D'ALEMBERT'S FUNCTIONAL EQUATION ON GROUPS

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I. Reduction to a representation theory problem. We consider a functional equation of the form

$$\phi_1(gh) + \phi_2(g^{-1}h) = \sum_{i=1}^{n} \kappa_i(g)\lambda_i(h) \qquad (*)$$

where ϕ_1, ϕ_2, κ_i, λ_i, $i = 1,\ldots,n$ are some (measurable) complex functions given on a locally compact group G. We can and shall assume that functions κ_i and λ_i, $i = 1,\ldots,n$ are linearly independent. The equation (*) can be viewed as a generalization of the known D'Alembert's functional equation

$$\phi(gh) + \phi(g^{-1}h) = 2\phi(g)\phi(h),$$

which was studied by many authors (cf 4-6). A particular case of (*) when $\phi_1 = -\phi_2$ arises in some statistical applications and was studied by one of the authors in the situation when G is a compact Lie group [6]. The solution of (*) in the case $G = R^1$ can be found in the Aczel's book [1] (pp. 171-176, 199).

In this paper we present the solution of (*) in the case when $\int|\phi_i(g)|^2 dv(g) < \infty$ $i = 1, 2$, where ν is the left Haar measure on G. Henceforth we denote the space of such functions as $L_2(G)$. We also consider the case when ϕ_i, $i = 1, 2$ can be represented as a finite combination of positive definite functions.

Note that if $\varphi(g) = \phi_1(g) + \phi_2(g)$, $\xi(g) = \phi_1(g) - \phi_2(g)$ then

$$\varphi(gh) + \varphi(g^{-1}h) = \sum_{i=1}^{n} [\kappa_i(g) + \kappa_i(g^{-1})]\lambda_i(h) = \sum_{i=1}^{m} \alpha_i(g)\varphi_i(h) \qquad (**)$$

and

$$\xi(gh) - \xi(g^{-1}h) = \sum_{j=1}^{n} [\kappa_i(g) - \kappa_j(g^{-1})]\lambda_j(h) = \sum_{j=1}^{r} \beta_j(g)\xi_j(h) \qquad (***)$$

with linearly independent functions α_i, φ_i, $i = 1,\ldots,m$ and β_j, ξ_j, $j = 1,\ldots,r$. Therefore we restrict our attention to the case $\phi_1 = \phi_2$ or $\phi_1 = -\phi_2$ in (*).

We reduce the solution of (*) to a certain problem in representation theory in the following way. Let us begin with the equation (**). If denotes the closure in $L_2(G)$ of the linear space spanned by the left shifts $\varphi(g\cdot)$, $g \in G$, of the function φ then the left regular representation U acts in H: $U(g)\eta(\cdot) = \eta(g^{-1}\cdot)$, $\eta \in H$ and $U(g)$ is a unitary operator. The relation (**) implies that

$$[U(g) + U(g^{-1})]\varphi = \sum_{i=1}^{m} \alpha_i(g)\varphi_i,$$

where φ denotes the vector of L corresponding to the function $\varphi(\cdot)$, and $\varphi_1,\ldots,\varphi_m$ are vectors from H. Because of the definition of H, U is a cyclic representation with a cyclic vector φ (i.e. the space spanned by vectors $U(g)\varphi$, $g \in G$ os dense in H). The following results is a corollary of these considerations and the theorem 2.1.

THEOREM 1.1. Let G be a locally compact group of type one such that the elements of the form g^4, $g \in G$, generate a dense subgroup of G. Every solution $\varphi \in L_2(G)$ of the equation (**) has the form

$$\varphi(g) = \langle U(g^{-1})\varphi, \eta\rangle$$

where U is a finite dimensional, unitary, cyclic (with a cyclic vector φ) representation of G such that the space spanned by vectors $[U(g) + U(g^{-1})]\varphi$, $g \in G$ has dimension m, and η is some vector of the representation space U.

Note that a formula for the dimension of the representation U can be obtained from the theorem 2.1.

It is immediately seen that the functional equation (***) is equivalent to the finite dimensionality of the space spanned by vectors $[U(g) - U(g^{-1})]\xi$, $g \in G$ where again U is a unitary representation with a cyclic vector ξ.

THEOREM 1.2. Let G be a locally compact group of type one such that the elements of the form g^2, $g \in G$ generate a dense subgroup of G. Every solution $\xi \in L_2(G)$ has the same form as indicated in theorem 1.1. with ξ instead of φ, $[U(g) - U(g^{-1})]\xi$ instead of $[U(g) + U(g^{-1})]\varphi$ and r instead of m.

The solution of the general equation (*) now follows easily from Theorems 1.1 and 1.2.

The content of these theorems is that if G is noncompact then D'Alembert's functional equation has few solutions, as non-compact groups, usually, have few finite dimensional unitary representations.

THEOREM 1.3. Under assumtions of the theorem 1.1. if there exists a non-zero solution of (**) then G is compact.

Proof. It follows from the theorem 1.1. that every solution of (**) has the form

$$\varphi(g) = \langle U(g^{-1})\varphi, \eta\rangle$$

with a finite dimensional representation U. However, such a matrix element cannot be square integrable unless G is compact. Indeed let K be the kernel of U. Clearly K is closed, normal subgroup of G and φ is constant on cosets of K in G. In order for φ to be square integrable K must have finite volume under Haar measure which implies compactness of K.

To prove that G is compact it suffices to show that G/K is compact. To this end we may assume that K = {e} so that U is injective. But then G is compactly injectible, and hence C is the product of a compact group and R^p for some p (cf. [2] s. 16.4.2). But R^p has no injective, finite dimensional unitary representations unless p = 0. Thus G is compact.

We give another version of theorem 1.1.

THEOREM 1.4. Theorem 1.1 holds if one assumes that φ is a linear combination of positive definite functions instead of $\varphi \in L_2(G)$.

Proof. It follows from Godement [3] that there exists a unitary representation U and vectors φ and η such that

$$\varphi(g) = \langle U(g^{-1})\varphi, \eta\rangle.$$

These vectors φ and η can be assumed to be cyclic for U (the latter since $\varphi(g) = \langle \varphi, U(g)\eta\rangle$). Moreover we can replace η by its projection onto the closed subspace spanned by the vectors $U(g^{-1})\varphi$. Because of (**) the space of functions $\langle [U(g) + U(g^{-1})]\varphi, U(\cdot)\eta\rangle$, $g \in G$ if finite dimensional. The cyclicity of $\hat{\eta}$ implies that the space spanned by the vectors $[U(g) + U(g^{-1})]\varphi$ is finite dimensional and the cyclicity of φ and theorem 2.1 imply that U is finite dimensional.

2. Symmetric and anit-symmetric interwining operators. Let U be a unitary representation of the topological group G in a Hilbert space H. Let H* be the continuous dual of H and let U* be the contragradient representation to U, i.e.

$$\langle \xi, U^*(g)\eta\rangle = \langle U(g^{-1})\xi, \eta\rangle$$

Clearly H* is a Hilbert space which is conjugate isomorphic with H and U* is a unitary operator.

Let I(U,U*) denote the set of all continuous operators A mapping H into U* such that

$$U^*(g)A = A\,U(g).$$

If A^* is the dual operator, $A^*: H^{**} = H \to H^*$, then A is said to be symmetric if $A^* = A$ and anti-symmetric if $A^* = -A$. The space of symmetric elements of $I(U,U^*)$ is denoted $I_s(U,U^*)$ and the space of anti-symmetric elements is denoted $I(U,U^*)$. Clearly

$$I(U,U^*) = I_s(U,U^*) + I_a(U,U^*).$$

Note that A^* is not the same as the Hilbert space adjoint of A which is a mapping of H^* to H.

We prove the following.

THEOREM 2.1. Let G be a locally compact group and U a type one unitary representation of G which possesses a cyclic vector φ. Let L_+ and L_- be the subspaces of H defined as

$$L_\pm = \text{span}\{[U(g) \pm U(g^{-1})]\varphi,\ g \in G\}.$$

Then

(a) If L_- is finite dimensional and the elements of the form g^2, $g \in G$, generate a dense subgroup of G, then U is finite dimensional and

$$\dim U = \dim L_- + \dim I_s(U,U^*).$$

(b) If L_+ is finite dimensional and the elements of the form g^4, $g \in G$, generate a dense subgroup of G then U is finite dimensional and

$$\dim U = \dim L_+ + \dim I_a(U,U^*).$$

Proof. Since the proofs of (a) and (b) are similar we prove only (b). Let L' be the annihilator of L_+ in H^*. We shall establish a one to one correspondence between L' and $I_a(U,U^*)$. Specifically, the correspondence will be obtained as follows. Let $\lambda \in L'$. For each vector of the form

$$\xi = \sum_i c_i U(g_i)\varphi \qquad \text{define} \qquad B_\lambda(\xi) = \sum_i c_i U^*(g_i)\lambda.$$

We shall show that this correspondence is well defined. Granting this B_λ becomes a densely defined interwining operator from H to H*. The main problem in proving our theorem is to demonstarte that B_λ is in fact a bounded operator. This will be achieved by expanding the domain of B as much as possible.

Now, let $V = U \oplus U^*$, and let $\mathfrak{a}$ be the von-Neumann algebra on $H \oplus H^*$ generated by the set of operators $\{V(g), g \in G\}$. Each element of $\mathfrak{a}$ is an operator of the form $A \oplus A'$ where

$$A = \lim_\alpha \sum_{g\in G} c^\alpha(g)U(g), \qquad A' = \lim_\alpha \sum_{g\in G} c^\alpha(g)U^*(g),$$

and c^α is a net of functions on G which are supported on finite sets on G. The limits are taken in the strong operator topology.

Lemma 2.1. Suppose $A \oplus A' \in \mathfrak{a}$ and $\lambda \in L'$. Then $A\varphi = 0$ implies that $A'\lambda = 0$.

Proof. Let A and A' be represented as described above and consider the function

$$f(h) = \langle U(h)\varphi, A'\lambda\rangle.$$

By definition

$$f(h) = \lim_\alpha \sum_{g\in G} \langle U(h)\varphi, c^\alpha(g)U^*(g)\lambda\rangle =$$

$$\lim_\alpha \sum_{g\in G} \langle U(g^{-1}h)\varphi, \lambda\rangle\, c^\alpha(g)$$

Since $\lambda \in L'$

$$\langle U(g)\varphi, \lambda\rangle = -\langle U(g^{-1})\varphi, \lambda\rangle.$$

Thus

$$f(h) = \lim_{\alpha} \sum_{g\in G} <U(h^{-1})c_{\alpha}(g)U(g)\varphi, \lambda = <U(h^{-1})A\varphi,\lambda> = 0$$

and $A'\lambda = 0$ because of the cyclicity of φ. The lemma is proven.

Let $\mathcal{C}$ be the subspace of H defined by

$$\mathcal{C} = \{A\varphi, A \quad A' \in \mathfrak{A} \text{ for some } A'\}.$$

For each $\lambda \in L'$ let B_{λ} be the mapping of $\mathcal{C}$ into H^* defined as $B_{\lambda}(A\phi) = A'\lambda$ where $A \oplus A' \in \mathfrak{A}$. By the Lemma 2.1 this definition makes sense.

Lemma 2.2. For all $v, w \in \mathcal{C}$

(i) $<v, B_{\lambda}w> = -<B_{\lambda}v,w>$

(ii) $B_{\lambda}U(g)v = U^*(g)B_{\lambda}v$

Conversely, any linear operator $B: \mathcal{C} \to H^*$ which satisfies (i) and (ii) has the form $B = B_{\lambda}$ where $\lambda = B\varphi \in L'$.

Proof. The proof of (i) is analogous to that of the lemma 2.1 and (ii) follows from the definition of B_{λ}. The last statement of lemma 2.2 is true since

$$<U(g)\varphi,B\varphi> = -<\varphi BU(g)\varphi> = -<\varphi,U^*(g)B\varphi> =$$
$$<-U(g^{-1})\varphi,B\varphi>$$

Now let π: $H \to H$ be a central projection for U, i.e. π commutes with U and with the commuting algebra of U. We call π balanced if $\pi \quad \pi^*$ is a central projection for V. More specifically, the general "matrix" form of an interwining operator for V is

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha \in I(U,U)$, $\delta \in I(U^*,U^*)$, $\beta \in I(U^*,U)$ and $\gamma \in I(U,U^*)$. Hence π is balanced if π is central for U and $\delta\pi = \pi^*\gamma$ for all $\gamma \in I(U,U^*)$. (The β-identity follows from γ-identity by transposition). Intuitively, "balanced central projection" means that if an irreducible representation U_0 "occurs" in π, then U_0^* also "occurs" in π given that U_0^* "occurs" in U. Of course, U might have no discrete spectrum, so this is only formal.

Lemma 2.3. If π is balanced, then $\pi L_+ \subset L_+$. If there is a non-zero balanced projection π such that $\pi L_+ = 0$ then the image of π is one-dimensional and U is trivial on the image of π.

Proof. It suffices to show that π^* maps L' into L'. Let $\lambda \in L'$. Then $\lambda = B_\lambda \varphi$ and $\pi^*\lambda = \pi^* B_\lambda \varphi$. But then

$$\langle U(g)\varphi, \pi^*\lambda\rangle = \langle U(g)\varphi, \pi^* B_\lambda \varphi\rangle = -\langle \varphi, B_\lambda \pi U(g)\varphi\rangle =$$

$$- \langle \varphi, \pi^* U^*(g)\lambda\rangle = - \langle U(g^{-1})\varphi, \pi^*\rangle$$

since $\pi U(g) \oplus \pi^* U^*(g) \in \mathfrak{a}$. Thus, as claimed, $\pi^*\lambda \in L'$.

Now, if $\pi(U(g) + U(g^{-1}))\varphi = 0$ then $U(g)\pi\varphi = -U(g^{-1})\pi\varphi$, so that $U(g^4)\pi\varphi = \pi\varphi$. Since the elements g^4 generate a dense subgroup, U is trivial on the image of π. Since a cyclic representation can contain the identity representation at most once, our lemma follows.

Corollary. There are only a finite number of disjoint balanced projections for U. Also $\sum_i^q \pi_i = 1$ if $\pi_1, \ldots, \pi_q$ is a maximal family of disjoint balanced projections.

Proof. Let π_1 be the unique balanced projection such that $\pi_1 L_+ = 0$ (it is possible that $\pi_1 = 0$). If $\{\pi_i\}$ is any family of disjoint balanced projections, then $\sum \oplus \pi_i L_+ \subset L_+$. By finite dimensionality there can be at most a finite number of such π_i. The equality $\sum_1^q \pi_i = I$ for a maximal family follows from the fact that π is balanced iff $I - \pi$ is balanced.

This corollary allows us to assume that in the proof of finite dimensionality of U, the identity is the only non-zero balanced projection. In this case U is "almost" a primary representation as the next lemma shows.

Lemma 2.4. U has at most two disjoint central projections.

Proof. Let π be a central projection for U. Let H_π be the image of π in H and let $H_{\tilde{\pi}}$ be the closure of the image of H_π under I(U,U*). Let $\tilde{\pi}$ be the projection in H* onto $H_{\tilde{\pi}}$. It is clear that $H_{\tilde{\pi}}$ is invariant under U* so that $\pi \oplus \tilde{\pi} \in \mathcal{U}'$. We claim also that $\pi \oplus \tilde{\pi} \in \mathcal{U}$. To see this it is sufficient to prove that $\pi \oplus \tilde{\pi}$ commutes with $\mathcal{U}'$. It is obvious that $H_{\tilde{\pi}}$ is invariant under any operator which commutes with U*, so it suffices to show that $\tilde{\pi}A = A\pi$ for all $A \in I(U,U^*)$ and $\pi A' = A'\tilde{\pi}$ for all $A' \in I(U^*,U)$. The first identity is proven as follows

$$\tilde{\pi}A = \tilde{\pi}A\pi + \tilde{\pi}A(I-\pi) = A\pi + \tilde{\pi}A(I-\pi)$$

Now, since π is central $U|\pi$ is disjoint from $U|(I-\pi)$ (i.e. they contain no equivalent sub-representations) and $\tilde{\pi}A(I-\pi) = 0$.

The second identity follows similarly using the inclusion $A'H_{\tilde{\pi}} \subset H_\pi$, which holds since A'A maps H_π into H_π for all $A \in I(U,U^*)$.

Therefore $\pi \oplus \tilde{\pi}$ and $(\tilde{\pi})^* \oplus \tilde{\pi}^*$ are central projections. Hence their product $\pi(\tilde{\pi})^* \oplus \pi^*\tilde{\pi}$ is central, so $\pi(\tilde{\pi})^*$ is balanced. Thus $\pi(\tilde{\pi})^* = I$ or 0. If $\pi(\tilde{\pi})^* = I$, then $\pi = I$. If $\pi(\tilde{\pi})^* = 0$, then $\pi + (\tilde{\pi})^*$ is a balanced projection and $\pi + (\tilde{\pi})^* = I$. But if $\sigma \leq \pi$ (i.e. $\sigma\pi = \sigma$), then $(\tilde{\sigma})^* \leq (\tilde{\pi})^*$, so that $\sigma + (\tilde{\sigma})^* = I$. The latter is impossible unless $\sigma = \pi$. Thus if $\pi \neq I$, then π has no smaller central projections. Since π was arbitrary this shows that π and $I - \pi$ are the only central projections for U. Q.E.D.

Now, let π be a minimal central projection for U. By the above lemma $I - \pi$ is also such a projection. By restricting U to the image of π we may assume that the only central projection for U is I. Hence U is a primary representation so U must be of the form nU_0 where U_0 is an irreducible representation and $n \in \{1,2,\ldots,\infty\}$. If one shows that U_0 is finite dimensional then it will follow that U is finite dimensional since a cyclic representation cannot contain any finite dimensional representation with infinite multiplicity (see [2]s, 15.5.3). Thus can take $U = U_0$.

We prove that in the irreducible case the set $\mathcal{C}$ (the domain of B_λ) equals H. To see this we use the fact that a topologically irreducible representation of a C*algebra is algebraically irreducible (cf [2], 2.8.4). Let C*(G) be the group C*algebra of G. Let U (resp. U*) denote the representation of C*(G) corresponding to U (resp. U*). If $f \in C^*(G)$, then $\tilde{U}(f) \oplus \tilde{U}^*(f) \in \mathcal{C}$. It follows that $\tilde{U}(f)\varphi \in \mathcal{C}$ for all $f \in C^*(G)$. But the set of $\tilde{U}(f)\varphi$ is invariant under U, and U is irrediucible. Hence $\mathcal{C} = H$ as claimed. Since B_λ has an adjoint operator, it follows from the closed graph theorem that B_λ is continuous. As U is irreducible the space I(U,U*) is at most one dimensional. The mapping $\lambda \to B_\lambda$ is injective from L' to I(U,U*), so L' is at most one dimensional. This shows that L_+ has finite co-dimension in H, what implies finite dimensionality of H. Thus the proof of the finite dimensionality of U in general is finished.

Since U is finite dimensional B_λ is a continuous operator from H* to H*. The mapping $\lambda \to B_\lambda$ is injective. It is also surjective for if $B \in I(U,U^*)$ then $\lambda = B\varphi \in L'$. It is easily seen that $B = B_\lambda$. Thus

$$\dim U = \dim L_+ + \dim L' = \dim L_+ + \dim I_a(U,U^*)$$

and the theorem 2.1 is proven.

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