ON A SELECTION PROBLEM IN RELIABILITY THEORY*

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1. INTRODUCTION

Suppose we have an ℓ-out-of-m system, where m units are to be placed and at least ℓ of them should function to make the system work, and the units are statistically independent (see Barlow and Proschan (1975)). In many situations several brands of units are available, from which we have to choose at most m brands and draw m units of the system from them. Note that it is permissible to draw units from a population more than once. We will find 'optimal' solutions for the series system (i.e. $\ell = m$) and the 1-out-of-2 system when each unit has an exponentially distributed lifelength.

Let $\pi_1,\ldots,\pi_k$ $(k \geq 2)$ denote the available brands and assume that each unit from the i-th brand has an exponentially distributed lifelengths with mean lifelength λ_i^{-1} $(i = 1,\ldots,k)$. Based on kn independent observations $\{X_{ij}\}_{j=1}^n$, $1 \leq i \leq k$, from $\pi_1,\ldots,\pi_k$, we want to find an 'optimal' solution. Because of sufficiency, the problem can be reduced to the one based on $\{X_i = \sum_{j=1}^n X_{ij}: 1 \leq i \leq k\}$, with X_i having Gamma distribution with mean $n\lambda_i^{-1}$ and variance $n\lambda_i^{-2}$.

For the series system the expected lifelength of the system, when we use brands $\pi_{i_1},\ldots,\pi_{i_m}$ $(1 \leq i_1 \leq \ldots \leq i_m \leq k)$ for the m units, is easily seen to be $(\sum_{j=1}^m \lambda_{i_j})^{-1}$, and for the 1-out-of-2 system it is given by $\lambda_i^{-1} + \lambda_j^{-1} - (\lambda_i + \lambda_j)^{-1}$ when we use brands π_i and π_j $(1 \leq i \leq j \leq k)$. We will consider a loss function which is inversely proportional to the expected lifelength corresponding to an action.

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In section 2, we will give the results for the series system. It is assumed that the loss incurred by using brands $\pi_{i_1},\ldots,\pi_{i_m}$ $(1 \leq i_1 \leq \cdots \leq i_m \leq k)$ for the series system is given by

$$L(\underline{\lambda},(i_1,\ldots,i_m)) = \sum_{j=1}^{m} \lambda_{i_j}. \tag{1.1}$$

Then it is shown that for the series system, the natural rule, which draws all the units from the population associated with the largest sample mean life time, or equivalently with $\max_{1 \leq i \leq k} x_i$, is uniformly best among the permutation invariant rules and, therefore, it is admissible and minimax among all the rules.

In section 3, the Bayes rule for the 1-out-of-2 system is given. Here, it is assumed that the loss function is given by

$$L(\underline{\lambda},(i,j)) = (\lambda_i^{-1} + \lambda_j^{-1} - (\lambda_i + \lambda_j)^{-1})^{-1}, \tag{1.2}$$

where (i,j) $(1 \leq i \leq j \leq k)$ denotes the action of drawing units from π_i and π_j. Furthermore, the prior distribution of $\underline{\lambda}$ is assumed to be the independent natural conjugate Gamma-2 distribution (see p. 54 Raiffa and Schlaifer (1961)).

The 1-out-of-2 system with $k = 2$ has been considered by Broström (1977). He constructed a loss function which depends on (λ_1,λ_2) only through λ_1/λ_2 to obtain the invariance under the scale change. However, it should be pointed out that the construction of such a loss function can not be done for $k > 2$, for the obvious reason; in fact, he used the loss function L given by (1.2) divided by $L(\underline{\lambda}, (1,2))$, but for $k > 2$ there are no 'intermediate' actions by which we can standardize the loss function without losing comparability of the different actions.

Finally, let us introduce some notations. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(k)}$ denote the ordered observations $(x_1, x_2, \ldots, x_k)$, and $\pi_{(i)}$ and $\lambda_{(i)}$ denote the π and the λ associated with $x_{(i)}$, $i = 1,\ldots,k$.

2. OPTIMAL SOLUTION FOR THE SERIES SYSTEM

In this section, it is assumed that the loss function is given by (1.1). The action space is denoted by $\mathcal{U} = \{(i_1,\ldots,i_m): 1 \leq i_1 \leq \cdots \leq i_m \leq k\}$ where $(i_1,\ldots,i_m)$ is interpreted as drawing the j-th unit from the brand π_{i_j} $(j = 1, \ldots,m)$. Given $\underline{x} = (x_1,\ldots,x_k)$, the posterior risk of a decision rule d, which selects an action $(i_1,\ldots,i_m) \in \mathcal{U}$ with probability 1, is denoted by

$$r(d,\underline{x}) = E\left[\sum_{j=1}^{m} \lambda_{i_j} \middle| \underline{x}\right], \tag{2.1}$$

when the prior distribution of $\underline{\lambda}$ is given.

In this section, when considering only Bayes rules, attention can be restricted to non-randomized decision rules (see Ferguson (1967), §1.8).

The following result considerably reduces the number of decision rules to be compared for the Bayes rule w.r.t. a symmetric prior of $\underline{\lambda}$.

Lemma 1. If the prior distribution of $\underline{\lambda}$ is permutation symmetric on $(0,\infty)^k$, then the Bayes rule d* is given by

$$r(d^*,\underline{x}) = \min_{1\leq s\leq k\wedge m} \ \min_{\underline{n}_s \in N_s} r(d_{\underline{n}_s},\underline{x}),$$

where $N_s = \{\underline{n}_s = (n_1,\ldots,n_s): n_1 \geq \cdots \geq n_s \geq 1, \sum_{j=1}^{s} n_j = m, n_i\text{'s are integers}\}$, $k\wedge m = \text{Min}\ (k,m)$ and $d_{\underline{n}_s}$ draws n_j units from $\pi_{(k-j+1)}$ for $j = 1,\ldots,s$,

Proof. For $s = 1,\ldots,k\wedge m$ let us define $\mathcal{U}_s$ to be

$$\mathcal{U}_s = \{(i_1,\ldots,i_s: n_1,\ldots,n_s): \underline{n}_s \in N_s, 1 \leq i_j \leq k, i_j \neq i_{j'}, \text{for } j\neq j'\},$$

where $(i_1,\ldots,i_s;\ n_1,\ldots,n_s)$ is interpreted as drawing n_j units from π_{i_j} $(j = 1,2,\ldots,s)$. Note that we are partitioning the action space $\mathcal{Q}$ into $k \wedge m$ components, $\mathcal{Q}_s(s = 1,\ldots,k\wedge m)$, where we should use s different brands. Again, $\mathcal{Q}_s$ can be written as

$$\mathcal{Q}_s = \bigcup_{\underline{n}_s \in N_s} \mathcal{Q}_{\underline{n}_s}, \text{ where } \mathcal{Q}_{\underline{n}_s} = \{(i_1,\ldots,i_s):(i_1,\ldots,i_s;\underline{n}_s) \in \mathcal{Q}_s\}.$$

Note that the loss function given by (2.1) can be written as $L(\underline{\lambda},a) = \sum_{j=1}^{s} n_j\lambda_{i_j}$ for $a \in \mathcal{Q}_{\underline{n}_s}$. Now consider a decision problem with the action space $\mathcal{Q}_{\underline{n}_s}$, the above loss function and the observation vector $\underline{x}$. Clearly, this problem is equivalent to partitioning k brands $\pi_1,\ldots,\pi_k$ into s+1 subsets $(\gamma_1,\ldots,\gamma_s,\gamma_{s+1})$ where γ_j is of size 1 for $j = 1,\ldots,s$, γ_{s+1} is of size $k - s$ and the action $(i_1,\ldots,i_s)$ corresponds to the action $(\{\pi_{i_1}\},\ldots,\{\pi_{i_s}\},\ \{\pi_i\colon 1 \le i \le k,\ i \ne i_1,\ldots,i_s\})$.

It is easy to see that this component decision problem is invariant under the permutation group, and that the loss function satisfies the monotonicity and the invariance properties of Eaton (1967) with parameters θ_i of his paper being λ_i^{-1} $(i = 1,\ldots,k)$. Since the density $f(x,\theta_i)$ of X_i has monotone likelihood property in x and $\theta_i = \lambda_i^{-1}$, it follows from Eaton's results that the rule which assigns $\pi_{(k-j+1)}$ to γ_j for $j = 1,\ldots,s$ and $\pi_{(k-s+1)},\ldots,\pi_{(1)}$ to γ_{s+1} is Bayes w.r.t. a permutation symmetric prior distribution of $\underline{\theta} = (\theta_1,\ldots,\theta_k)$. This completes the proof.

The following lemma has a result which is of interest in itself.

Lemma 2. Assume that $X_1,\ldots,X_k$, given $\underline{\theta} = (\theta_1,\ldots,\theta_k) \in \Theta^k \subset (-\infty,\infty)^k$, are independently distributed random variables with X_i having p.d.f. $f(x,\theta_i)$. If $f(x,\theta)$ has the monotone likelihood ratio (MLR) property in x and θ, and if the prior distribution, $\tau(\underline{\theta})$, of $\underline{\theta} = (\theta_1,\ldots,\theta_k)$ is permutation symmetric on Θ^k, then

$$E[\theta_{(i)}|\underline{x}] \geq E[\theta_{(j)}|\underline{x}] \qquad \text{for } i \geq j,$$

where $E[\theta_{(i)}|\underline{x}]$ denotes the posterior expectation of θ associated with $x_{(i)}$.

Proof. Let $\Omega_0 = \{\underline{\theta} \in \Theta^k: \ \theta_{(i)} \geq \theta_{(j)}\}$.

Then

$$\int_{\Theta^k} [\theta_{(i)}-\theta_{(j)}]f(\underline{x},\underline{\theta})d\tau(\underline{\theta}) = \int_{\Omega_0} [\theta_{(i)}-\theta_{(j)}]f(\underline{x},\underline{\theta})d\tau(\underline{\theta}) + \int_{\Omega_0^c} [\theta_{(i)}-\theta_{(j)}]f(\underline{x},\underline{\theta})d\tau(\underline{\theta})$$

$$= \int_{\Omega_0} [\theta_{(i)}-\theta_{(j)}][f(\underline{x},\underline{\theta})-f(\underline{x},\underline{\theta}')]d\tau(\underline{\theta}),$$

where $f(\underline{x},\underline{\theta}) = \prod_{i=1}^{k} f(x_i,\theta_i)$ and $\underline{\theta}'$ is obtained from $\underline{\theta}$ by interchanging $\theta_{(i)}$ and $\theta_{(j)}$, keeping other components fixed. Thus $E[\theta_{(i)}-\theta_{(j)} \ |\underline{x}] = n(\underline{x}) \int_{\Omega_0} [\theta_{(i)}-\theta_{(j)}][f(\underline{x},\underline{\theta})-f(\underline{x},\underline{\theta}')]d\tau(\underline{\theta})$, where $n(x)$ is a normalizing factor. The result follows from the MLR property of $f(x,\theta)$ and the fact that $\theta_{(i)}-\theta_{(j)} \geq 0$ for $\underline{\theta} \in \Omega_0$.

REMARK 1. The MLR property of $f(x_i,\theta_i)$ in Lemma 2 can be replaced either by the M property of $f(\underline{x},\underline{\theta})$ in Eaton (1967), or by the DT property of $f(\underline{x},\underline{\theta})$ in Hollander, Proschan and Sethuraman (1977).

REMARK 2. If $\theta_1,\ldots,\theta_k$ are, a priori, positive random variables, then it is easy to see that $E[\theta_{(i)}^{-1}|\underline{x}] \leq E[\theta_{(j)}^{-1}|\underline{x}]$ for $i \geq j$. Therefore in our problem $E[\lambda_{(i)}|\underline{x}] \leq E[\lambda_{(j)}|\underline{x}]$ for $i \geq j$, if the prior distribution of $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k)$ is permutation symmetric on $(0,\infty)^k$.

The next result follows from Lemma 1 and Lemma 2.

THEOREM 1. For any permutation symmetric prior distribution of $\underline{\lambda}$ on $(0,\infty)^k$, the Bayes rule is $d_1 = d_{\underline{n}_1}$, as defined in Lemma 1; namely the Bayes rule draws all m units from $\pi_{(k)}$.

Proof. It follows from (2.1) that $r(d_{\underline{n}_s},\underline{x})$ can be written as

$$r(d_{\underline{n}_s},\underline{x}) = E[\sum_{j=1}^{s} n_j \lambda_{(k-j+1)} | \underline{x}] \quad \text{for } \underline{n}_s \in N_s.$$

Therefore

$$r(d_{\underline{n}_s},\underline{x}) - E[(m-s+1)\lambda_{(k)} + \sum_{j=2}^{s} \lambda_{(k-j+1)} | \underline{x}]$$

$$= E[(m-\sum_{j=2}^{s} n_j)\lambda_{(k)} + \sum_{j=2}^{s} n_j \lambda_{(k-j+1)} | \underline{x}] - E[(m-s+1)\lambda_{(k)} + \sum_{j=2}^{s} \lambda_{(k-j+1)} | \underline{x}]$$

$$= E[\sum_{j=2}^{s} (n_j - 1)(\lambda_{(k-j+1)} - \lambda_{(k)}) | \underline{x}] \geq 0 \text{ as pointed out in Remark 2.}$$

Thus $\underset{\underline{n}_s \in N_s}{\text{MIN}} r(d_{\underline{n}_s},x) = r(d_s,x)$ where $d_s = d_{\underline{n}^*_s}$ with $\underline{n}^*_s = (m-s+1,1,\ldots,1) \in N_s$, i.e.

(2.2) $\quad d_s$ draws $(m-s+1)$ units from $\pi_{(k)}$ and one unit from each $\pi_{(k-j+1)}$ $(j = 2,\ldots,s)$.

And for any s: $2 \leq s \leq k \wedge m$,

$$r(d_s,\underline{x}) - r(d_1,\underline{x}) = E[\sum_{j=2}^{s} (\lambda_{(k-j+1)} - \lambda_{(k)})] \geq 0.$$

Therefore the result follows from Lemma 1.

Cor. 1. The 'natural' rule d_1 is uniformly best among the permutation invariant rules.

Proof. This follows from considering a permutation symmetric prior τ which gives mass $\frac{1}{k!}$ at each permutation of components of a fixed vector $\underline{\lambda} \in (0,\infty)^k$.

Cor. 2. The natural rule d_1 is admissible and minimax within the class of all decision rules.

Proof. Since the permutation group is finite, the result follows from theorems in Ferguson (1967), §4-3.

REMARK 3. Alternative proof of Theorem 1 has been suggested by Panchapakesan (1978) after this paper was partially prepared. His idea is to treat the above decision problem as a product of m decision problems; namely $\mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_m$, where $i_j \in \mathcal{U}_j$ means to draw the j-th unit from the i-th brand. This can be done since the loss function $L(\underline{\lambda},a)$ for $a \in \mathcal{U}$ can be written as

$$L(\underline{\lambda},a) = \sum_{j=1}^{m} \lambda_{i_j} = \sum_{j=1}^{m} L_j(\underline{\lambda},a_j) \quad \text{for } a = (i_1,\ldots,i_m) \in \mathcal{U} \text{ and } a_j=\{i_j\} \in \mathcal{U}_j.$$

Then using Lemma 2, a simpler proof of Theorem 1 can be done. We have chosen to retain our method of proof for other independent interest since this alternative approach can not be done if the loss function is not additive.

REMARK 4. If we consider a loss function $L_1(\underline{\lambda},(i_1,\ldots,i_m)) = (m \min_{1\le i\le k} \lambda_i)^{-1} - (\sum_{j=1}^{m} \lambda_{i_j})^{-1}$, it is easy to verify the monotonicity and invariance properties of Eaton (1967). Therefore Lemma 1 holds for the loss function L_1. Assuming an exchangeable prior of $\underline{\lambda}$ on $(0,\infty)^k$, namely, given $B = \beta$, $\lambda_1,\ldots,\lambda_k$ are positive iid random variables with p.d.f. $g(\cdot,\beta)$ and the distribution of B is known, we can prove that the Bayes rule d^* for the loss function L_1 satisfies

$$r(d^*,\underline{x}) = \min_{1\le s\le k\wedge m} r(d_s,\underline{x}), \tag{2.3}$$

where the rule d_s is defined in (2.2). Note that this can not be achieved by the alternative method in Remark 3. Even though (2.3) is a considerable reduction in a number of candidates for the Bayes rule, specification of the Bayes rule seems very difficult except when m = 2. One interesting exchangeable prior is assuming an inverted Dirichlet prior distribution with p.d.f. $\tau(\underline{\lambda}) = \frac{\Gamma(\alpha k+a)}{\Gamma(\alpha)^k\Gamma(a)} \cdot \frac{\prod_{i=1}^{k} \lambda_i^{\alpha-1}}{(\sum_1^k \lambda_i+1)^{\alpha k+a}}$ where $\alpha > 0$, $a > 0$ are known constants, which is equivalent to assuming that $\lambda_1,\ldots,\lambda_k$, given $B = \beta$, are iid Gamma random variables with mean α/β and α/β^2, and B has Gamma distribution with mean a and variance a (see Johnson and Kotz, 1972, Page 239). Another simplest way of assuming an exchangeable prior is specifying that $\lambda_1,\ldots,\lambda_k$ are, a priori, positive iid random variables. Some numerical results in this direction for m > 2 would be interesting.

3. BAYES SOLUTION FOR THE 1-OUT-OF-2 SYSTEM

In this section it is assumed that the loss function is given by (1.2). The action space is denoted by $\mathcal{Q} = \{(i,j): 1 \le i \le j \le k\}$, where (i,j) is interpreted as drawing one unit each from the brands π_i and π_j, respectively. Furthermore, the prior distribution of $\underline{\lambda}$ is assumed to be independent natural conjugate Gamma-2 distribution. Then the joint a priori p.d.f. of $\underline{\lambda}$ is

$$(3.1) \qquad \tau(\underline{\lambda}) = \prod_{i=1}^{k} [\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda_i^{\alpha-1}e^{-\beta\lambda_i}], \ \alpha > 0 \text{ and } \beta > 0.$$

Then it is easy to see that the posterior p.d.f. of $\underline{\lambda}$, given $\underline{X}=\underline{x}$, is

$$(3.2) \qquad \tau(\underline{\lambda}|\underline{x}) = \prod_{i=1}^{k} [\frac{(x_i+\beta)^{n+\alpha}}{\Gamma(n+\alpha)} \lambda_i^{n+\alpha-1} e^{-(x_i+\beta)\lambda_i}].$$

It follows from this that $\lambda_{(1)},\ldots,\lambda_{(k)}$ are, a posteriori, independently distributed Gamma random variables with mean $(n+\alpha)/(x_{(i)}+\beta)$ and variance $(n+\alpha)/(x_{(i)}+\beta)^2$. Let $r(d,\underline{x})$ denote the posterior risk of rule d, given $\underline{x}$.

By considering $\mathcal{U}_1 = \{(i,i);\ i = 1,\ldots,k\}$ and $\mathcal{U}_2 = \{(i,j);\ 1 \le i < j \le k\}$ and using similar arguments to the one in Lemma 1, we have the next result.

Lemma 3. For any permutation symmetric prior of $\underline{\lambda}$ on $(0,\infty)^k$, the Bayes rule d* is given by

$$r(d^*,\underline{x}) = \operatorname{Min}\{r(d_1,\underline{x}),\ r(d_2,\underline{x})\},$$

where d_1 chooses 2 units from $\pi_{(k)}$ and d_2 chooses one unit from $\pi_{(k)}$ and another from $\pi_{(k-1)}$.

Now we state and prove a theorem which gives the Bayes solution.

THEOREM 2. The Bayes rule d* w.r.t. the prior given by (3.1) is given by

$$d^* = \begin{cases} d_1 & \text{if } x_{(k-1)} + \beta \le c(x_{(k)} + \beta), \\ d_2 & \text{if } x_{(k-1)} + \beta > c(x_{(k)} + \beta), \end{cases} \tag{3.3}$$

where $c = H^{-1}_{\alpha,n}(0) \in (0,1)$, $H_{\alpha,n}(c) = E[\frac{UV(U+cV)}{U^2+c^2V^2+cUV}] - \frac{2}{3}(n+\alpha)$ for $c > 0$ and U, V are iid Gamma random variables with mean $(n+\alpha)$ and variance $(n+\alpha)$.

Proof: It follows form (1.2) and (3.2) that

$$r(d_1,\underline{x}) = \tfrac{2}{3}E[\lambda_{(k)}|\underline{x}] = \frac{2}{3}\ \frac{n+\alpha}{x_{(k)}+\beta}$$

and

$$r(d_2,\underline{x}) = E[\lambda_{(k)}\lambda_{(k-1)}(\lambda_{(k)}+\lambda_{(k-1)})(\lambda^2_{(k)}+\lambda_{(k)}\lambda_{(k-1)}+\lambda^2_{(k-1)})^{-1}|\underline{x}]$$

$$= \frac{1}{x_{(k)}+\beta}\quad E[\frac{UV(U+r_k\ V)}{U^2+r_kUV+r_kV^2}] \text{ for } r_k = \frac{x_{(k-1)}+\beta}{x_{(k)}+\beta},$$

where U and V are iid Gamma random variables with mean $(n+\alpha)$ and variance $(n+\alpha)$. Thus $r(d_1,\underline{x}) > r(d_2,\underline{x})$ if and only if $H_{\alpha,n}(\frac{x_{(k-1)}+\beta}{x_{(k)}+\beta})<0$, which is equivalent to $\frac{x_{(k-1)}+\beta}{x_{(k)}+\beta} > H^{-1}_{\alpha,n}(0)$ since $H_{\alpha,n}(t)$ is a decreasing function of $t > 0$. Furthermore it is easy to see that

$$H_{\alpha,n}(1) = E[\frac{UV(U+V)}{U^2+UV+V^2}] - \frac{2}{3}(n+a) < 0$$

which implies $0 < H^{-1}_{\alpha,n}(0) < 1$. Hence the result follows from Lemma 3.

It is easy to see that $\frac{x_1}{x_1+\beta} \cdots \frac{x_k}{x_k+\beta}$ are marginally independent Beta random variables with mena $n/(n+\alpha)$. It follows from this that the Bayes risk of the rule d_1 satisfies

$$(3.4) \qquad r(d_1) = \frac{2}{3}(n+\alpha)\beta^{-1}E[Z_{(1)}]$$
$$\leq \frac{2}{3} \quad \frac{\alpha}{\beta}$$

where $Z_{(1)}$ is the smallest order statistic from a sample of size k from Beta distributions with mean $\alpha/(\alpha+n)$. It follows from (3.4) that the Bayes risk of the Bayes rule d* in (3.3) is finite. Furthermore, the distribution function of $\underline{X}$, given $\underline{\lambda} \in (0,\infty)^k$, is absolutely continuous with respect to the marginal distribution function of $\underline{X}$.

The next result follows from the above fact and a well known theorem (see, for example, Brown (1974), Theorem 3.14).

Corollary 3. The Bayes rule d* in (3.3) is admissible.

Similarly, it is easy to see that the generalized Bayes rule w.r.t. $d\underline{\lambda} = \prod_{i=1}^{k} d \log \lambda_i$ which corresponds to the vague prior $\alpha = \beta \to 0$, is given by (3.3) with $\alpha = \beta = 0$.

REMARK 5. If we consider a loss function $L_2(\underline{\lambda},(i,j)) = \frac{3}{2}(\min_{1\le i\le k} \lambda_i)^{-1} - [\lambda_i^{-1}+\lambda_j^{-1}-(\lambda_i+\lambda_j)^{-1}]$, it follows from the same method that the Bayes rule w.r.t. the prior specified in (3.1) is given by (3.3) with $c = G_{\alpha,n}^{-1}(0) \in (0,1)$ and $G_{\alpha,n}(t) = \frac{1}{2}\frac{1}{n+\alpha-1} - \frac{t}{n+\alpha-1} + E[\frac{t}{tU+V}]$ for $t > 0$, where U and V are iid Gamma random variables with mean and variance equal to $(n+\alpha)$. The remaining analogous results can also be obtained.

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In this paper, the methodology of subset selection is applied to a reliability problem of choosing units for the series system and the 1-out-of-2 system from k available brands $\pi_1, \pi_2, \ldots, \pi_k$. The problem is treated from a decision-theoretic (Bayesian) point of view.

It is assumed that units from the i-th brand have exponentially distributed

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) lifelengths with mean lifelength λ_1^{-1}, and that there are n independent observations from each of $\pi_1, \pi_2, \ldots, \pi_k$. The loss functions are assumed to be inversely proportional to the expected lifelength of the corresponding action.

For the series system, the natural rule, which selects all the units from the population associated with the largest sample mean life, is shown to be uniformly best within the class of permutation invariant decision rules, admissible and minimax.

For the 1-out-of-2 system the Bayes rule w.r.t. the natural independent conjugate Gamma-2 prior is given, and is shown to be admissible. Also the Bayes rule w.r.t. the same prior when the loss function is the relative regret in terms of the expected lifelength, is also given.