COMPARISONS OF THREE MINIMIAX

SUBSET SELECTION PROCEDURES

by

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Abstract

The subset selection problem is considered for normal populations. Minimax considerations suggest three procedures as the main contenders for this problem. Two of these are the "average-type" procedure and the classical "maximum-type" procedure. The third procedure has not before been considered as a serious contender. Numerical comparisons of the performance of the three procedures are made, which indicate that the new and the "maximum-type" procedure are quite comparable, although the new procedure seems to have better optimality properties. The "average-type" procedure appearsto be clearly inferior to the other two.

Key Words: Subset selection, minimax rules, normal populations.

Author's Footnote

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1. INTRODUCTION

This paper is concerned with the problem of selecting good populations out of k possible populations. The populations may be varieties of grain, treatments of a disease, competing manufacturing processes for producing a certain article, stocks from different companies, candidates for admission to a certain university program, etc.

We will discuss the subset selection approach to the selection problem, i.e. the size of the selected subset is determined by the data.

The k populations $\Pi_1,\dots,\Pi_k$ are characterized by $\theta_1,\dots,\theta_k$ respectively. Let $\underline{\theta} = (\theta_1,\dots,\theta_k)$ and let Ω be the parameter space. Let X_i be the observation from population Π_i. $\underline{X}=(X_1,\dots,X_k)$. It is assumed that X_i is $\mathcal{N}(\theta_i,1)$ and $X_1,\dots,X_k$ are independent. We are interested in selecting populations with large θ_i. Let the ordered θ_i be denoted by $\theta_{[1]}\leq \cdots \leq \theta_{[k]}$,and let $\Pi_{(i)},X_{(i)}$ correspond to $\theta_{[i]}$.

Definition 1.1. Π_i is called a best population if $\theta_i=\theta_{[k]}$, and a non-best population if $\theta_i < \theta_{[k]}$.

We are also interested in the concept of good and bad populations which will be defined in the following way.

Definition 1.2. Π_i is said to be a good population if $\theta_i > \theta_{[k]} - \Delta$ and a bad population if $\theta_i \leq \theta_{[k]} - \Delta$. Δ is a given positive constant.

We will later present procedures which are minimax with respect to certain risk functions. For the risk functions considered, two subset selection procedures are equivalent if their individual selection probabilities are the same. Therefore we can define a subset selection procedure by:

$$\psi(\underline{x}) = [\psi_1(\underline{x}),\ldots,\psi_k(\underline{x})]$$

where $\psi_i(\underline{x}) = P(\text{selecting } \Pi_i | \underline{X} = \underline{x})$.

A correct selection (CS) is defined to be a selection that includes the best population $\Pi_{(k)}$. Usually in a given subset selection problem the first thing to decide is what kind of control condition we want the procedures to satisfy. We will consider four control conditions:

$$\inf_{\underline{\theta}\in\Omega} P_{\underline{\theta}}(CS|\psi) \geq \gamma \tag{1.1}$$

$$\inf_{\underline{\theta}\in\Omega(\Delta)} P_{\underline{\theta}}(CS|\psi) \geq \gamma \tag{1.2}$$

$$\inf_{\underline{\theta}\in\Omega} R(\underline{\theta},\psi) \geq \gamma \tag{1.3}$$

$$\sup_{\underline{\theta}\in\Omega} S(\underline{\theta},\psi) \leq \beta \tag{1.4}$$

Here

$$S(\underline{\theta},\psi) = \sum_{i=1}^{k} E_{\underline{\theta}}(\psi_i) \tag{1.5}$$

i.e. $S(\underline{\theta},\psi)$ is expected size of the selected subset.

$$\Omega(\Delta) = \{\underline{\theta}:\theta_{[k]}-\theta_{[k-1]} \geq \Delta\} \tag{1.6}$$

and

$$R(\underline{\theta},\psi) = \sum_{i\in I_\Delta} E_{\underline{\theta}}(\psi_i) \tag{1.7}$$

where $I_\Delta = \{i : \theta_i > \theta_{[k]} - \Delta\}$,i.e. R is the expected number of good populations selected. We assume $1/k < \gamma < 1$.

Gupta and Studden (1966) and Berger (1977) showed that the following procedure, ψ^m, has certain minimax properties.

$$\psi_i^m = 1 \text{ iff } X_i \geq \max_{1\leq j\leq k} X_j - d$$

Bjørnstad (1978b) showed that also the following two procedures, ψ^e, ψ^a have several minimax properties.

$$\psi_i^e = 1 \text{ iff } Ce^{\Delta X_i} \geq \sum_{j\neq i} e^{\Delta X_j}$$

$$\psi_i^a = 1 \text{ iff } X_i \geq (k-1)^{-1}\sum_{j\neq i} X_j - c$$

Here d,c,C are determined according to what control conditions we want the procedures to satisfy. E.g. ψ^m satisfies (1.1) with equality if $d = d(\gamma)$, where $d(\gamma)$ is tabulated by Gupta (1956) and Gupta (1963), and ψ^m satisfies (1.2) and (1.3) with equality if $d = d(\gamma) - \Delta$. For

ψ^a we easily see that (1.1) is satisfied with equality if $c = (k/k-1)^{\frac{1}{2}} z(\gamma)$, and (1.2) is satisfied with equality if $c = (k/k-1)^{\frac{1}{2}} z(\gamma) - \Delta$. Here $z(\gamma)$ is the γ- quantile in the $\mathcal{N}(0,1)$-distribution, i.e. $\Phi(z(\gamma)) = \gamma$, where Φ is the $\mathcal{N}(0,1)$ distribution function. It can also be seen that if Δ is not too large then ψ^a satisfies (1.3) with equality if $c = (k/k-1)^{\frac{1}{2}} z(\gamma) - \Delta$. (See Bjørnstad (1978b).)

Values of C such that ψ^e satisfies (1.2) and (1.3) with equality are tabulated in Bjørnstad (1978b). We also see that if instead of C we use $e^{\Delta^2} \cdot C$, then ψ^e satisfies (1.1).

The procedure ψ^m was first proposed by Gupta (1956) and Seal (1957). Paulson (1949) considered also ψ^m, using a slightly different control condition. The procedure ψ^a was suggested by Seal (1955). The third procedure ψ^e was studied in a different context by Studden (1967).

In order to describe the minimax properties of the three above mentioned procedures, we need to define the different risk criteria that are considered. They are:

$$B(\underline{\theta},\psi) = \sum_{i \in I_\Delta^c} E_{\underline{\theta}}(\psi_i)$$

B is the expected number of bad populations selected.

$$S'(\underline{\theta},\psi) = \sum_{i=1}^{k-1} E_{\underline{\theta}}(\psi_{(i)})$$

Here $\psi_{(i)}$ corresponds to $\theta_{[i]}$. S' is the expected number of non-best populations selected when $\theta_{[k-1]} < \theta_{[k]}$.

$$\ell(\underline{\theta},\psi) = \sum_{i \in I_\Delta^c} \log\{E_{\underline{\theta}}(\psi_i)\}$$

$$L(\underline{\theta},\psi) = \sum_{i=1}^{k-1} \log\{E_{\underline{\theta}}(\psi_{(i)})\} \quad .$$

Also $S(\underline{\theta},\psi)$ will be used as a risk criterion.

It is clear that not all these criteria are equally appropriate or meaningful. The following arguments were also mentioned in Bjørnstad (1978b).

A criterion for measuring performance of subset selection procedures should indicate how well a procedure excludes inferior populations. Since S includes the probability of selecting the best population, clearly S' is a more appropriate measure of performance than S. Also, we do not need to protect against populations that are "close" to the best population, i.e. populations Π_i where $\theta_i > \theta_{[k]} - \Delta$. Hence B seems to be a natural criterion. So of the criteria B, S' and S we regard B as the most meaningful. The criteria ℓ, L clearly do not have such a nice intuitive appeal as B or S'. However, as shown in Bjørnstad (1978a), they have the nice feature of placing more weights on the worst populations than on those closest to the best.

As discussed in Bjørnstad (1978b) we regard (1.1)-(1.3) as our main control conditions, and of these, (1.3) as the most appropriate.

In Section 2 the minimax properties of ψ^m, ψ^e and ψ^a for normal populations are summarized. In Section 3 numerical comparisons of the three procedures are performed, with B as the risk criterion and (1.2) as control condition, for slippage configurations of $\underline{\theta}$.

2. MINIMAX PROPERTIES

Let the class of procedures satisfying (1.1), (1.2), (1.3), (1.4) be denoted by $\mathcal{D}_1(\gamma)$, $\mathcal{D}'(\gamma,\Delta)$, $\mathcal{D}(\gamma,\Delta)$, $\mathcal{D}_2(\beta)$ respectively. $\mathcal{D}_I$ denotes the class of permutation-invariant procedures.

Define the following slippage-subset of the whole parameter space Ω.

$$\Omega_p(\Delta) = \{\underline{\theta}\in\Omega : \theta_{[k]}-\theta_{[p]} < \Delta \ \& \ \theta_{[k]}-\theta_{[p-1]} \geq \Delta + \sum_{i=1}^{p-2} (\theta_{[p-1]}-\theta_{[i]})\} \quad (2.1)$$

for $p = 1,\ldots,k$.

Let

$$\Omega_1 = \bigcup_{p=1}^{k} \Omega_p(\Delta) \qquad (2.2)$$

Ω_1 consists of the cases where the good populations have "slipped" from the bad populations.

Let $\underline{\theta}^\Delta = (0,\ldots,0,\Delta)$ and let Δ_γ be determined by

$$E_{\underline{\theta}^{\Delta_\gamma}}(\psi_1^e) = b_k(\gamma) \qquad (2.3)$$

where

$$b_k(\gamma) = \gamma/9 \qquad \text{if } k = 4$$
$$= \gamma/7 \qquad \text{if } k = 5 \qquad (2.4)$$
$$= (11/75)\gamma \quad \text{if } k \geq 6$$

Then from Bjørnstad (1978b) we have the following result.

Theorem 2.1 $X_1,\ldots,X_k$ are independent. X_i is $\mathcal{N}(\theta_i,1)$ for $i = 1,\ldots,k$. Assume $C \geq k-1$, i.e. at least one population is selected by ψ^e. If $k \geq 4$ and $\Delta \leq \Delta_\gamma$ or $k \leq 3$, then ψ^e minimizes

(i) $$\sup_{\underline{\theta}\in\Omega_1} B(\underline{\theta},\psi) \quad \text{and} \quad \sup_{\underline{\theta}\in\Omega_s(\Delta)} S'(\underline{\theta},\psi)$$

for all $\psi\in\mathcal{D}(\gamma,\Delta)$, and

(ii) $$\sup_{\underline{\theta}\in\Omega} \ell(\underline{\theta},\psi) \quad \text{and} \quad \sup_{\underline{\theta}\in\Omega(\Delta)} L(\underline{\theta},\psi)$$

for all $\psi\in\mathcal{D}(\gamma,\Delta) \cap \mathcal{D}_I$.

Remark. Values of Δ_γ can be found from Table 2 in Bjørnstad(1978b). Let now

$$\Delta_2(\beta)=(k/k-1)^{\frac{1}{2}} \{z(\tfrac{\beta}{k}) + z(\tfrac{k-1}{k})\}$$

and

$$\Delta_a(\gamma) = (k/k-1)^{\frac{1}{2}} \min\{[\tfrac{1}{2}z(\gamma) + \tfrac{1}{2}z(\tfrac{k-1}{k})], [z(\gamma) - z(\tfrac{k-1}{k})]\} .$$

The minimax properties of ψ^a are given by the next result, from Bjørnstad (1978b).

Theorem 2.2. $X_1,\ldots,X_k$ are independent. X_i is $\mathcal{N}(\theta_i,1)$ for $i = 1,\ldots,k$.

(a) Let $c = (k/k-1)^{\frac{1}{2}} z(\gamma)$, such that $\psi^a \in \mathcal{D}_1(\gamma)$. Then ψ^a minimizes for all $\psi \in \mathcal{D}_1(\gamma)$

$$\sup_{\underline{\theta}\in\Omega} S'(\underline{\theta},\psi) \qquad (\sup_{\underline{\theta}\in\Omega} S(\underline{\theta},\psi))$$

if and only if

$$\gamma \geq (k-2)/(k-1) \quad (k-1/k) .$$

(b) Let $c = (k/k-1)^{\frac{1}{2}} z(\beta/k)$, such that $\psi^a \in \mathcal{D}_2(\beta)$. Let $\beta \geq k-1$ and assume $k \geq 4$ and $\Delta \leq \Delta_2(\beta)$ or $k \leq 3$. Then ψ^a maximizes for all $\psi \in \mathcal{D}_2(\beta)$

$$\inf_{\underline{\theta}\in\Omega} R(\underline{\theta},\psi) .$$

(c) Let $c = (k/k-1)^{\frac{1}{2}} z(\gamma) - \Delta$. Assume $\Delta \leq \Delta_a(\gamma)$. Then $\psi^a \in \mathcal{D}(\gamma,\Delta)$ and minimizes for all $\psi \in \mathcal{D}(\gamma,\Delta)$

$$\sup_{\underline{\theta}\in\Omega} S(\underline{\theta},\psi) .$$

The minimax properties of ψ^m are given in the following result.

Theorem 2.3. $X_1,\ldots,X_k$ are independent. X_i is $\mathcal{N}(\theta_i,1)$ for $i = 1,\ldots,k$.

(a) Let $d = d(\gamma)$, such that $\psi^m \in \mathcal{D}_1(\gamma)$. Then ψ^m minimizes for all $\psi \in \mathcal{D}_1(\gamma)$

$$\sup_{\underline{\theta}\in\Omega} S(\underline{\theta},\psi) \quad \text{and} \quad \sup_{\underline{\theta}\in\Omega} S'(\underline{\theta},\psi).$$

(b) Let $d = d(\gamma) - \Delta$, and assume $\Delta < d(\gamma)$. Then $\psi^m \in \mathcal{D}(\gamma,\Delta)$.

Let $\psi \in \mathcal{D}(\gamma,\Delta)$ or $\mathcal{D}'(\gamma,\Delta)$. Then

$$\lim_{k\to\infty} \inf\{ \sup_{\underline{\theta}\in\Omega} B_k(\underline{\theta},\psi)\} \geq \lim_{k\to\infty}\{ \sup_{\underline{\theta}\in\Omega} B_k(\underline{\theta},\psi^m)\}$$

where $B_k = B/k$. I.e. ψ^m is asymptotically minimax in the classes $\mathcal{D}'(\gamma,\Delta)$ and $\mathcal{D}(\gamma,\Delta)$ as $k\to\infty$ with respect to the whole parameter space Ω, for the standardized risk B/k.

Remarks. (1) Part (a) of Theorem 2.3 follows from Gupta and Studden (1966), who showed the same result in the case of a monotone likelihood ratio (MLR) location-family. Also Berger (1977) has similar results for other families of distributions.

(2) Part (b) of Theorem 2.3 is proved in Bjørnstad (1978b)

(3) Theorem 2.2(a) is generalized to MLR location-families in Bjørnstad (1978b).

(4) The three theorems indicate that from a minimax point of view, ψ^m, ψ^a and ψ^e are the main competitors in this problem.

3. PERFORMANCE - EVALUATION OF THE THREE MAIN CONTENDERS

From the remarks in Section 1, it follows that we consider $B(\underline{\theta},\psi)$ and $\ell(\underline{\theta},\psi)$ to be the most suitable measures of performance, subject to the control condition (1.3). For this problem Theorem 2.1 tells us that ψ^e is minimax, while ψ^m and ψ^a are not minimax. In this section we will give a more comprehensive comparison of the three procedures, using $B(\underline{\theta},\psi)$ as the criterion and assume d, c, C are determined so that (1.2) holds. There are two things we will look at. The first is to find out how much better ψ^c performs under the least favorable configuration $\theta_1=\ldots=\theta_{k-1}=\theta_k-\Delta$. Secondly we compare the three procedures under other configurations of $\underline{\theta}$. The configurations that will be considered are of the slippage-type: $\theta_1=\ldots=\theta_{k-1}=\theta_k-\delta$ for some selected values of k, Δ and $\delta \geq \Delta$.

Let us first, however, use some of the numerical evaluations that have been made in the literature. Deely and Gupta (1968) tabulated $S'(\underline{\theta},\psi^m)/(k-1)$ for the slippage configurations: $\theta_1=\ldots=\theta_{k-1}=\theta_k-\delta$, for selected values of δ, γ, and k. They assumed that ψ^m satisfies (1.1). Now, let us assume ψ^a satisfies (1.1). Then under the slippage-configuration given above we have that

$$S'(\underline{\theta},\psi^a)/(k-1) = \Phi(z(\gamma) - \frac{\delta}{\sqrt{k(k-1)}}). \qquad (3.1)$$

Using (3.1) and the tables in Deely and Gupta (1968) we find that for small δ ($\leq .5$), ψ^m and ψ^a perform equally well, but for large δ ψ^m performs a lot better. The same conclusion is reached if we consider the equally-spaced configurations: $\theta_{i+1}-\theta_i = \delta$, for $i=1,\ldots,k-1$, by using tables in Gupta (1965). For the criteria where ψ^a is minimax and ψ^m is not (see Theorem 2.2), we find that ψ^a is only slightly better than ψ^m.

We should also mention that Chernoff and Yahav (1977), Gupta and Hsu (1977b) and Hsu (1977) have performed Monte-Carlo studies to compare the procedures ψ^m and ψ^a with Bayes procedures for certain loss functions, in the case of normal populations and normal exchangeable priors. They all found that the Bayes procedures can be approximated well by a procedure of the type ψ^m. Gupta and Hsu (1977b) and Hsu (1977) showed that while this is true also for ψ^a if the prior is concentrated, the best ψ^a (compared to the Bayes procedure) can do very badly if the prior is diffuse.

From the above observations it follows that ψ^m overall seems quite superior to ψ^a. We also note that ψ^e has not been considered before.

We now consider the problem of comparing the three procedures ψ^e, ψ^m and ψ^a, using $B(\underline{\theta},\psi)$ as a criterion and assuming d, c, C are determined so that (1.2) holds with equality. The comparisons are made under the slippage configurations $\underline{\theta}_\delta$ given by

$$\underline{\theta}_\delta: \quad \theta_1 = \ldots = \theta_{k-1} = \theta_k - \delta \quad \text{for} \quad \delta \geq \Delta.$$

In this case $B(\underline{\theta}_\delta,\psi) = (k-1)E_{\underline{\theta}_\delta}(\psi_1)$. Let

$$P_\delta(\psi) = E_{\underline{\theta}_\delta}(\psi_1).$$

$P_\delta(\psi)$ is tabulated in Table 1 for ψ^e, ψ^m and ψ^a. The probabilities for ψ^e and ψ^m are obtained by Monte Carlo methods. For ψ^a we have

$$P_\delta(\psi^a) = \Phi\{z(\gamma) - \sqrt{\frac{k-1}{k}}\,\Delta - \frac{\delta}{\sqrt{k(k-1)}}\}. \qquad (3.2)$$

For all cases we let $\gamma = .90$. For each combination of (k,Δ,δ) one simulation of 5000 interactions of $\underline{X} = (X_1,\ldots,X_k)$ was carried out to estimate $P_\delta(\psi^e)$ and $P_\delta(\psi^m)$. The top entry for each k is $P_\delta(\psi^e)$, the middle entry is $P_\delta(\psi^m)$ and the bottom entry is $P_\delta(\psi^a)$.

The critical constant C in ψ^e is determined in Bjørnstad (1978b) by Monte Carlo methods. Hence the actual control-level for ψ^e may not be exactly $\gamma = .90$. Although we can use the tables in Gupta (1956) and Gupta (1963) to find the critical constant d for ψ^m, d is here determined by Monte Carlo methods to make $P_\delta(\psi^e)$ and $P_\delta(\psi^m)$ comparable. By using the 25,000 observations we have for each pair (k,Δ) we estimated the actual control levels for ψ^e. They all lie in the range: .894-.911. To get $P_\delta(\psi^a)$ comparable to the other two probabilities, we have, in those cases where the estimated control levels of ψ^e turn out to be significantly different from .90, calculated $P_\delta(\psi^a)$ from (3.2), using these estimated levels.

The comparisons in Table 1 confirm earlier comparisons of ψ^a and ψ^m in that ψ^m seems to do a lot better than ψ^a. For small Δ and δ

not too large we see that ψ^e and ψ^a perform equally well, which is not surprising since ψ^e and ψ^a are approximately the same procedures when Δ is small. However, we observe that ψ^e otherwise performs almost uniformly better than ψ^a over slippage configurations, and the improvement is considerable for $\Delta \geq 1$.

Comparison of ψ^m and ψ^e does not give such a unique answer as the comparisons above did. We know that $P_\delta(\psi^e)$ is less than $P_\delta(\psi^m)$ when $\delta = \Delta$. However, for most cases the differences are quite small. Only for $k = 10$ and $\Delta \geq .5$ does ψ^e perform quite a bit better than ψ^m under the least favorable configuration.

For $\delta > \Delta$ we see that ψ^m is usually better than ψ^e. The improvement is substantial for small Δ. However for $\Delta \geq 1$, the improvement is not much different from the difference between ψ^e and ψ^m for $\delta = \Delta$. So, we can say that for small Δ, ψ^m seems to perform, on the whole, better than ψ^e. For $\Delta \geq 1.5$ the procedures seem quite comparable.

Let us now briefly summarize the impressions we are left with. ψ^e seems to have the most desirable minimax properties of the three procedures. Also the procedure ψ^a has quite a few nice minimax properties. However, as has also been observed by other authors, ψ^a seems to behave very poorly compared to ψ^m under configurations of $\underline{\theta}$ different from the least favorable. Our evaluations here indicate that the same is true relative to ψ^e. This leaves ψ^e and ψ^m as the principle procedures. Although ψ^e has more desirable minimax properties than ψ^m, the comparisons here seem to suggest that the two procedures are quite comparable. For small Δ, ψ^m appears to perform better on the whole. However, for Δ in the range we are usually interested in, it is seen that neither of the procedures seems to be clearly better than the other.

So to summarize, we have derived a procedure, ψ^e, that appears to be comparable to the classical procedure ψ^m in most cases, and that has some optimality properties not possessed by ψ^m.

Table 1

The probability of selecting population Π_1 when $\theta_1=\ldots=\theta_{k-1}=\theta_k-\delta$ for ψ^e, ψ^m, ψ^a satisfying control condition (1.2).

$\Delta = .1,\ \gamma = .90$

k	δ	.1	.6	1.1	2.1	3.1
	ψ^e	.881	.848	.800	.629	.446
3	ψ^m	.884	.839	.763	.511	.248
	ψ^a	.888	.845	.791	.656	.497
	ψ^e	.882	.860	.839	.759	.656
5	ψ^m	.881	.858	.794	.600	.322
	ψ^a	.884	.860	.834	.772	.699
	ψ^e	.876	.866	.845	.790	.723
7	ψ^m	.885	.863	.820	.634	.366
	ψ^a	.880	.864	.846	.806	.761
	ψ^e	.879	.868	.857	.816	.781
10	ψ^m	.888	.867	.836	.684	.423
	ψ^a	.880	.869	.858	.833	.805

Table 1, contd.

$\Delta = .5,\ \gamma = .90$

k	δ	.5	1.0	1.5	2.5	3.5
	ψ^e	.761	.685	.580	.356	.167
3	ψ^m	.769	.674	.560	.286	.110
	ψ^a	.761	.694	.619	.458	.304
	ψ^e	.767	.733	.662	.501	.328
5	ψ^m	.772	.718	.615	.373	.162
	ψ^a	.774	.739	.701	.620	.532
	ψ^e	.765	.739	.687	.559	.408
7	ψ^m	.779	.742	.643	.415	.199
	ψ^a	.762	.738	.712	.657	.599
	ψ^e	.763	.749	.714	.620	.509
10	ψ^m	.786	.751	.678	.469	.240
	ψ^a	.765	.748	.731	.695	.657

Table 1, contd.
$\Delta = 1.0,\ \gamma = .90$

k	δ	1.0	1.5	2.0	3.0	4.0
	ψ^e	.524	.419	.312	.137	.037
3	ψ^m	.525	.414	.298	.117	.028
	ψ^a	.549	.465	.385	.242	.134
	ψ^e	.548	.482	.394	.211	.075
5	ψ^m	.555	.466	.360	.158	.045
	ψ^a	.579	.535	.490	.402	.319
	ψ^e	.562	.510	.438	.260	.105
7	ψ^m	.581	.499	.399	.193	.057
	ψ^a	.580	.549	.519	.457	.397
	ψ^e	.560	.524	.473	.311	.139
10	ψ^m	.593	.527	.442	.232	.070
	ψ^a	.581	.561	.540	.498	.456

Table 1, contd.
$\Delta = 1.5$, $\gamma = .90$

k	δ	1.5	2.0	2.5	3.5	4.5
	ψ^e	.276	.190	.112	.035	.006
3	ψ^m	.271	.184	.107	.029	.004
	ψ^a	.314	.246	.186	.097	.044
	ψ^e	.321	.233	.148	.055	.013
5	ψ^m	.331	.227	.140	.043	.008
	ψ^a	.355	.314	.276	.206	.149
	ψ^e	.335	.266	.197	.079	.018
7	ψ^m	.341	.257	.185	.058	.011
	ψ^a	.367	.339	.311	.259	.211
	ψ^e	.343	.278	.214	.100	.026
10	ψ^m	.366	.273	.200	.075	.015
	ψ^a	.382	.362	.343	.305	.269

Table 1, contd.

$\Delta = 2.0,\ \gamma = .90$

k	δ	2.0	2.5	3.0	4.0	5.0
	ψ^e	.093	.056	.029	.004	.0002
3	ψ^m	.094	.053	.028	.003	.0002
	ψ^a	.135	.096	.065	.027	.0100
	ψ^e	.125	.084	.049	.010	.001
5	ψ^m	.128	.081	.045	.008	.001
	ψ^a	.170	.143	.119	.080	.052
	ψ^e	.135	.099	.057	.013	.002
7	ψ^m	.135	.096	.048	.010	.001
	ψ^a	.201	.180	.161	.126	.097
	ψ^e	.158	.110	.077	.019	.002
10	ψ^m	.164	.111	.073	.015	.002
	ψ^a	.204	.190	.176	.150	.127

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