \mu, \mu+\Delta)$  for some  $\mu \in \mathbb{R}$ , respectively,  $\mu \geq \mu_0^{-\Delta}$  holds. Then (2.1) reduces to

$$\begin{aligned} P_k(S, d_1) &= \sum_{\tilde{s} \subseteq \{1, \dots, k-1\}} P\{S(\underline{X}) = s\} \int_{\mathbb{R}} G(n+\Delta)^{|\tilde{s}|} dG(n) \\ &= \sum_{r=1}^k P\{k \in S(\underline{X}), |S(\underline{X})| = r\} \int_{\mathbb{R}} G(n+\Delta)^{r-1} dG(n) \end{aligned} \quad (2.3)$$

or, alternatively,

$$\begin{aligned} P_k(S, d_1) &= P\{k \in S(\underline{X})\} \int_{\mathbb{R}} G(n+\Delta)^{k-1} dG(n) \\ &+ \sum_{r=1}^{k-1} P\{k \in S(\underline{X}), |S(\underline{X})| \leq r\} \int_{\mathbb{R}} G(n+\Delta)^{r-1} [1-G(n+\Delta)] dG(n). \end{aligned} \quad (2.4)$$

And (2.2) reduces to

$$P_k(S, d_2) = \sum_{\tilde{s} \subseteq \{1, \dots, k-1\}} P\{S(\underline{X}) = s; X_i + U_i < X_k + \Delta + U_k, i \in \tilde{s}\} \quad (2.5)$$

where  $U_1, \dots, U_k$  are independently and identically distributed random variables with distribution function  $G$ , which are also independent of  $X_1, \dots, X_k$ . (End of proof's first part.)

Formulae (2.3)-(2.5) and the next lemma will be used repeatedly in



Sections 3 (and 4) when we give the second part of the proof, consisting of four versions corresponding to the four procedures under consideration.

Lemma: For every  $A \subseteq B \subseteq \{1, \dots, k-1\}$ ,  $r \in \{0, 1, \dots, |A|\}$ ,

$$a_1, \dots, a_{|A|} \in \mathbb{R} \text{ and } b_1, \dots, b_{|B|} \in \mathbb{R}$$

$$P\{|\{i | i \in A, X_i \geq a_i\}| \leq r; \quad X_j \leq b_j, j \in B\} \quad (2.6)$$

is nonincreasing in  $\theta_\ell$ ,  $\ell = 1, \dots, k-1$ .

Proof: For  $r = 0$  the assertion is clearly true. For  $r > 0$  and  $\ell \in A$ ,

(2.6) is equal to

$$\begin{aligned} & P\{|\{i | i \in A, i \neq \ell, X_i \geq a_i\}| \leq r-1; \quad X_j \leq b_j, j \in B, j \neq \ell\} P\{X_\ell \leq b_\ell\} \\ & + P\{|\{i | i \in A, i \neq \ell, X_i \geq a_i\}| = r; \quad X_j \leq b_j, j \in B, j \neq \ell\} P\{X_\ell \leq \min(a_\ell, b_\ell)\} \end{aligned}$$

which obviously is nonincreasing in  $\theta_\ell$ . Similarly one can prove the assertion in case of  $\ell \in B \setminus A$ , whereas in case of  $\ell \notin B$  it is trivially true, since in that case (2.6) does not even depend on  $\theta_\ell$ .

### 3. THE SECOND PART OF THE PROOF

#### 3.1 Case $(S_2, d_1)$

For every fixed  $t \in \{2, \dots, k-1\}$ , we have  $|S_2(\underline{X})| = t$  with probability one, and thus (2.3) reduces to

$$P_k(S_2, d_1) = P\{k \in S_2(\underline{X})\} \int_{\mathbb{R}} G(\eta + \Delta)^{t-1} dG(\eta). \quad (3.1)$$

Moreover, we have

$$\begin{aligned}
P\{k \in S_2(X)\} &= P\{|\{i | X_k < X_i\}| \leq t-1\} & (3.2) \\
&= \int_{\mathbb{R}} P\{|\{i | X_i > \xi, i \neq k\}| \leq t-1 | X_k = \xi\} dP\{X_k = \xi\} \\
&= \int_{\mathbb{R}} P\{|\{i | X_i > \xi + \theta_k, i \neq k\}| \leq t-1\} dF(\xi).
\end{aligned}$$

Since the integrand obviously is nondecreasing in  $\theta_k$  and by the Lemma is nonincreasing in  $\theta_1, \dots, \theta_{k-1}$ , the proof for  $(S_2, d_1)$  is completed.

### 3.2. Case $(S_2, d_2)$

Let  $t \in \{2, \dots, k-1\}$  be fixed. Then using the fact that  $(U_1 - U_k, \dots, U_{k-1} - U_k)$  is symmetrically distributed, from (2.5) we get

$$\begin{aligned}
P_k(S_2, d_2) &= & (3.3) \\
&= \sum_{1 \leq i_1 < i_2 < \dots < i_{t-1} \leq k-1} P\{X_{i_j} < X_i; \quad U_j - U_k + X_{i_j} < X_k + \Delta, j=1, \dots, t-1\} \\
&\quad \substack{\ell \notin S \quad i \in S \\ \tilde{s} = \{i_1, \dots, i_{t-1}\}} \\
&= \int_{\mathbb{R}} \int_{t-1} \sum_{1 \leq i_1 < i_2 < \dots < i_{t-1} \leq k-1} P\{X_{i_j} < X_i, \xi + \theta_k; \\
&\quad \substack{\ell \notin S \quad i \in \tilde{s} \\ \tilde{s} = \{i_1, \dots, i_{t-1}\}} \\
&\quad a_j + X_{i_j} < \xi + \theta_k + \Delta, j=1, \dots, t-1\} dF(\xi) dP\{U_j - U_k = a_j, j=1, \dots, t-1\}.
\end{aligned}$$

Since every probability term obviously is nondecreasing in  $\theta_k$ , it follows that  $P_k(S_2, d_2)$  also has this property.

Now we show that for every fixed  $\xi \in \mathbb{R}$  and  $a_1, \dots, a_{t-1} \in \mathbb{R}$  the integrand is nonincreasing in  $\theta_1, \dots, \theta_{k-1}$ . For sake of simplicity we prove it for  $\theta_1$ . Let  $b_j = \xi + \theta_k + \Delta - a_j$ ,  $j=1, \dots, t-1$  and  $\xi_k = \xi + \theta_k$ .

$$\sum_{1 \leq i_1 < i_2 < \dots < i_{t-1} \leq k-1} P\{X_\ell < X_{i_j}, \xi_k; \quad X_{i_j} < b_j, j=1, \dots, t-1\} \quad (3.4)$$

$$\ell \notin \tilde{s} \quad i \in \tilde{s}$$

$$\tilde{s} = \{i_1, \dots, i_{t-1}\}$$

$$= \int_{\mathbb{R}^{k-2}} \sum_{1 \leq i_1 < i_2 < \dots < i_{t-1} \leq k-1} P\{X_\ell < X_{i_j}, \xi_k; \quad X_{i_j} < b_j, j=1, \dots, t-1 | X_2 = \xi_2, \dots, X_{k-1} = \xi_{k-1}\} dP\{X_2 = \xi_2, \dots, X_{k-1} = \xi_{k-1}\}.$$

$$\ell \notin \tilde{s} \quad i \in \tilde{s}$$

$$\tilde{s} = \{i_1, \dots, i_{t-1}\}$$

Let now  $\xi_2, \dots, \xi_{k-1} \in \mathbb{R}$  be fixed and assume, without loss of generality, that  $\xi_2 < \xi_3 < \dots < \xi_{k-1}$  holds. Then the integrand reduces to

$$P\{X_1, X_{k-t} < X_{k-t+1}, \xi_k; \quad X_{k-t+1} \leq b_1, \quad (3.5)$$

$$X_{k-t+2} \leq b_2, \dots, X_{k-1} \leq b_{t-1} | X_2 = \xi_2, \dots, X_{k-1} = \xi_{k-1}\}$$

$$+ P\{X_{k-t+1} < X_1, \xi_k; \quad X_1 \leq b_1,$$

$$X_{k-t+2} \leq b_2, \dots, X_{k-1} \leq b_{t-1} | X_2 = \xi_2, \dots, X_{k-1} = \xi_{k-1}\}.$$

Finally we have to distinguish between two cases according to whether  $\xi_{k-t+1} < \xi_k$  or not. In case of  $\xi_{k-t+1} < \xi_k$ , (3.5) reduces to

$$P\{X_1 < X_{k-t+1} \leq b_1 ; \quad (3.6)$$

$$X_{k-t+2} \leq b_2, \dots, X_{k-1} \leq b_{t-1} \mid X_2 = \xi_2, \dots, X_{k-1} = \xi_{k-1}\}$$

$$+ P\{X_{k-t+1} < X_1 \leq b_1 ;$$

$$X_{k-t+2} \leq b_2, \dots, X_{k-1} \leq b_{t-1} \mid X_2 = \xi_2, \dots, X_{k-1} = \xi_{k-1}\}$$

$$= P\{X_1, X_{k-t+1} \leq b_1 ;$$

$$X_{k-t+2} \leq b_2, \dots, X_{k-1} \leq b_{t-1} \mid X_2 = \xi_2, \dots, X_{k-1} = \xi_{k-1}\},$$

whereas if  $\xi_{k-t+1} \geq \xi_k$ , (3.5) reduces to

$$P\{X_1, X_{k-t} < \xi_k; \quad X_{k-t+1} \leq b_1, \quad (3.7)$$

$$X_{k-t+2} \leq b_2, \dots, X_{k-1} \leq b_{t-1} \mid X_2 = \xi_2, \dots, X_{k-1} = \xi_{k-1}\}.$$

Since now these last terms are nonincreasing in  $\theta_1$ , the proof for  $(S_2, d_2)$  is completed.

### 3.3. Case $(S_3, d_1)$

For every fixed  $c_1, \dots, c_k \in \mathbb{R}$  and using (2.4), it suffices to show that for every  $r \in \{1, \dots, k\}$   $P\{k \in S_3(\underline{X}), |S_3(\underline{X})| \leq r\}$  is nonincreasing in  $\theta_1, \dots, \theta_{k-1}$  and nondecreasing in  $\theta_k$ . Now

$$P\{k \in S_3(\underline{X}), |S_3(\underline{X})| \leq r\} = \quad (3.8)$$

$$= P\{|\{i \mid X_i \geq c_i, i \neq k\}| \leq r-1\} P\{X_k \geq c_k\}, \quad r = 1, \dots, k.$$

The first factor does not depend on  $\theta_k$  and by the lemma is nonincreasing

in  $\theta_1, \dots, \theta_{k-1}$ , whereas the second factor does not depend on  $\theta_1, \dots, \theta_{k-1}$  and is nondecreasing in  $\theta_k$ . Thus the proof for  $(S_3, d_1)$  is completed.

### 3.4. Case $(S_3, d_2)$

For every fixed  $c_1, \dots, c_k$ , from (2.5) we have

$$P_k(S_3, d_2) = \sum_{\tilde{s} \subseteq \{1, \dots, k-1\}} P\{X_i \geq c_i, X_j < c_j; \quad (3.9)$$

$$X_i + U_i < X_k + \Delta + U_k\}$$

$$i \in \tilde{s} \quad j \notin \tilde{s}$$

which clearly is nondecreasing in  $\theta_k$  since in every summand we have  $s = \tilde{s} \cup \{k\}$  by our convention. Again, for sake of simplicity, it will be shown that  $P_k(S_3, d_2)$  is nonincreasing in  $\theta_1$ . Now

$$P_k(S_3, d_2) = \sum_{\tilde{s} \subseteq \{2, \dots, k-1\}} P\{X_i \geq c_i, X_1 < c_1, X_j < c_j; \quad (3.10)$$

$$i \in \tilde{s} \quad j \notin \tilde{s}$$

$$j \neq 1$$

$$X_i + U_i < X_k + \Delta + U_k\}$$

$$i \in \tilde{s}$$

$$+ \sum_{\tilde{s} \subseteq \{2, \dots, k-1\}} P\{X_i \geq c_i, X_1 \geq c_1, X_j < c_j; \quad (3.10)$$

$$i \in \tilde{s} \quad j \notin \tilde{s}$$

$$j \neq 1$$

$$X_1 + U_1, X_i + U_i < X_k + \Delta + U_k\}$$

$$i \in \tilde{s}$$

$$= \sum_{\tilde{s} \subseteq \{2, \dots, k-1\}} P\{X_i \geq c_i, X_1 < c_1, X_j < c_j; \quad (3.10)$$

$$i \in \tilde{s} \quad j \notin \tilde{s}$$

$$j \neq 1$$

$$X_i + U_i < X_k + \Delta + U_k \leq c_1 + U_1\}$$

$$i \in \tilde{s}$$

$$+ \sum_{\tilde{s} \subseteq \{2, \dots, k-1\}} P\{X_i \geq c_i, X_j < c_j; \\ i \in s \quad j \notin s \\ j \neq 1 \\ c_1 + U_1, X_1 + U_1, X_{i+U_i} < X_k + \Delta + U_k\}. \\ i \in \tilde{s}$$

Clearly, every summand is nonincreasing in  $\theta_1$  and, therefore, the proof for  $(S_3, d_2)$  is completed.

### 3.5. Concluding Remarks

In the case of normal populations as described in Section 1 we have  $X_i \sim N(nv_i, n\sigma^2)$  and  $Y_i \sim N(mv_i, m\sigma^2)$ ,  $i = 1, \dots, k$ , for some  $\sigma^2 > 0$ . Thus for every  $\alpha \in \{1, 2, 3\}$  and  $\beta \in \{1, 2\}$  the probability that  $(S_\alpha, d_\beta)$  finally leads to a decision in favor of population  $\pi_k$  can be represented by a certain function  $H_{\alpha, \beta}(v_1, v_2, \dots, v_k)$ . To prove the conjectures we could, alternatively, have tried to show that  $H_{\alpha, \beta}$  is nonincreasing in  $v_1, \dots, v_{k-1}$  and nondecreasing in  $v_k$ . But this turns out to be a very difficult and cumbersome way.

### 4. A LOWER BOUND FOR $(S_1, d_1)$

To prove the conjecture for  $(S_1, d_1)$  in view of (2.4) it would suffice to show that for every  $r \in \{1, \dots, k\}$   $P\{k \in S_1(\underline{X}), |S_1(\underline{X})| \leq r\}$  is nonincreasing in  $\theta_1, \dots, \theta_{k-1}$  and non-decreasing in  $\theta_k$ . For  $r = 1$  and  $r = k$  this probability is equal to  $P\{X_1, \dots, X_{k-1} < X_k - c\}$  and  $P\{X_1, \dots, X_{k-1} \leq X_k + c\}$ , respectively, where each of them clearly has the desired property. Thus the conjecture for  $k = 2$  is proved.

For  $k > 2$  it turns out to be rather difficult to prove the conjecture. Therefore, we derive a lower bound for the probability of a correct

selection, which assumes its minimal value at the desired parameter configuration.

For  $k > 2$  and  $r \in \{2, \dots, k-1\}$  we have

$$\begin{aligned}
 & P\{k \in S_1(\underline{X}), |S_1(\underline{X})| \leq r\} = \tag{4.1} \\
 &= \int_{\mathbb{R}} P\{k \in S_1(X_1, \dots, X_{k-1}, \xi), |S_1(X_1, \dots, X_{k-1}, \xi)| \leq r | X_k = \xi\} \\
 & \quad dP\{X_k = \xi\} \\
 & \geq \int_{\mathbb{R}} P\{|\{i | X_i \geq \xi + \theta_k - c, i \neq k\}| \leq r-1, \\
 & \quad X_1, \dots, X_{k-1} \leq \xi + \theta_k + c\} dF(\xi).
 \end{aligned}$$

The preceding inequality holds also for  $r = 1$  and follows from the fact that we have

$$\begin{aligned}
 & \{k \in S(\underline{X}), |S(\underline{X})| \leq r\} = \tag{4.2} \\
 &= \{|\{i | X_i \geq X_k - c, i \neq k\}| \leq r-1, X_k = \max_{j=1, \dots, k} X_j\} \\
 & \cup_{\ell=1}^{k-1} \{|\{i | X_i \geq X_\ell - c, i \neq k, \ell\}| \leq r-2, X_k \geq X_\ell - c, X_\ell = \max_{j=1, \dots, k} X_j\}.
 \end{aligned}$$

Now the integrand in the last integral of (4.1) clearly is nondecreasing in  $\theta_k$  and by our lemma it is nonincreasing in  $\theta_1, \dots, \theta_{k-1}$ . Thus, using (2.4), we get

$$\begin{aligned}
 & P_k(S_1, d_1) \geq P\{Z_1, \dots, Z_{k-1} \leq Z_k + \delta + c\} \int_{\mathbb{R}} G(n+\Delta)^{k-1} dG(n) \tag{4.3} \\
 & + \sum_{r=1}^{k-1} P\{|\{i | Z_i \geq Z_k + \delta - c, i \neq k\}| \leq r-1,
 \end{aligned}$$

$$Z_1, \dots, Z_{k-1} \leq Z_k + \delta + c \int_{\mathbb{R}} G(\eta + \Delta)^{r-1} [1 - G(\eta + \Delta)] dG(\eta)$$

where  $Z_1, \dots, Z_k$  are independently and identically distributed random variables with distribution function  $F$ . Now the right hand side of (4.3), being in a form similar to (2.4), can be brought into a form similar to that of (2.3). Then it is equal to

$$\begin{aligned} \sum_{r=1}^k P\{\{i | Z_i \geq Z_k + \delta - c, i \neq k\} = r-1, \\ Z_1, \dots, Z_{k-1} \leq Z_k + \delta + c\} \int_{\mathbb{R}} G(\eta + \Delta)^{r-1} dG(\eta) \\ = \sum_{i=0}^{k-1} \int_{\mathbb{R}} \binom{k-1}{i} F(\xi + \delta - c)^{k-i-1} [F(\xi + \delta + c) - F(\xi + \delta - c)]^i dF(\xi) \int_{\mathbb{R}} G(\eta + \Delta)^i dG(\eta). \end{aligned} \quad (4.4)$$

Thus we finally arrive at the following result:

Corollary: For  $k \geq 2$

$$\begin{aligned} P_k(S_1, d_1) &\geq \\ &\geq \int_{\mathbb{R}} \int_{\mathbb{R}} \{F(\xi + \delta - c) + [F(\xi + \delta + c) - F(\xi + \delta - c)]G(\eta + \Delta)\}^{k-1} dF(\xi) dG(\eta). \end{aligned} \quad (4.5)$$

Note that for  $k = 2$  this lower bound for the probability of a correct selection is exact.

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It is shown that this conjecture is true for four of these procedures. For a fifth procedure it is proved that at least a certain lower bound of the probability of a correct selection has this property which is analogous to the result of Tamhane and Bechhofer (1979) concerning the sixth procedure.

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