An Extension of Kazamaki's Results on BMO Differentials*

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ABSTRACT

Kazamaki has shown that if $(M^n)_{n \geq 1}$, M are BMO martingales with continuous paths and $\lim M^n = M$ in BMO, then $\mathcal{E}(M^n)$ converges in $\underline{\underline{H}}^1$ to $\mathcal{E}(M)$, where $\mathcal{E}(M)$ denotes the stochastic exponential of M. While Kazamaki's result does not extend to the right continuous case, it does extend "locally." It is shown here that if M^n, M are semimartingales and M^n converges locally in $\underline{\underline{H}}^\omega$ (a semimartingale BMO-type norm) to M then X^n converges locally in $\underline{\underline{H}}^p$ $(1 \leq p < \infty)$ to X, where X^n, X are respectively solutions of stochastic integral equations with Lipschitz-type coefficients and differentials dM^n, dM. (The coefficients are also allowed to vary.) This is a stronger stability than usually holds for solutions of stochastic integral equations, reflecting the strength of the $\underline{\underline{H}}^\omega$ norm.

1. INTRODUCTION

Recently Kazamaki [5] and Kazamaki and Sekiguchi [6] showed that if M is a continuous martingale in $\underline{\underline{BMO}}$ then the stochastic exponential $\mathcal{E}(M)$ is in $\underline{\underline{H}}^1$, and if M^n converges in the $\underline{\underline{BMO}}$ martingale norm to a martingale M in $\underline{\underline{BMO}}$ then $\mathcal{E}(M^n)$ converges to $\mathcal{E}(M)$ in the $\underline{\underline{H}}^1$ martingale norm. Simple examples show that these results do not extend to the right continuous case: $M \in \underline{\underline{BMO}}$ does not necessarily imply that $\mathcal{E}(M) \in \underline{\underline{H}}^1$ (cf. Remarks (3,7), #3). One does have, of course, that $M \in \underline{\underline{BMO}}$ implies that $\mathcal{E}(M)$ is locally in $\underline{\underline{H}}^1$, but this is not surprising since every local martingale is locally in $\underline{\underline{H}}^1$!

A consequence of the results presented here is that if $M \in \underline{\underline{BMO}}$ then $\mathcal{E}(M)$ is locally in $\underline{\underline{H}}^p$ for all p, $1 \leq p < \infty$. Moreover we show that if M^n converges locally in $\underline{\underline{BMO}}$ to M then $\mathcal{E}(M^n)$ converges locally in $\underline{\underline{H}}^p$ to $\mathcal{E}(M)$ for all $p (1 \leq p < \infty)$. We extend Kazamaki's results further, however, by working with semimartingales and general stochastic differential equations. Suppose X, X^n are respectively solutions of

$$X_t = J_t + \int_0^t (FX)_{s-} dM_s \tag{1.1}$$

$$X^n_t = J^n_t + \int_0^t (F^n X^n)_{s-} dM^n_s \tag{1.2}$$

where M, M^n, J, J^n are semimartingales and where F, $F^n \in \mathrm{Lip}\,(K)$. (Precise definitions are given in section 2.) Meyer [9] has extended the notion of $\underline{\underline{BMO}}$ martingales to semimartingales. Such a semimartingale is said to be in $\underline{\underline{H}}^\omega$. We show that if $M \in \underline{\underline{H}}^\omega$ and $J \in \underline{\underline{H}}^p$ for some $p (1 \leq p < \infty)$, then X is locally in $\underline{\underline{H}}^p$ for the same p. Moreover we show that if M^n converges locally in $\underline{\underline{H}}^\omega$ to M, and if F^n converges to F and J^n converges to J in appropriate ways, then X^n converges locally in $\underline{\underline{H}}^p$ to X ($1 \leq p < \infty$ with p depending on the convergence of F^n, J^n to F and J). This principal result is the content of Theorem (3.4).

By insisting that the differentials converge locally in $\underline{\underline{H}}^{\omega}$ we obtain local convergence of the solutions: that is, we get $(X^n)^{T_k}$ converging to X^{T_k} in $\underline{\underline{H}}^p$ for stopping times T_k increasing to ∞ a.s. (X and X^n are the solutions respectively of (1.1) and (1.2)). The previous best results (Emery [3] and Protter [12]) obtained only the <u>weak-local</u> convergence of a <u>subsequence</u>.

Section 2 consists of preliminaries including some recent developments not contained in Meyer [7]. One innovation is the generalization of Emery's idea of "carving" a semimartingale into small $\underline{\underline{H}}^{\infty}$ slices; by requiring only that the slices be in $\underline{\underline{H}}^p$, we show in Lemma (2.15) that if M^n converges to M in $\underline{\underline{H}}^p$ then there exists an N such that for all $n > N$ the same stopping times that carve M are also carving times for M^n. The convergence theorem described above (Theorem (3.4)) is the content of section 3. In section 4 we extend a result of Garcia, Maillard, and Peltraut [4] by constructing a local martingale with a given <u>random</u> "multiplicative jump" at a given totally inaccessible stopping time. We then apply Theorem (3.4) to obtain a continuity theorem for martingales with multiplicative jumps.

I wish to thank C. Doléans-Dade for her careful reading of an earlier version of this paper. Her comments have lead (I hope) to a much more readable and correct version. She also pointed out a gap in the proof of Theorem (3.4) which is now filled.

2. PRELIMINARIES

We use the notation of and assume the reader is familiar with the theory of the semimartingale calculus as given in Meyer [7]. Let $(\Omega, \underline{\underline{F}}, P)$ be a complete probability space and let $(\underline{\underline{F}}_t)_{t \geq 0}$ be a right continuous filtration with $\underline{\underline{F}}_{\infty} = \underline{\underline{F}}$ and where $\underline{\underline{F}}_0$ contains all P-null sets.

Let $\underline{\underline{C}}$ denote the adapted processes whose paths are right continuous with left limits (cadlag). For $J \in \underline{\underline{C}}$ let

$$||J||_{\underline{\underline{S}}^p} = || \sup_{s<\infty} |J_s| \, ||_{L^p} \qquad (1 \leq p \leq \infty).$$

For a stopping time T we say a process $X \in \underline{\underline{C}}$ <u>is stopped at</u> T- if

$$X_t = X_t^{T-} = X_t 1_{[0,T[} + X_{T-} 1_{[T,\infty[}$$

where $X_{t-} = \lim_{s \to t, s<t} X_s$. We let

$$\delta X_T = X_T - X_{T-},$$

the jump at T, and we make the notational conventions that

$$||X||_{\underline{\underline{S}}^p(T)} = ||X^T||_{\underline{\underline{S}}^p}; \quad ||X||_{\underline{\underline{S}}^p(T-)} = ||X^{T-}||_{\underline{\underline{S}}^p} .$$

Emery [1] and Meyer [9] have proposed $\underline{\underline{H}}^p$ norms ($1 \leq p \leq \infty$ and $p = \omega$) for semimartingales. (Here ω represents the first limit ordinal and is not a point in Ω.) Meyer has shown that the $\underline{\underline{H}}^p$ norms for semimartingales are equivalent to the martingale $\underline{\underline{H}}^p$ norms when the process in question is a martingale ($1 \leq p \leq \infty$ or $p = \omega$; $p = \omega$ corresponds to the BMO martingales).

For a semimartingale M with a decomposition $M = N + A$ where N is a local martingale and A is a VF process, we define ($1 \leq p \leq \infty$)

$$j_p(N,A) = || [N,N]_\infty^{\frac{1}{2}} + \int_{0-}^{\infty} |dA_s| \, ||_{L^p}.$$

For $p = \omega$, $j_\omega(N,A)$ is the smallest constant c such that for any stopping time T

$$E\{([N,N]_\infty - [N,N]_{T-})^{\frac{1}{2}} + \int_{T-}^{\infty} |dA_s| \, | \underline{\underline{F}}_T\} \leq c \quad \text{a.s.}$$

We let

$$||M||_{\underline{\underline{H}}^p} = \inf_{M=N+A} j_p(N,A), \quad 1 \leq p \leq \infty \text{ or } p = \omega,$$

where the infimum is taken over all possible decompositions of M. For notational convenience we write

$$||X||_{\underline{\underline{H}}^p(T)} = ||X^T||_{\underline{\underline{H}}^p}; \; ||X||_{\underline{\underline{H}}^p(T-)} = ||X^{T-}||_{\underline{\underline{H}}^p}.$$

We caution the reader, however, that Emery in [2] uses the notation $||\cdot||_{\underline{\underline{H}}^p(T-)}$ differently: $||X||_{\underline{\underline{H}}^p(T-)}$ denotes (in [2]) the infimum of the $\underline{\underline{H}}^p$ norms of all semimartingales L such that $L^{T-} = X^{T-}$.

The next proposition is elementary but it may give the reader some feeling for the $||\cdot||_{\underline{\underline{H}}^\omega}$ norm, which can be thought of as an extension to semimartingales of the $\underline{\underline{BMO}}$ norm for martingales.

(2.1) PROPOSITION. Let X and Y be semimartingales with $||X||_{\underline{\underline{H}}^\omega} < \infty$ and $||Y||_{\underline{\underline{H}}^\omega} < \infty$, and let T be a stopping time. Then the following hold:

(2.2) $$||\delta X_T 1_{[T,\infty[}||_{\underline{\underline{H}}^\omega} \leq ||X||_{\underline{\underline{H}}^\omega(T)}$$

(2.3) $$||X||_{\underline{\underline{H}}^\omega(T-)} \leq 2||X||_{\underline{\underline{H}}^\omega(T)}$$

(2.4) $$||X||_{\underline{\underline{H}}^\omega(T)} \leq ||X||_{\underline{\underline{H}}^\omega}$$

(2.5) $$||X+Y||_{\underline{\underline{H}}^\omega(T-)} \leq ||X||_{\underline{\underline{H}}^\omega(T-)} + ||Y||_{\underline{\underline{H}}^\omega(T-)}.$$

PROOF. Let X = N+A be any decomposition of X. Then $|\delta X_T| \leq |\delta N_T| + |\delta A_T|$. For any stopping time S we have

$$|\delta X_T| 1_{\{S\leq T\}} \leq ([N,N]_T - [N,N]_{S-})^{\frac{1}{2}} + \int_{S-}^{T} |dA_s|,$$

and (2.2) follows upon conditioning with respect to $\underline{\underline{F}}_S$.

To establish (2.3) note that $||X||_{\underline{\underline{H}}^\omega(T-)} = ||X-\delta X_T 1_{[T,\infty[}||_{\underline{\underline{H}}^\omega(T)} \leq 2||X||_{\underline{\underline{H}}^\omega(T)}$. Inequality (2.4) is clear and (2.5) is merely the triangle inequality applied to the semimartingales X^{T-} and Y^{T-}. □

We make repeated use of the following inequalities. A proof can be found in Meyer [9].

(2.6) EMERY-MEYER INEQUALITIES. *Let* $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, *or* $1 \leq p \leq \infty$ *and* $q = \omega$, *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. *Let* X *be predictable and* M *be a semimartingale such that the stochastic integral* $X \cdot M$ *exists. Then*

(2.7) $$||X \cdot M||_{\underline{\underline{H}}^r} \leq ||X||_{\underline{\underline{S}}^p} ||M||_{\underline{\underline{H}}^q}$$

(2.8) $$||X \cdot M||_{\underline{\underline{H}}^p} \leq h_p ||X||_{\underline{\underline{S}}^p} ||M||_{\underline{\underline{H}}^\omega}.$$

If $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *but* $r < \infty$, *then*

(2.9) $$||X \cdot M||_{\underline{\underline{S}}^r} \leq c_r ||X||_{\underline{\underline{S}}^p} ||M||_{\underline{\underline{H}}^\omega}$$

and if $1 \leq p < \infty$, *then*

(2.10) $$||X \cdot M||_{\underline{\underline{S}}^p} \leq s_p ||X||_{\underline{\underline{S}}^p} ||M||_{\underline{\underline{H}}^\omega}.$$

where h_p, c_r, s_p *are universal constants.*

We record here a trivial but useful observation.

(2.11) PROPOSITION. *Let* X *be predictable, let* M *be a semimartingale and suppose the stochastic integral* $X \cdot M$ *exists. For any stopping times* S, T *with* $S < T$ a.s. *we have for* $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$:

$$||X \cdot M - X \cdot M^S||_{\underline{\underline{H}}^r(T-)} \leq ||X||_{\underline{\underline{S}}^p(T)} ||M - M^S||_{\underline{\underline{H}}^q(T-)}$$

$$||X \cdot M - X \cdot M^S||_{\underline{\underline{H}}^p(T-)} \leq h_p ||X||_{\underline{\underline{S}}^p(T)} ||M - M^S||_{\underline{\underline{H}}^\omega(T-)}$$

If $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *but* $r < \infty$, *then*

$$||X \cdot M - X \cdot M^S||_{\underline{\underline{S}}^r(T-)} \leq c_r ||X||_{\underline{\underline{S}}^p(T)} ||M - M^S||_{\underline{\underline{H}}^q(T-)}$$

and if $1 \leq p \leq \infty$, *then*

$$||X \cdot M - X \cdot M^S||_{\underline{\underline{S}}^p(T-)} \leq s_p ||X||_{\underline{\underline{S}}^p(T)} ||M - M^S||_{\underline{\underline{H}}^\omega(T-)}.$$

PROOF. Let $N = (M-M^S)^{T-}$. The proposition then follows by an application of the Emery-Meyer inequalities (2.6) □

We shall also use a technique developed by Doléans-Dade, Meyer, and Emery, the idea of which is contained in the next definition, which was first given in Emery [1].

(2.11) DEFINITION. Let $\varepsilon > 0$ and M be a semimartingale. M is said to be *carved in slices smaller than* ε if there exists a *finite* sequence of stopping times (*carving times*) $0 = T_0 < T_1 < \dots < T_k$ such that $M = M^{T_k^-}$, $M \in \underline{\underline{H}}^\infty$ and for $1 \leq i \leq k$,

$$\text{(2.12)} \qquad \| M - M^{T_{i-1}} \|_{\underline{\underline{H}}^\infty (T_i -)} < \varepsilon .$$

We write $M \in D^\infty(\varepsilon)$ if M is cut in slices smaller than ε. We also write $M \in D^\infty(\varepsilon, k)$ to signify the number k of non-zero carving times needed to cut M into slices smaller than ε.

(2.13) DEFINITION. For $\varepsilon > 0$ we say that M *is in* $D^p(\varepsilon,k)$, $1 \leq p \leq \infty$ *or* $p = \omega$ if there exists a *finite* sequence of stopping times $0 = T_0 < T_1 < \dots < T_k$ such that $M = M^{T_k -}$, $M \in \underline{\underline{H}}^p$, and for $1 \leq i \leq k$,

$$\text{(2.14)} \qquad \| M - M^{T_{i-1}} \|_{\underline{\underline{H}}^p (T_i -)} < \varepsilon .$$

Note that $M \in D^q(\varepsilon,k)$ implies that $M \in D^p(\varepsilon,k)$ for $q \geq p$, or $q = \infty$ and $p = \omega$.

The next lemma is due to Emery ([1] or [3]).

(2.15) LEMMA. *Let* M *be a semimartingale. For each* $\varepsilon > 0$ *there exists an arbitrarily large stopping time* T *and a constant* k *depending on* T *and* ε *such that* $M^{T-} \in D^p(\varepsilon, k)$, $1 \leq p \leq \infty$ *and* $p = \omega$.

PROOF. It suffices to prove the result for the $\underline{\underline{H}}^\infty$ norm since it is stronger than the $\underline{\underline{H}}^p$ norm, $1 \leq p \leq \infty$ or $p = \omega$. Let $M = N+A$ be a decomposition of M. By letting $C_t = \sum_{s \leq t} \delta N_s 1_{\{|\delta N_s| \geq \gamma/2\}}$ and $\tilde{C}_t$ be its dual predictable projection (also called its "compensator") we have that $\hat{N} = N_t + (C_t - \tilde{C}_t)$ is a martingale. It is a simple matter to check that C_t has locally integrable variation and that γ is a bound for the jumps of $\hat{N}$. (See Meyer [8] for the details). So we assume that $M=N+A$ with γ bounding the jumps of N. Let $R_0=0$ and inductively define:

$$R_{k+1} = \inf\{t \geq R_k : \int_{]R_k,t]} |dA_s| \geq \gamma \text{ or } \int_0^t |dA_s| \geq k\}.$$

For each k, $A^{R_k-} \in D^\infty(\gamma)$. Let $S_0 = 0$ and inductively define:

$$S_{k+1} = \inf\{t \geq S_k : [N,N]_t - [N,N]_{T_k} \geq \gamma^2 \text{ or } [N,N]_t \geq k\}.$$

For each k, N^{S_k-} is in $\underline{\underline{H}}^\infty$ and since

$$(N-N^{S_k})_t^{S_{k+1}-} = (N_t^{S_{k+1}} - N_t^{S_k}) - \delta N_{S_k} 1_{[S_k,\infty[}$$

we have

$$||N-N^{S_k}||_{\underline{\underline{H}}^\infty(S_{k+1}-)} \leq ||([N,N]_{S_{k+1}} - [N,N]_{S_k})^{\frac{1}{2}} + |\delta N_{S_{k+1}}|\,||_{L^\infty}$$

$$\leq ||((\delta N_{S_{k+1}})^2 + [N,N]_{S_{k+1}-} - [N,N]_{S_k})^{\frac{1}{2}} + |\delta N_{S_{k+1}}|\,||_{L^\infty}$$

$$(\gamma^2+\gamma^2)^{\frac{1}{2}} + \gamma = (1+\sqrt{2})\gamma.$$

Thus taking $\gamma = \varepsilon/2(1+\sqrt{2})$, we have $N \in D^\infty(\varepsilon/2)$ and $A \in D^\infty(\varepsilon/2)$. But it is a simple matter to check that the sum of two elements of $D^\infty(\varepsilon/2)$ is in $D^\infty(\varepsilon)$. □

The next lemma will be used in the proof of Lemma (3.25), which in turn is crucial to the proof of Theorem (3.4). It is important because we can have $M^n \in D^p(\alpha,k)$ for $n \geq N$ where the constant k does not depend on n. The case of interest for us is $p = \omega$.

(2.16) PROPOSITION. *Fix a* p, $1 \leq p \leq \infty$ *or* $p = \omega$ *and let* M *be a semimartingale such that* $||M||_{\underline{\underline{H}}^p} < \infty$. *Suppose* M^n *is a sequence of semimartingales such that* $\lim_{n\to\infty} ||M-M^n||_{\underline{\underline{H}}^p} = 0$. *Then for any* $\varepsilon > 0$ *there exists an arbitrarily large stopping time* T *and constants* N *and* k *such that* $M \in D^p(\varepsilon,k)$ *and* $M^n \in D^p(\varepsilon,k)$ *for all* $n > N$.

PROOF. Lemma (2.15) assurs us of the existence of an arbitrarily large T such that $M \in D^p(\varepsilon/2,k)$. Choose $\gamma = \varepsilon/8$ and choose N so that $n > N$ implies $||M-M^n||_{\underline{\underline{H}}^p} < \gamma$. Let $0 = T_0 < T_1 < \ldots < T_k = T$ be the stopping times that carve M into slices. Then

$$\begin{aligned} ||M^n-(M^n)^{T_{i-1}}||_{\underline{\underline{H}}^p(T_i-)} &= ||(M^n-M)-(M^n-M)^{T_{i-1}} + (M-M^{T_{i-1}})||_{\underline{\underline{H}}^p(T_i-)} \\ &\leq 2||M^n-M||_{\underline{\underline{H}}^p(T_{i+1}-)} + ||(M-M^{T_{i-1}})||_{\underline{\underline{H}}^p(T_i-)} \\ &\leq 4||M^n-M||_{\underline{\underline{H}}^p} + \varepsilon/2 < \varepsilon. \end{aligned}$$

Thus the same times $T_0,\ldots,T_k$ carve each M^n into k slices, each smaller than ε, for $n > N$. □

The most general coefficients used in scalar stochastic differential equations for which unique solutions exist are those which satisfy the conditions described in the next definition.

(2.17) DEFINITION. Let $K > 0$ and let F be an operator mapping $\underline{\underline{C}}$ into itself. F *is said to be in* Lip(K) if the following two conditions are satisfied:

(2.18) For X, Y in $\underline{\underline{C}}$ and each stopping time T, $X^{T-} = Y^{T-}$ implies $(FX)^{T-} = (FY)^{T-}$;

(2.19) $(FX-FY)^* \leq K(X-Y)^*$ as processes, where $X_t^* = \sup_{s \leq t} |X_s|$.

We state our results in terms of local convergences. Processes X^n converge locally in a norm $||\cdot||$ to X if there exists a sequence of stopping times $(T_k)_{k \geq 1}$ increasing to ∞ a.s. such that for each fixed k, $\lim_{n\to\infty} ||(X^n-X)^{T_k}|| = 0$.

We remark that in [12] we used a convergence designated "weak-local", and Emery in [2] uses simply the term "local" to denote weak-local convergence. The processes X^n converge to X weak-locally if $\lim_{n\to\infty} ||(X^n-X)^{T_k-}|| = 0$ for each k. We will not need this type of convergence, but see Remark (3.7), #4, in section 3.

3. STOCHASTIC DIFFERENTIAL EQUATIONS.

We consider the following type of equation:

$$(3.1) \qquad X_t = J_t + \sum_{i=1}^{k} \int_0^t (F_i X)_{s-} dM_s^i$$

where M^i are semimartingales, F_i are in Lip(K), and J is either a semimartingale or $J \in \underline{\underline{C}}$ $(1 \leq i \leq k)$. For simplicity of notation, we assume k=1. We are interested only in the case $||M||_{\underline{\underline{H}}^\infty} < \infty$. We refer the reader to Emery [2,3] for an account of recent results concerning this equation. In particular, unique solutions exist.

Whether $J \in \underline{\underline{C}}$ or J is a semimartingale governs what norm we use for the solution X. The semimartingale-norm $||\cdot||_{\underline{\underline{H}}^p}$ is a stronger norm than the $\underline{\underline{C}}$-norm $||\cdot||_{\underline{\underline{S}}^p}$, however the following lemma allows us to work exclusively with $||\cdot||_{\underline{\underline{S}}^p}$ and still deduce the results for $||\cdot||_{\underline{\underline{H}}^p}$, should J be a semimartingale.

(3.2) LEMMA. Suppose M, J, M^n, and J^n are all semimartingales for $n \geq 1$. Let X and X^n be solutions respectively of

$$X_t = J_t + \int_0^t (FX)_{s-} dM_s$$

$$X_t^n = J_t^n + \int_0^t (F^n X^n)_{s-} dM_s^n$$

where F and F^n are in Lip(K), some K independent of n. Suppose that (1) J^n tends to J locally in $\underline{\underline{H}}^p$, (2) $F^n X$ tends to FX locally in $\underline{\underline{S}}^p$, (3) $||M||_{\underline{\underline{H}}^\omega} < \infty$ and M^n tends to M locally in $\underline{\underline{H}}^\omega$. Then X^n tends to X locally in $\underline{\underline{H}}^p$ if and only if X^n tends to X locally in $\underline{\underline{S}}^p$ $(1 \leq p \leq \infty)$.

PROOF. Necessity is simply a consequence of the fact that the $\underline{\underline{H}}^p$ norm is stronger than the $\underline{\underline{S}}^p$ norm. Without loss of generality we may assume that the convergence hypothesized in conditions (1), (2), and (3) above is global, not local. Further, by stopping at an arbitrarily large time T if necessary, we may assume that $||FX_-||_{\underline{\underline{S}}^p} < \infty$. Then by hypothesis (2), the assumption that $||X^n - X||_{\underline{\underline{S}}^p}$ tends to 0, and the inequality

$$||F^n X_-^n||_{\underline{\underline{S}}^p} \leq ||F^n X_-^n - F^n X_-||_{\underline{\underline{S}}^p} + ||F^n X_- - FX_-||_{\underline{\underline{S}}^p} + ||FX_-||_{\underline{\underline{S}}^p}$$

$$\leq K||X_-^n - X_-||_{\underline{\underline{S}}^p} + ||F^n X - FX||_{\underline{\underline{S}}^p} + ||FX_-||_{\underline{\underline{S}}^p},$$

we may assume $||F^n X_-^n||_{\underline{\underline{S}}^p}$ is bounded uniformly in n. Using the Emery-Meyer inequalities we have:

$$||X^n - X||_{\underline{\underline{H}}^p} \leq ||J^n - J||_{\underline{\underline{H}}^p} + ||FX_- - F^n X_-||_{\underline{\underline{S}}^p} ||M||_{\underline{\underline{H}}^\omega} \tag{3.3}$$

$$+ ||F^n X_- - F^n X_-^n||_{\underline{\underline{S}}^p} ||M||_{\underline{\underline{H}}^\omega} + ||F^n X_-^n||_{\underline{\underline{S}}^p} ||M - M^n||_{\underline{\underline{H}}^\omega}$$

and since $||F^nX_- - F^nX_-||_{\underline{\underline{S}}^p} \leq K||X_- - X^n_-||_{\underline{\underline{S}}^p}$ and $||F^nX^n_-||_{\underline{\underline{S}}^p}$ is bounded uniformly in n, the right side of (3.3) tends to 0 as n tends to ∞. □

In view of Lemma (3.2) we need state the next theorem, our chief result, only for the case where $J \in \underline{\underline{C}}$.

(3.4) THEOREM. Let M, $(M^n)_{n\geq 1}$ be semimartingales. Let F, $(F^n)_{n\geq 1}$ be in Lip(K) for some K and all n. Suppose J, $(J^n)_{n\geq 1}$ are in $\underline{\underline{C}}$ and that X, $(X^n)_{n\geq 1}$ are solutions respectively of

$$(3.5) \qquad X_t = J_t + \int_0^t (FX)_{s-} dM_s$$

$$(3.6) \qquad X^n_t = J^n_t + \int_0^t (F^nX^n)_{s-} dM^n_s.$$

Fix a p $(1 \leq p < \infty)$ and suppose (1) J^n converges locally in $\underline{\underline{S}}^p$ to J, (2) F^nX_- converges locally in $\underline{\underline{S}}^p$ to FX_- (3) $||M||_{\underline{\underline{H}}^\omega} < \infty$ and M^n converges locally in $\underline{\underline{H}}^\omega$ to M. Then X^n converges locally in $\underline{\underline{S}}^p$ to X.

(3.7) REMARKS.

(1) Because of the local nature of stochastic integrals (cf. Meyer [7,p.307]), one can replace J_t and J^n_t in (3.5) and (3.6) with $\tilde{J}_t = J_t 1_\Lambda$ and $\tilde{J}^n_t = J^n_t 1_\Lambda$, for $\Lambda \in \underline{\underline{F}}_0$. Then it is elementary that $t \to X_t$ and $t \to \tilde{X}_t$ ($\tilde{X}$ is the solution of (3.5) with J replaced by $\tilde{J}$) agree on Λ a.s. (cf., e.g., Protter [11,p.48]). This gives us a way to handle the situation if the initial conditions J_0 and J^n_0 are not in L^p: if there exist sets $\Lambda_k \in \underline{\underline{F}}_0$ increasing to Ω such that $||(J^n-J)1_{\Lambda_k}||_{\underline{\underline{S}}^p}$ tends to 0 for each k, one can define $T_k = k1_{\Lambda_k}$ and one concludes that the modified stopped processes $X^n_{t\wedge T_k}1_{\{T_k>0\}}$ converge to $X_{t\wedge T_k}1_{\{T_k>0\}}$ in $\underline{\underline{S}}^p$ for each k. Of course, the T^k increase to ∞ a.s.

(2) If we know that J, $(J^n)_{n\geq 1}$ are also semimartingales and that J^n tends locally in $\underline{\underline{H}}^p$ to J, then Lemma (3.2) allows us to conclude that X^n tends to X locally in $\underline{\underline{H}}^p$. We also remark that local $\underline{\underline{H}}^p$ convergence does not in general

imply local $\underline{\underline{H}}^q$ convergence for $q > p$. See [12,p.344] for a simple example of processes which converge in $\underline{\underline{H}}^1$ but do not converge locally in $\underline{\underline{H}}^p$ for any $p > 1$.

(3) Neither the $\underline{\underline{H}}^\omega$ nor the stronger $\underline{\underline{H}}^\infty$ norm can be used to extend Kazamaki's result: if N_t is the compensated Poisson process of intensity one and jump size α, and if T is the first jump time, then $M_t = N_{t\wedge T}$ is in $\underline{\underline{H}}^\infty$ and $\underline{\underline{H}}^\omega$, but $\mathcal{E}(M)$ is in $\underline{\underline{H}}^1$ if and only if $\alpha < 1$.

(4) As will be clear from the proof, the hypothesis that F^nX converges locally in $\underline{\underline{S}}^p$ to FX can be weakened to F^nX converges weak-locally in $\underline{\underline{S}}^p$ to FX (or in Émery's terminology, F^nX converges "locally" to FX in $\underline{\underline{S}}^p$.)

(5) One need not state the convergence of the coefficients in terms of X. For example, if as in [10] we assume F^n is of the form $F^n(\omega,t,x)$ with $F^n(\cdot,0,x) = 0$ a.s., and if we require

$$\lim_{n\to\infty} P\left\{\sup_{|x|\le m} |F^n(\cdot,t,x) - F(\cdot,t,x)| > \varepsilon\right\} = 0$$

for each $\varepsilon > 0$ and each $m \in N$, then the conclusion of Theorem (3.12) still holds.

PROOF of Theorem (3.4). By stopping at a large stopping time if necessary, we may assume without loss of generality that:

(a) $\lim_{n\to\infty} ||J^n-J||_{\underline{\underline{S}}^p} = 0$

(b) $\lim_{n\to\infty} ||F^nX-FX||_{\underline{\underline{S}}^p} = 0$

(c) $\lim_{n\to\infty} ||M^n-M||_{\underline{\underline{H}}^\omega} = 0.$

We choose and fix an α such that $0 < \alpha < 1/s_pK$. We know (Lemma (2.15)) that there exists an arbitrarily large stopping time T such that $M^{T-} \in D^\omega(\alpha/2,k)$

for some $k < \infty$. By proposition (2.16) and reduction (c) above we know there exists an N such that for all $n > N$ we have $(M^n)^{T-} \in D^{\omega}(\alpha,k)$, where k does not depend on n. Let T be such a stopping time, and let $0 = T_0 < T_1 < \ldots < T_k = T$ be the times that carve M (and also M^n for $n > N$) into slices less than α. We then have

(d) $M^{T-} \in D(\frac{\alpha}{2},k)$ and $(M^n)^{T-} \in D(\alpha,k)$ for all $n > N$, $k < \infty$, and $0 < \alpha < 1/s_pK$.

Observe that for any stopping time R we have that:

$$(3.8)\qquad \begin{aligned} ||F^nX^n||_{\underline{\underline{S}}^p(R-)} &\le ||F^nX^n_- - F^n0_-||_{\underline{\underline{S}}^p(R)} \\ &+ ||F^n0_- - F^nX_-||_{\underline{\underline{S}}^p(R)} + ||F^nX_- - FX_-||_{\underline{\underline{S}}^p(R)} \\ &+ ||FX_- - F0_-||_{\underline{\underline{S}}^p(R)} + ||F0_-||_{\underline{\underline{S}}^p(R)} \\ &\le K||X^n_-||_{\underline{\underline{S}}^p(R)} + 2K||X_-||_{\underline{\underline{S}}^p(R)} \\ &+ ||F^nX_- - FX_-||_{\underline{\underline{S}}^p(R)} + ||F0_-||_{\underline{\underline{S}}^p(R)} . \end{aligned}$$

Since $F0_-$ is left continuous we let $R_n = \inf\{t > 0: \ |F0_-| \ge n\}$ and $\lim_{n\to\infty} R_n = \infty$ a.s. and $||F0_-||_{\underline{\underline{S}}^p(R_n-)} \le n$. Thus without loss of generality we can assume

(e) $||F0_-||_{\underline{\underline{S}}^p} < \infty$.

Since $||F^nX_- - FX_-||_{\underline{\underline{S}}^p}$ tends to 0 by reduction (b), it is bounded, and hence (3.8) implies that $||F^nX^n||_{\underline{\underline{S}}^p(R-)}$ is bounded <u>if</u> $||X||_{\underline{\underline{S}}^p(R-)} < \infty$ and $\sup_n ||X^n_-||_{\underline{\underline{S}}^p(R-)} < \infty$. That there exists an arbitrarily large stopping time R such that $||X||_{\underline{\underline{S}}^p(R)} < \infty$ and $\sup_n ||X^n||_{\underline{\underline{S}}^p(R)} < \infty$ is perhaps the crux of proof. Given (d) this fact is the content of Lemma (3.25), the statement and proof of which follow this proof. Thus by (e), inequality (3.8), and Lemma (3.25) we may assume without loss of generality that

(f) $\sup_n ||F^n X^n_-||_{\underline{\underline{S}}^p} < \infty$.

From equations (3.4) and (3.5) we have that

$$(3.9)\quad X^n_t - X_t = J^n_t - J_t + \int_0^t (F^n X_{s-} - FX_{s-})dM_s + \int_0^t (F^n X^n_{s-} - F^n X_{s-})dM_s$$

$$+ \int_0^t F^n X^n_{s-} d(M^n - M)_s .$$

By the Emery-Meyer inequalities and Proposition (2.11) we have for any stopping time R > 0 a.s. that equation (3.9) yields:

$$(3.10)\quad ||X^n - X||_{\underline{\underline{S}}^p(R-)} \leq ||J^n - J||_{\underline{\underline{S}}^p} + s_p ||F^n X_- - FX_-||_{\underline{\underline{S}}^p} ||M||_{\underline{\underline{H}}^\omega}$$

$$+ s_p K ||X^n - X||_{\underline{\underline{S}}^p(R-)} ||M||_{\underline{\underline{H}}^\omega(R-)}$$

$$+ s_p ||F^n X^n_-||_{\underline{\underline{S}}^p} ||M^n - M||_{\underline{\underline{H}}^\omega}$$

$$= \gamma_n + s_p K ||M||_{\underline{\underline{H}}^\omega(R-)} ||X^n - X||_{\underline{\underline{S}}^p(R-)}$$

where s_p is given in (2.11). Note that by reductions (a), (b), (c) and (f) we know that $\lim_{n\to\infty} \gamma_n = 0$.

Let $0 = T_0 < T_1 < \ldots < T_k = T$ be the "carving times" whose existence is assured in (d). Then for $R = T_1$ and letting $\beta(1,n) = \gamma_n$, inequality (3.10) yields:

$$||X^n - X||_{\underline{\underline{S}}^p(T_1-)} \leq \beta(1,n) + r||X^n - X||_{\underline{\underline{S}}^p(T_1-)}$$

where $r = s_p K\alpha < 1$. Subtraction yields

$$(3.11)\quad ||X^n - X||_{\underline{\underline{S}}^p(T_1-)} \leq \beta(1,n)/(1-r).$$

Recall that $\delta X_t = X_t - X_{t-}$, the jump at t. One easily sees that:

$$(3.12)\quad \|\delta X^n_{T_1} - \delta X_{T_1}\|_{L^p} \leq \|\delta J^n_{T_1} - \delta J_{T_1}\|_{L^p}$$
$$+ \|F^n X - FX\|_{\underline{\underline{S}}^p(T_1-)} \|\delta M_{T_1} 1_{[T_1,\infty[}\|_{\underline{\underline{H}}^\omega}$$
$$+ K\|X^n - X\|_{\underline{\underline{S}}^p(T_1-)} \|\delta M_{T_1} 1_{[T_1,\infty[}\|_{\underline{\underline{H}}^\omega}$$
$$+ \|F^n X^n_{T_1-}\|_{\underline{\underline{S}}^p(T_1-)} \|\delta(M^n-M)_{T_1} 1_{[T_1,\infty[}\|_{\underline{\underline{H}}^\omega}$$

and since $\|\delta M_{T_1} 1_{[T_1,\infty[}\|_{\underline{\underline{H}}^\omega} \leq \|M\|_{\underline{\underline{H}}^\omega}$ (Proposition (2.1)), combining (3.11) and (3.12) gives us that

$$\lim_{n\to\infty} \|X^n - X\|_{\underline{\underline{S}}^p(T_1)} = 0.$$

Now suppose we have established that $\lim_{n\to\infty} \|X^n - X\|_{\underline{\underline{S}}^p(T_i)} = 0$, and consider T_{i+1}. We have that X_t and X^n_t are solutions respectively of (3.4) and (3.5) with J_t and J^n_t replaced respectively with $X_{t\wedge T_i} + J_t - J_{t\wedge T_i}$ and $X^n_{t\wedge T_i} + J^n_t - J^n_{t\wedge T_i}$ and with M and M^n replaced respectively with $M - M^{T_i}$ and $M^n - (M^n)^{T_i}$. Inequality (3.10) then gives us, taking $R = T_{i+1}$,

$$\|X^n - X\|_{\underline{\underline{S}}^p(T_{i+1}-)} \leq \beta(i,n) + s_p K\|M - M^{T_i}\|_{\underline{\underline{H}}^\omega(T_{i+1}-)} \|X^n - X\|_{\underline{\underline{S}}^p(T_{i+1}-)}$$
$$\leq \beta(i,n) + r\|X^n - X\|_{\underline{\underline{S}}^p(T_{i+1}-)}$$

where $r = s_p K\alpha < 1$. Thus

$$(3.13)\quad \|X^n - X\|_{\underline{\underline{S}}^p(T_{i+1}-)} \leq \beta(i,n)/(1-r)$$

where $\lim_{n\to\infty} \beta(i,n) = 0$. A calculation analogous to (3.12) shows that

$$\lim_{n\to\infty} \|X^n - X\|_{\underline{\underline{S}}^p(T_{i+1})} = 0,$$

and hence by a (finite) induction we conclude that $\lim_{n\to\infty} ||X^n - X||_{\underline{\underline{S}}^p(T)} = 0$ and the proof is complete. □

The next two lemmas were used in the proof of Theorem (3.4). Lemma (3.14) has some interest in its own right, since it gives a bound on the $\underline{\underline{S}}^p$ norm of the solution of a stochastic differential equation in terms of other norms which are (at least theoretically) known.

(3.14) LEMMA. Let M be a semimartingale, let $F \in \mathrm{Lip}(K)$ and let $J \in \underline{\underline{C}}$. Let X be the (unique) solution of

$$(3.15) \qquad Z_t = J_t + \int_0^t (FZ)_{s-}\,dM_s.$$

Suppose that (1) $||M||_{\underline{\underline{H}}^\omega} = m < \infty$, (2) $||J||_{\underline{\underline{S}}^p} = j < \infty$, (3) $||F0_-||_{\underline{\underline{S}}^p} = \tau < \infty$. Let α be a constant such that $0 < \alpha < 1/s_p K$, where s_p is given in (2.10). Let T be a stopping time such that $M^{T-} \in D^\omega(\alpha,k)$. Then

$$||X||_{\underline{\underline{S}}^p(T)} \leq C < \infty$$

where $C = C(p,j,K,\tau,m,\alpha,k)$ is a constant depending on the seven parameters in its argument.

PROOF. First suppose that $||M||_{\underline{\underline{H}}^\omega} \leq \alpha$. Then by the Emery-Meyer inequalities (2.6) we have

$$(3.16) \qquad \begin{aligned} ||X||_{\underline{\underline{S}}^p} &\leq ||J||_{\underline{\underline{S}}^p} + s_p ||FX_-||_{\underline{\underline{S}}^p} ||M||_{\underline{\underline{H}}^\omega} \\ &\leq j + s_p K\alpha(||X||_{\underline{\underline{S}}^p} + ||F0_-||_{\underline{\underline{S}}^p}) \\ &\leq j + r\tau + r||X||_{\underline{\underline{S}}^p} \end{aligned}$$

where $0 < r = s_p K\alpha < 1$. Then by subtraction we have

$$(3.17) \qquad ||X||_{\underline{\underline{S}}^p} \leq (j + r\tau)/(1-r)$$

provided that $||X||_{\underline{\underline{S}}^p} < \infty$. We show this by successive approximation. Let $X_t^0 = J_t$ and recursively define

$$X_t^{n+1} = J_t + \int_0^t (FX^n)_{s-} dM_s .$$

Then clearly $||X^n||_{\underline{\underline{S}}^p} < \infty$ for each n using induction and the Emery-Meyer inequalities. Moreover one easily checks that for $\ell \geq 1$:

$$(3.18) \qquad ||X^\ell - X^0||_{\underline{\underline{S}}^p} \leq (j+\tau) \sum_{i=1}^{\ell} r^i$$

where $0 < r = s_p K\alpha < 1$. Fix a large N. Then for $m > n > N$ we have:

$$(3.19) \qquad ||X^m - X^n||_{\underline{\underline{S}}^p} \leq r^N ||X^{m-N} - X^{n-N}||_{\underline{\underline{S}}^p}$$

and combining (3.18) and (3.19) yields

$$(3.20) \qquad ||X^m - X^n||_{\underline{\underline{S}}^p} \leq r^N [2(j+\tau)/(1-r)]$$

for $m > n > N$, and the right side of (3.20) tends to 0 as $N \to \infty$. Thus $(X^n)_{n \geq 1}$ is Cauchy in the Banach space $(\underline{\underline{C}}, ||\cdot||_{\underline{\underline{S}}^p})$. It is a simple matter to check that X^n converges in $||\cdot||_{\underline{\underline{S}}^p}$ to a solution Z of (3.15), and then $Z = X$ by the uniqueness of the solution.

Now remove the assumption that $||M||_{\underline{\underline{H}}^\omega} \leq \alpha$. We know that $M^{T-} \in D^\omega(\alpha,k)$. Let $0 = T_0 < T_1 < \ldots < T_k = T$ be the "carving times" of Definition (2.13). Let $N^1 = M^{T_1-}$. By the unicity of solutions we know $X_t^{T_1-}$ is the unique solution of

$$Z_t = J_t + \int_0^t (FX)_{s-} dN_s^1 .$$

The preceeding reduction gives

$$(3.21) \qquad ||X||_{\underline{\underline{S}}^p(T_1-)} \leq C(p,j,K,\tau,\alpha) < \infty .$$

Since $X_{T_1} = X_{T_1-} + \delta J_{T_1} + FX_{T_1-}\delta M_{T_1}$, we have that

(3.22) $\|X\|_{\underline{\underline{S}}^p(T_1)} \leq \|X\|_{\underline{\underline{S}}^p(T_1-)} + 2j + s_p mK\|X\|_{\underline{\underline{S}}^p(T_1-)} + s_p mK\tau.$

Inequalities (3.21) and (3.22) imply

(3.23) $$\|X\|_{\underline{\underline{S}}^p(T_1)} \leq C(p,j,K,\tau,m,\alpha) < \infty.$$

Now set $N_t^{i+1} = M_t^{T_{i+1}-} - M_t^{T_i}$ for $1 \leq i \leq k-1$, and note that $X_t^{T_{i+1}-}$ is the solution of the following equation:

$$Z_t = X_{t\wedge T_i} + (J_t^{T_{i+1}-} - J_t^{T_i}) + \int_0^t (FZ)_{s-} dN_s^{i+1}.$$

Since $\|N^{i+1}\|_{\underline{\underline{H}}^\omega} \leq \alpha$ we get as in (3.17) that

(3.24) $$\|X^{T_{i+1}-}\|_{\underline{\underline{S}}^p} \leq (\|X\|_{\underline{\underline{S}}^p(T_i)} + 2j + r\tau)/(1-r)$$

and since

$$X_{T_{i+1}} = X_{T_{i+1}-} + \delta J_{T_{i+1}} + FX_{T_{i+1}-}\delta M_{T_{i+1}}$$

it follows as in (3.21) through (3.23) that

$$\|X_{T_{i+1}}\| \leq C(p,j,K,\tau,m,\alpha,i+1) < \infty.$$

Since this is true for $1 \leq i \leq k$ and $T_k = T$, we have the result. □

(3.25) LEMMA. *Let* M, $(M^n)_{n\geq 1}$ *be semimartingales; let* J, $(J^n)_{n\geq 1}$ *be in* $\underline{\underline{C}}$; *let* F, $(F^n)_{n\geq 1}$ *be in* Lip(K). *Let* X, $(X^n)_{n\geq 1}$ *be solutions respectively of*:

$$X_t = J_t + \int_0^t (FX)_{s-} dM_s$$

$$X_t^n = J_t^n + \int_0^t (F^n X^n)_{s-} dM_s^n.$$

Assume that (1) $\|M\|_{\underline{\underline{H}}^\omega} < \infty$ *and* $\lim_{n\to\infty} \|M^n - M\|_{\underline{\underline{H}}^\omega} = 0$; (2) $\|J\|_{\underline{\underline{S}}^p} < \infty$ *and* $\lim_{n\to\infty} \|J^n - J\|_{\underline{\underline{S}}^p} = 0$; (3) $\lim_{n\to\infty} \|F^n X - FX\|_{\underline{\underline{S}}^p} = 0$. *Let* T *be a stopping time*

<u>such that</u> $||F0_-||_{\underline{\underline{S}}^p(T)} < \infty$ <u>and</u> $M^{T-} \in D^\omega(\alpha/2,k)$ <u>for some</u> α, $0 < \alpha < 1/s_p K$, <u>where</u> s_p <u>is given in</u> (2.10). <u>Then</u> $\inf_N \sup_{n>N} ||X^n||_{\underline{\underline{S}}^p(T)} < \infty$.

PROOF. By Proposition (2.16) we know that there exists an N such that for $n > N$ we have $M^n \in D^\omega(\alpha,k)$ with k not depending on n. We can assume without loss of generality that $||M^n||_{\underline{\underline{H}}^\omega} \leq 2m$ for $n > N$. Note that

$$||F^n 0_-||_{\underline{\underline{S}}^p(T)} \leq ||F^n 0 - F^n X||_{\underline{\underline{S}}^p(T-)} + ||F^n X - FX||_{\underline{\underline{S}}^p(T-)} + ||FX - F0||_{\underline{\underline{S}}^p(T-)}$$

$$+ ||F0||_{\underline{\underline{S}}^p(T-)},$$

$$\leq 2K||X||_{\underline{\underline{S}}^p(T-)} + ||F^n X - FX||_{\underline{\underline{S}}^p(T-)} + ||F0||_{\underline{\underline{S}}^p(T-)},$$

hence $\sup_{n>N} ||F^n 0_-||_{\underline{\underline{S}}^p(T)} = \tau < \infty$ provided that $||X||_{\underline{\underline{S}}^p(T-)} < \infty$. But $||X||_{\underline{\underline{S}}^p(T)} < \infty$ by Lemma (3.14). Applying Lemma (3.14) to X^n as well, we see that for $n > N$:

$$||X^n||_{\underline{\underline{S}}^p(T)} \leq C(p, j_n, K, \tau, 2m, \alpha, k)$$

where $j_n = ||J^n||_{\underline{\underline{S}}^p(T)}$, which is bounded as well by hypothesis (2). This completes the proof. □

4. AN AMUSING APPLICATION.

Recently Garcia, Maillard, and Peltraut [4] have shown that given a totally inaccessible stopping time T and a constant K, there exists a martingale L with precisely one jump (at T) such that $L_T/L_{T-} = K$. We extend this result in Lemma (4.2) to allow random jumps in L^1 and $\underline{\underline{F}}_{T-}$ measurable, and in Theorem (4.4) we establish a "continuity" result for local martingales with (random) "multiplicative jumps".

Let T denote a finite totally inaccessible stopping time, let $A_t = 1_{[T,\infty[}$, and let (M_t) denote the $\underline{\underline{BMO}}$ martingale $M_t = A_t - \tilde{A}_t$, where $\tilde{A}$ is the dual predictable projection of A (also called the compensator of A). Let $\Lambda \in L^1(\underline{\underline{F}}_{T-})$, and $N_t = E\{\Lambda - 1 | \underline{\underline{F}}_t\}$. Define F mapping $\underline{\underline{C}}$ to $\underline{\underline{C}}$ by $(FC)_t = N_t C_t$. Suppose X is the unique solution of

$$X_t = 1 + \int_0^t FX_{s-} dM_s. \tag{4.1}$$

Then X is the stochastic exponential of the local martingale $Y_t = \int_0^t N_{s-} dM_s$, and so

$$\frac{X_T}{X_{T-}} = 1 + N_{T-}(M_T - M_{T-})$$

which implies that $X_T / X_{T-} = \Lambda$.

(4.2) LEMMA. <u>The local martingale</u> X <u>of</u> (4.1) <u>is locally in</u> $\underline{\underline{H}}^p$ <u>for</u> $1 < p < \infty$.

PROOF. Let $S^\ell = \inf\{t: |N_t| \geq \ell\} 1_{\{|N_0| \leq \ell\}}$. Define F^ℓ by $F^\ell C = (FC)^{S^\ell -}$. Then $F^\ell \in \mathrm{Lip}(\ell)$. Since M is in $\underline{\underline{BMO}}$, it is in also in $\underline{\underline{H}}^\omega$ when considered as a semimartingale. By Lemma (2.15) we can find a stopping time T^ℓ with $P(T^\ell < S^\ell) < 1/2^\ell$ and such that (1) $M^{T^\ell -} \in D^\omega(\alpha, k)$ for some $k < \infty$ and $0 < \alpha < 1/s_p \ell$; and such that (2) $||F^\ell 0_-||_{\underline{\underline{S}}^p(T^\ell)} < \infty$. Thus by Lemma (3.14) we conclude

$$||X||_{\underline{\underline{S}}^p(T^\ell)} < \infty .$$

Since $\lim_{\ell \to \infty} S^\ell = \infty$ a.s., also $\lim_{\ell \to \infty} T^\ell = \infty$, and the proof is complete. □

Let Λ^n be a sequence of random variables in $L^1(\underline{\underline{F}}_{T-})$ and let $N_t^n = E\{\Lambda^n - 1 | \underline{\underline{F}}_t\}$. Define $(F^n C)_t = N_t^n C_t$ for $C \in \underline{\underline{C}}$. Suppose X^n are solutions of

$$X_t^n = 1 + \int_0^t F^n X_{s-}^n dM_s \quad (n \geq 1). \tag{4.3}$$

Then each X^n has multiplicative jump of size Λ^n at time T.

(4.4) THEOREM. *If Λ^n converges to Λ in $L^q (1 < q < \infty)$ with $|\Lambda^n| \leq Y \in L^1$ for $n \geq 1$, then X^n converges locally in $\underline{\underline{H}}^p$ $(1 \leq p < q)$ to X.*

PROOF. Let $Y_t = E(Y|\mathcal{F}_t)$ and let $S^k = \inf\{t > 0: Y_t \geq k\}$. Then by stopping M at S^k we can assume without loss of generality that F and $F^n \in \mathrm{Lip}(k)$ for all n. By Hölder's inequality we have $||F^n X - FX||_{\underline{\underline{S}}^p} = ||(N^n - N)X||_{\underline{\underline{S}}^p} \leq ||N^n - N||_{\underline{\underline{S}}^q} ||X||_{\underline{\underline{S}}^r}$ for $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$. By Lemma (3.14) we know there exists an arbitrarily large stopping time R such that $||X||_{\underline{\underline{S}}^r(R)} < \infty$. By Doob's inequality $||N^n - N||_{\underline{\underline{S}}^q} \leq q(q-1)^{-1} ||\Lambda^n - \Lambda||_{L^q}$ which tends to 0 as n tends to ∞. An application of Theorem (3.4) gives the result. □

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