Some Complex Variable Transformations and Exact Power Comparisons of Two-Sided Tests of Equality of Two Hermitian Covariance Matrices

by

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Abstract

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Power studies of tests of equality of covariance matrices of two p-variate complex normal populations $\Sigma_1 = \Sigma_2$ against two-sided alternatives have been made based on the following five criteria: 1) Roy's largest root, 2) Hotelling's trace, 3) Pillai's trace, 4) Wilks' criterion and 5) Roy's largest and smallest roots. Some theorems on transformations and Jacobians in the two-sample complex Gaussion case have been proved in order to obtain a general theorem for establishing the local unbiasedness conditions connecting the two critical values for tests 1) to 5). Extensive unbiased power tabulations have been made for p=2, for various values of n_1, n_2, λ_1 and λ_2 where n_i is the df of the SP matrix from the i-th sample and λ_i is the i-th latent root of $\Sigma_1\Sigma_2^{-1}$ (i=1,2). Equal tail areas approach has also been used further to compute powers of tests 1) to 4) for p=2 for studying the bias and facilitating comparisons with powers in the unbiased case. The inferences have been found similar to those in the real case. (Chu and Pillai, Mimeograph Series No. 500, Department of Statistics, Purdue University).

Some Complex Variable Transformations and Exact Power Comparisons of Two-Sided Tests of Equality of Two Hermitian Covariance Matrices[1]

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1. Introduction and Summary

In the complex multivariate normal theory [Goodman, 3], the study of distribution problems concerning MANOVA, canonical correlation and equality of two covariance matrices was made by several authors, notably by Khatri [6] and James [4]. The non-central distributions of the characteristic roots concerning the various test procedures were explicitly given by James [4] in terms of zonal polynomials of complex hermitian matrices. Here, we consider the problem of testing the equality of covariance matrices of two complex normal populations.

Let $\underset{\sim}{X}_1(p\mathrm{x}n_1)$ and $\underset{\sim}{X}_2(p\mathrm{x}n_2)$ $p \leq n_1, n_2$, be independent complex matrix variates, columns of $\underset{\sim}{X}_1$ being independently distributed as $CN(\underset{\sim}{0},\Sigma_1)$ and those of $\underset{\sim}{X}_2$ independently distributed as $CN(\underset{\sim}{0},\underset{\sim}{\Sigma}_2)$. Let $0 \leq c_1 \leq \cdots \leq c_p < \infty$ be the characteristic roots of $|X_1\bar{X}_1' - cX_2\bar{X}_2'|=0$ and $\lambda_1,\ldots,\lambda_p$, the characteristic roots of $|\underset{\sim}{\Sigma}_1 - \lambda\underset{\sim}{\Sigma}_2| = 0$. To test $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ or equivalently $\lambda_1 = \ldots = \lambda_p = 1$ against $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$ (two-sided), the following five criteria are considered:

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1) Roy's largest root, c_p, [Roy, 13] or $L_p^{(p)} = c_p/(1+c_p)$,

2) Hotelling's trace, $U^{(p)} = \sum_{i=1}^{p} c_i$, [Pillai, 9],

3) Pillai's trace, $V^{(p)} = \sum_{i=1}^{p} [c_i/(1+c_i)]$, [Pillai, 9],

4) Wilks' criterion, $W^{(p)} = \prod_{i=1}^{p} (1+c_i)^{-1}$, [Wilks, 16],

5) Roy's largest-smallest roots, $c_1(L_1^{(p)} = c_1/(1+c_1))$ and $c_p(L_p^{(p)} = c_p/(1+c_p))$, to be denoted by $LS^{(p)}$ defined in terms of $L_p^{(p)}$ and $L_1^{(p)}$, [Roy, 14].

Power studies of tests of $\Sigma_1 = \Sigma_2$ against the alternative of a one-sided nature:

$$\lambda_i \geq 1, \quad \sum_{i=1}^{p} \lambda_i > p, \quad i = 1,\ldots,p,$$

were carried out by Pillai and Hsu [10] based on the first four criteria. Exact power tabulations were made in the two-roots case for various (λ_1,λ_2) and different degrees of freedom n_1 and n_2 (actually in terms of $m = n_1-2$ and $n = n_2-2$).

In this paper, a power comparison study has been attempted for tests of the hypothesis $\Sigma_1 = \Sigma_2$ against $\Sigma_1 \neq \Sigma_2$ (two-sided) based on the above five criteria. A theorem similar to the real case [Chu and Pillai, 2] is proved first obtaining the condition of local unbiasedness for a class of tests of which 1) to 5) are special cases. Using the theorem, relations between the two critical values for each of the five tests are obtained as special cases for tests 1) to 5) for the two-roots case. Further, critical values for level α = .05 (five percent points) for the five tests are computed for p = 2 and values of n_1 = 2, 3, 4, 7 and n_2 = 7, 17, 32, 62 and are given in Table 1. Also, powers of the criteria 1) to 5) have been tabulated for various values of (λ_1,λ_2), n_1 = 2, 3, 4, 7 and n_2 = 7, 17, 32, 62 and these are presented in Table 2. In addition, power tabulations have also been carried out from the equal tail areas point of view, of tests 1) to 4) which are observed to be biased. These tabulations are also available in Table 2 for the same values of (λ_1,λ_2), n_1 and n_2 as before facilitating comparisons with powers in the unbiased case. The critical values in this case are also given in Table 1. (Table 1 is available in Chu [1].)

A few findings seem to emerge from the numerical results of powers tabulated and there is a general agreement with those discussed in the real case [Chu and Pillai, 2]. In general it is observed that the largest root has some power advantage over the other criteria studied. These findings are presented in Section 6.

Some results on transformations and Jacobians in the two-sample complex Gaussian case are proved in the next section which are needed in the sequel.

2. Some results on transformations and Jacobians

Lemma 1. If $\underset{\sim}{M}$(pxp) is hermitian and at least positive semi-definite of rank r($\leq$ p), then there exists a unitary matrix $\underset{\sim}{A}$ with real first row such that $\underset{\sim}{M} = \underset{\sim}{A}\, \underset{\sim}{D}_{c_k}\, \bar{\underset{\sim}{A}}'$ where $\underset{\sim}{D}_{c_k}$ is diagonal with elements the characteristic roots c_k's of $\underset{\sim}{M}$.

Proof. From linear algebra, we know that there exists a unitary matrix $\underset{\sim}{B}$ such that $\underset{\sim}{M} = \underset{\sim}{B}\, \underset{\sim}{D}_{c_k}\, \bar{\underset{\sim}{B}}'$ where the columns of $\underset{\sim}{B}$ are the unit characteristic vectors of $\underset{\sim}{M}$. If each column of $\underset{\sim}{B}$ is multiplied by some constant of absolute value unity, the resulting matrix preserves the properties of $\underset{\sim}{B}$, i.e. the resulting matrix is unitary and diagonalizes $\underset{\sim}{M}$. Therefore, if we multiple the jth column of $\underset{\sim}{B} = (b_{ij})$ by $\pm \bar{b}_{1j}/|b_{1j}|$ for $j = 1,\ldots,p$, and denote the resulting matrix by $\underset{\sim}{A} = (a_{ij})$, $(a_{ij} = \pm b_{ij}\bar{b}_{1j}/|b_{1j}|)$, then $\underset{\sim}{A}$ is unitary and diagonalizes $\underset{\sim}{M}$, and moreover, the first row of $\underset{\sim}{A}$ is real. Hence the lemma.

Lemma 2. If $\underset{\sim}{M}_1(p\mathrm{x}p)$ is hermitian and at least positive semi-definite of rank $r(\leq p)$ and $\underset{\sim}{M}_2(p\mathrm{x}p)$ is hermitian positive definite, then there exists a non-singular matrix $\underset{\sim}{A}$ with real first row such that $\underset{\sim}{M}_1 = \underset{\sim}{A}\, \underset{\sim}{D}_{c_k}\, \bar{\underset{\sim}{A}}'$ and $\underset{\sim}{M}_2 = \underset{\sim}{A}\bar{\underset{\sim}{A}}'$, where c_k' are the roots of the equation $|\underset{\sim}{M}_1 - c\underset{\sim}{M}_2| = 0$.

Proof. Since $\underset{\sim}{M}_2$ is hermitian positive definite, there exists a lower triangular matrix $\underset{\sim}{T}$ with real diagonal elements such that $\underset{\sim}{M}_2 = \underset{\sim}{T}\bar{\underset{\sim}{T}}'$. Now $\underset{\sim}{T}^{-1}\underset{\sim}{M}_1\bar{\underset{\sim}{T}}'^{-1}$ is hermitian and at least positive semi-definite. By Lemma 1, there exists a unitary matrix $\underset{\sim}{B}$ with real first row such that $\underset{\sim}{T}^{-1}\underset{\sim}{M}_1\bar{\underset{\sim}{T}}'^{-1} = \underset{\sim}{B}\, \underset{\sim}{D}_{c_k}\bar{\underset{\sim}{B}}'$ where c_k's are the characteristic roots of $\underset{\sim}{M}_1\underset{\sim}{M}_2^{-1}$. Let $\underset{\sim}{A} = \underset{\sim}{T}\underset{\sim}{B}$. $\underset{\sim}{A}$ is non-singular with real first row. Then $\underset{\sim}{M}_1 = \underset{\sim}{T}\underset{\sim}{B}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{B}}'\bar{\underset{\sim}{T}}' = \underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}'$, $\underset{\sim}{M}_2 = \underset{\sim}{T}\bar{\underset{\sim}{T}}' = \underset{\sim}{T}\underset{\sim}{B}\bar{\underset{\sim}{B}}'\bar{\underset{\sim}{T}}' = \underset{\sim}{A}\bar{\underset{\sim}{A}}'$.

Lemma 3. The matrix $\underset{\sim}{A}$ of Lemma 2 will be unique, except for a post-factor $\underset{\sim}{D}_k = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$, if $\underset{\sim}{M}_1$ is positive definite and all the characteristic roots c_k's are distinct.

Proof. Suppose there are two non-singular $\underset{\sim}{A}$'s with real first row, say $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$, satisfying the conditions of Lemma 2. Then we have

$$\underset{\sim}{M}_1 = \underset{\sim}{A}_1 \underset{\sim}{D}_{c_k} \bar{\underset{\sim}{A}}_1' = \underset{\sim}{A}_2 \underset{\sim}{D}_{c_k} \bar{\underset{\sim}{A}}_2',$$

$$\underset{\sim}{M}_2 = \underset{\sim}{A}_1 \bar{\underset{\sim}{A}}_1' = \underset{\sim}{A}_2 \bar{\underset{\sim}{A}}_2'.$$

Thus $\quad \underset{\sim}{A}_2^{-1} \underset{\sim}{A}_1 \underset{\sim}{D}_{c_k} = \underset{\sim}{D}_{c_k} \bar{\underset{\sim}{A}}_2' \bar{\underset{\sim}{A}}_1'^{-1} = \underset{\sim}{D}_{c_k} \underset{\sim}{A}_2^{-1} \underset{\sim}{A}_1,$

i.e. $\quad \underset{\sim}{B} \underset{\sim}{D}_{c_k} = \underset{\sim}{D}_{c_k} \underset{\sim}{B}$ where $\underset{\sim}{B} = \underset{\sim}{A}_2^{-1} \underset{\sim}{A}_1 = (\beta_{ij})$.

And this implies $\beta_{ij} c_j = c_i \beta_{ij}$.

So we have $\quad \beta_{ij} = 0$ if $i \neq j$ and $c_i \neq c_j$.

Thus $\quad \underset{\sim}{B} = \underset{\sim}{D}_{\alpha_k}$ (say) where $\alpha_k = a_k + ib_k$, $k = 1, \ldots, p$.

And $\quad \underset{\sim}{D}_{|\alpha_k|^2} = \underset{\sim}{D}_{\alpha_k} \underset{\sim}{D}_{\bar{\alpha}_k} = \underset{\sim}{B} \bar{\underset{\sim}{B}}' = \underset{\sim}{A}_2^{-1} \underset{\sim}{A}_1 \bar{\underset{\sim}{A}}_1' \bar{\underset{\sim}{A}}_2'^{-1} = \underset{\sim}{A}_2^{-1} \underset{\sim}{A}_2 \bar{\underset{\sim}{A}}_2' \bar{\underset{\sim}{A}}_2'^{-1} = \underset{\sim}{I}.$

So $\quad |\alpha_k| = 1$ for $k = 1, \ldots, p$.

Therefore $\underset{\sim}{A}_1 = \underset{\sim}{A}_2 \underset{\sim}{D}_{\alpha_k}$ where $|\alpha_k| = 1$, $k = 1, \ldots, p$.

Since the first row elements of both $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are real, so α_k must be real, i.e. $b_k = 0$ and $a_k = \pm 1$, $k = 1, \ldots, p$. Thefore $\underset{\sim}{A}_1 = \underset{\sim}{A}_2 \underset{\sim}{D}_k$ where $\underset{\sim}{D}_k = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$. We note that $\underset{\sim}{A}$ can thus be made unique by choosing the real first row positive. The transformation is now one to one.

Lemma 4. If $\underset{\sim}{X}_1(p \times n_1)$, $\underset{\sim}{X}_2(p \times n_2)$, $(p \leq n_1, n_2)$ are each of rank p, then there exists a transformation $\underset{\sim}{X}_1(p \times n_1) = \underset{\sim}{A}(p \times p) \underset{\sim}{D}_{\sqrt{c_k}}(p \times p) \underset{\sim}{L}_1(p \times n_1)$

and $\underset{\sim}{X}_2(pxn_2) = \underset{\sim}{A}(pxp)\underset{\sim}{L}_2(pxn_2)$ where $\underset{\sim}{A}$ is non-singular with real first row, c_k's are the roots (all positive) of the equation $|\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1' - c\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2'| = 0$ and $\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1' = \underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2' = \underset{\sim}{I}$. If all c_k's are distinct, then this transformation is unique except for a post-factor $\underset{\sim}{D}_k$ to go with $\underset{\sim}{A}$.

Proof. By Lemma 2, there exists a non-singular $\underset{\sim}{A}$ with real first row such that $\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1' = \underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}'$ and $\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2' = \underset{\sim}{A}\bar{\underset{\sim}{A}}'$. We now define $\underset{\sim}{L}_1(pxn_1)$ and $\underset{\sim}{L}_2(pxn_2)$ by $\underset{\sim}{X}_1 = \underset{\sim}{A}\underset{\sim}{D}_{\sqrt{c_k}}\underset{\sim}{L}_1$ and $\underset{\sim}{X}_2 = \underset{\sim}{A}\underset{\sim}{L}_2$ and note that, given $\underset{\sim}{X}_1, \underset{\sim}{X}_2$ and c_k's and $\underset{\sim}{A}$, $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ are uniquely solvable. Also $\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1' = \underset{\sim}{D}_{c_k^{-\frac{1}{2}}}\underset{\sim}{A}^{-1}\underset{\sim}{X}_1 \bar{\underset{\sim}{X}}_1'\bar{\underset{\sim}{A}}'^{-1}\underset{\sim}{D}_{c_k^{-\frac{1}{2}}} = \underset{\sim}{I}$ and $\underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2' = \underset{\sim}{A}^{-1}\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2'\bar{\underset{\sim}{A}}'^{-1} = \underset{\sim}{I}$. This proves the existence of the transformation. Notice that if all c_k's are distinct, then by Lemma 3, $\underset{\sim}{A}$ is unique except for a post-factor $\underset{\sim}{D}_k$ and that $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ will go with $\underset{\sim}{A}$ being defined by $\underset{\sim}{L}_1 = \underset{\sim}{D}_{c_k^{-\frac{1}{2}}}\underset{\sim}{A}^{-1}\underset{\sim}{X}_1$ and $\underset{\sim}{L}_2 = \underset{\sim}{A}^{-1}\underset{\sim}{X}_2$.

Lemma 5. If $\underset{\sim}{X}_1(pxn_1)$, $\underset{\sim}{X}_2(pxn_2)(p \leq n_1, n_2)$ have the joint density:

$$\pi^{-p(n_1+n_2)}|\underset{\sim}{\Sigma}_1|^{-n_1}|\underset{\sim}{\Sigma}_2|^{-n_2}\exp[-\mathrm{tr}(\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1'+\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')],$$

where $\underset{\sim}{\Sigma}_1$, $\underset{\sim}{\Sigma}_2$ are hermitian positive definite, then the distribution of the characteristic roots of $(\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1')(\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')^{-1}$ involves as parameters only the characteristic roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ (to be called λ_k's).

The proof follows from Lemma 2 and is available in Chu [1].

Lemma 6. The Jacobian of the transformation $\underset{\sim}{X}_1(p\times n_1) = \underset{\sim}{A}(p\times p)\underset{\sim}{D}_{\sqrt{c_k}}(p\times p)\underset{\sim}{L}_1(p\times n_1)$, $\underset{\sim}{X}_2(p\times n_2) = \underset{\sim}{A}(p\times p)\underset{\sim}{L}_2(p\times n_2)$ where $\underset{\sim}{A}$ is non-singular with positive and real first row and $\underset{\sim}{L}_1\underset{\sim}{\bar{L}}_1' = \underset{\sim}{L}_2\underset{\sim}{\bar{L}}_2' = \underset{\sim}{I}$ is given by

$$J(\underset{\sim}{X}_1,\underset{\sim}{X}_2: \underset{\sim}{A},\ c_k\text{'s},\ \underset{\sim}{L}_{1I},\underset{\sim}{L}_{2I})$$

$$= \left|\frac{\partial(\underset{\sim}{X}_1,\underset{\sim}{X}_2,\underset{\sim}{L}_1\underset{\sim}{\bar{L}}_1',\underset{\sim}{L}_2\underset{\sim}{\bar{L}}_2')}{\partial(\underset{\sim}{A},c_k\text{'s},\underset{\sim}{L}_1,\underset{\sim}{L}_2)}\right|_{\underset{\sim}{A},c_k\text{'s},\underset{\sim}{L}_{1I},\underset{\sim}{L}_{2I}} \Bigg/ \left|\frac{\partial(\underset{\sim}{L}_1\underset{\sim}{\bar{L}}_1')}{\partial(\underset{\sim}{L}_{1D})}\right|_{\underset{\sim}{L}_{1I}} \left|\frac{\partial(\underset{\sim}{L}_2\underset{\sim}{\bar{L}}_2')}{\partial(\underset{\sim}{L}_{2D})}\right|_{\underset{\sim}{L}_{2I}}$$

$$= 2^p|\det(\underset{\sim}{A})|^{2(n_1+n_2-p+\frac{1}{2})}\prod_{j=1}^{p} c_j^{n_1-p}\prod_{j>k}(c_j-c_k)^2 \Bigg/ \left|\frac{\partial(\underset{\sim}{L}_1\underset{\sim}{\bar{L}}_1')}{\partial(\underset{\sim}{L}_{1D})}\right|_{\underset{\sim}{L}_{1I}} \left|\frac{\partial(\underset{\sim}{L}_2\underset{\sim}{\bar{L}}_2')}{\partial(\underset{\sim}{L}_{2D})}\right|_{\underset{\sim}{L}_{2I}}.$$

The proof is omitted here and is available in Chu [1]. It may be noted that, when comparing with the proof of Roy [15] for an analogous theorem in the real case, in the power of $|\det(\underset{\sim}{A})|$, there is a factor $\frac{1}{2}$, in view of the fact that for the uniqueness of the transformation, the first row of $\underset{\sim}{A}$ was assumed positive and real.

3. The condition of local unbiasedness

The acceptance regions based on criteria 1) to 5) with local unbiasedness property and α level of significance can be written in one form:

$$R:\quad a(p,n_1,n_2) \le \omega(c_1,\dots,c_p) \le b(p,n_1,n_2),$$

where a and b are so chosen to satisfy

$$(3.1)\quad \text{(i)}\quad P(a \le \omega(c_1,\dots,c_p) \le b|\lambda_1 = \dots = \lambda_p = 1) = 1-\alpha,$$

$$(3.2)\quad \text{(ii)}\quad \left.\frac{\partial P(a \le \omega(c_1,\dots,c_p) \le b|\lambda_1,\dots,\lambda_p)}{\partial\lambda_i}\right|_{\lambda_1=\dots=\lambda_p=1} = 0, i=1,\dots,p,$$

where $\omega(c_1,\ldots,c_p) = c_p/(1+c_p)$ for test 1), $\sum_{i=1}^{p} c_i$ for 2), $\sum_{i=1}^{p} [c_i/(1+c_i)]$ for 3), $\prod_{i=1}^{p} (1+c_i)^{-1}$ for 4) and $c_1/(1+c_1)$, $c_p/(1+c_p)$ for 5).

In this section, we will show that the p equations given in (ii) are really equivalent to one equation and are in turn equivalent to

$$(3.3) \qquad (ii') \quad \left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0.$$

We call (ii) or equivalently (ii') "the condition of local unbiasedness".

Theorem 1. The p equations given in (ii) are equivalent to one equation and are in turn equivalent to (ii').

Proof. The joint density of $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ is given by

$$\pi^{-p(n_1+n_2)}|\underset{\sim}{\Sigma}_1|^{-n_1}|\underset{\sim}{\Sigma}_2|^{-n_2}\exp[-\mathrm{tr}(\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1'+\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')].$$

By Lemma 5, we may, without loss of generality, start directly from the following canonical form:

$$\pi^{-p(n_1+n_2)}\prod_{i=1}^{p}\lambda_i^{-n_1}\exp[-\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1' + \underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')],$$

where $\underset{\sim}{D}_{\lambda_k^{-1}} = \mathrm{diag}\,(1/\lambda_k)$. Then

$$P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)$$
$$= \pi^{-p(n_1+n_2)}\int_{a\le\omega(c_1,\ldots,c_p)\le b}\prod_{i=1}^{p}\lambda_i^{-n_1}\exp[-\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1'+\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')]d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Hence

$$(3.4) \qquad \frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial\lambda_i^{-1}}$$

$$= \pi^{-p(n_1+n_2)} \int_{a \le \omega(c_1,\ldots,c_p) \le b} \prod_{j=1}^{p} \lambda_j^{-n_1} [n_1\lambda_i - (\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1')_{ii}] \exp[-\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}} \underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1' + \underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')] d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

By Lemma 4, we may transform $\underset{\sim}{X}_1 = \underset{\sim}{A}\underset{\sim}{D}_{\sqrt{c_k}}\underset{\sim}{L}_1$ and $\underset{\sim}{X}_2 = \underset{\sim}{A}\underset{\sim}{L}_2$ where $\underset{\sim}{A} = (a_{jk}+ib_{jk})$ is non-singular with real first row and $\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1' = \underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2' = \underset{\sim}{I}$. And by Lemma 6, we have

$$J(\underset{\sim}{X}_1,\underset{\sim}{X}_2 : \underset{\sim}{A}, c_k\text{'s}, \underset{\sim}{L}_{1I}, \underset{\sim}{L}_{2I}) = 2^p |(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j - c_{j'})^2$$

$$\div \left| \frac{\partial(\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1')}{\partial(\underset{\sim}{L}_{1D})} \right|_{\underset{\sim}{L}_{1I}} \left| \frac{\partial(\underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2')}{\partial(\underset{\sim}{L}_{2D})} \right|_{\underset{\sim}{L}_{2I}}.$$

Therefore, (3.4) becomes

$$(3.5)\quad 2^p \pi^{-p(n_1+n_2)} \int_{R^*} \prod_{j=1}^{p} \lambda_j^{-n_1} [n_1\lambda_i - (\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}')_{ii}] \exp[-\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}} \underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}' + \underset{\sim}{A}\bar{\underset{\sim}{A}}')]$$

$$|(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} d\underset{\sim}{A} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j - c_{j'})^2 \prod_{j=1}^{p} dc_j$$

$$\int_{\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1' = \underset{\sim}{I}} \frac{d\underset{\sim}{L}_{1I}}{\left| \frac{\partial(\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1')}{\partial(\underset{\sim}{L}_{1D})} \right|_{\underset{\sim}{L}_{1I}}} \int_{\underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2' = \underset{\sim}{I}} \frac{d\underset{\sim}{L}_{2I}}{\left| \frac{\partial(\underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2')}{\partial(\underset{\sim}{L}_{2D})} \right|_{\underset{\sim}{L}_{2I}}}$$

$$= \frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{R^*} \prod_{j=1}^{p} \lambda_j^{-n_1} [n_1\lambda_i - (\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}')_{ii}] \exp[-\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}} \underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}' + \underset{\sim}{A}\bar{\underset{\sim}{A}}')]$$

$$|(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} d\underset{\sim}{A} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j - c_{j'})^2 \prod_{j=1}^{p} dc_j,$$

where R^* is $a \le \omega(c_1,\ldots,c_p) \le b$ and $-\infty <$ all $a_{jk}, b_{jk} < \infty$ and $\tilde{\Gamma}_p(n) = \pi^{\frac{1}{2}p(p-1)} \prod_{j=1}^{p} \Gamma(n-j+1)$. Thus we have

$$(3.6)\quad \left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial\lambda_i^{-1}}\right|_{\lambda_1=\ldots=\lambda_p=1}$$

$$= \frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{R^*} [n_1-(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}')_{ii}]\exp[-\mathrm{tr}(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}'+\underset{\sim}{A}\bar{\underset{\sim}{A}}')]$$

$$|(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} d\underset{\sim}{A} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j.$$

The only term in the above integrand which depends on i is $(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}')_{ii} = (a_{i1}^2c_1+b_{i1}^2c_1+\ldots+a_{ip}^2c_p+b_{ip}^2c_p)$ and the integral is taken over R*. Thus the integral is invariant under a change of the subscript i for i = 2,...,p. For i = 1, the integral is half of the integral for i = 2 (note that $b_{1k} = 0$ for k = 1,...,p). Hence the p equations $\left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial\lambda_i^{-1}}\right|_{\lambda_1=\ldots=\lambda_p=1} = 0$ are really equivalent to one equation. Now adding up integrals like (3.6) over i = 1,...,p, we have

$$(3.7)\quad \frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{R^*} [n_1p-\mathrm{tr}(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}')]\exp[-\mathrm{tr}(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}'+\underset{\sim}{A}\bar{\underset{\sim}{A}}')]$$

$$|(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} d\underset{\sim}{A} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j.$$

It is easy to see that (3.7) is actually equal to

$$\left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}.$$

Hence, (ii) is equivalent to (ii').

Theorem 2. Condition (ii') can be written as

$$C(p)[\int_{a\le\omega(c_1,\ldots,c_p)\le b} n_1 p \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)} \prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j -$$

$$\int_{a\le\omega(c_1,\ldots,c_p)\le b} (n_1+n_2)[\sum_{j=1}^{p} c_j/(1+c_j)] \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)}$$

$$\prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j] = 0,$$

where $C(p) = \dfrac{\pi^{p(p-1)}\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)}$.

Proof. Starting from (3.7) and transforming $\underset{\sim}{A} = \underset{\sim}{B}\underset{\sim}{D}_{(1+c_k)^{-\frac{1}{2}}}$, where $\underset{\sim}{B}$ is non-singular with real and positive first row if the matrix $\underset{\sim}{A}$ in Theorem 1 is taken with real and positive first row. $J(\underset{\sim}{A}: \underset{\sim}{B}) = |\underset{\sim}{D}_{(1+c_k)^{-\frac{1}{2}}}|^{2p-1} = \prod_{j=1}^{p}(1+c_j)^{-p+\frac{1}{2}}$. Then (3.7) becomes

$$(3.8)\quad \frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{R^{**}} [n_1 p - \mathrm{tr}(\underset{\sim}{B}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\bar{\underset{\sim}{B}}')]\exp[-\mathrm{tr}(\underset{\sim}{B}\bar{\underset{\sim}{B}}')] |(\det \underset{\sim}{B})|^{2n_1+2n_2-2p+1} d\underset{\sim}{B}$$

$$\times \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)} \prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j,$$

where R** is $a \le \omega(c_1,\ldots,c_p) \le b$ and $\underset{\sim}{B}$ is non-singular with real and positive first row.

Further, transform $\underset{\sim}{B} = \underset{\sim}{T}\underset{\sim}{L}$ where $\underset{\sim}{T}$ is lower triangular with real and positive diagonal and $\underset{\sim}{L}$ is semi-unitary with real and positive first row. We shall now find the Jacobian $J(\underset{\sim}{B}: \underset{\sim}{T},\underset{\sim}{L})$. Differentiating both sides of $\underset{\sim}{B} = \underset{\sim}{T}\underset{\sim}{L}$, we have $(d\underset{\sim}{B}) = (d\underset{\sim}{T})\underset{\sim}{L} + \underset{\sim}{T}(d\underset{\sim}{L})$. Pre-multiplying by $\underset{\sim}{T}^{-1}$, we get $\tilde{T}^{-1}(d\tilde{B}) = \tilde{T}^{-1}(d\tilde{T})\tilde{L} + (d\tilde{L})$. Putting $\tilde{U} = \tilde{T}^{-1}(d\tilde{B})$ and $\tilde{V} = \tilde{T}^{-1}(d\tilde{T})$, we have $\tilde{U} = \tilde{V}\tilde{L} + (d\tilde{L})$.

Hence
$$J(\tilde{B}: \tilde{T},\tilde{L}) = J(d\tilde{B}: d\tilde{T},d\tilde{L})$$
$$= J(d\tilde{B}: \tilde{U})J(\tilde{U}: \tilde{V},(d\tilde{L}))J(\tilde{V},(d\tilde{L}): (d\tilde{T}), (d\tilde{L}))$$
$$= |\tilde{T}|^{2p-1}2^p\left(d\tilde{L}_I\Big/\left|\frac{\partial(\tilde{L}\bar{\tilde{L}}')}{\partial(\tilde{L}_D)}\right|_{\tilde{L}_I}\right)\prod_{i=1}^{p} t_{ii}^{-2i+1}$$
$$= \left(\prod_{i=1}^{p} t_{ii}^{2p-2i}\right)2^p\left(d\tilde{L}_I\Big/\left|\frac{\partial(\tilde{L}\bar{\tilde{L}}')}{\partial(\tilde{L}_D)}\right|_{\tilde{L}_I}\right).$$

Then
$$\frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)}\int_{\tilde{B}}[n_1p-\mathrm{tr}(\tilde{B}\tilde{D}_{c_k(1+c_k)^{-1}}\bar{\tilde{B}}')]\exp[-\mathrm{tr}(\tilde{B}\bar{\tilde{B}}')]\,|(\det\tilde{B})|^{2n_1+2n_2-2p+1}\,d\tilde{B}$$

$$(3.9)\quad = \frac{2^{2p}}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)}\int_{\tilde{L}\bar{\tilde{L}}'=\tilde{I}}\int_{\tilde{T}}[n_1p-\mathrm{tr}(\tilde{T}\tilde{L}\tilde{D}_{c_k(1+c_k)^{-1}}\bar{\tilde{L}}'\bar{\tilde{T}}')]\exp[-\mathrm{tr}(\tilde{T}\bar{\tilde{T}}')]\,|\tilde{T}\bar{\tilde{T}}'|^{n_1+n_2-p+\frac{1}{2}}\prod_{i=1}^{p} t_{ii}^{2p-2i}\,d\tilde{T}\,d\tilde{L}_I\Big/\left|\frac{\partial(\tilde{L}\bar{\tilde{L}}')}{\partial(\tilde{L}_D)}\right|_{\tilde{L}_I}$$

$$= \frac{2^{2p}}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)}\int_{\tilde{L}\bar{\tilde{L}}'=\tilde{I}}\int_{\tilde{T}}[n_1p-\mathrm{tr}(\tilde{T}\tilde{L}\tilde{D}_{c_k(1+c_k)^{-1}}\bar{\tilde{L}}'\bar{\tilde{T}}')]\exp[-\mathrm{tr}(\tilde{T}\bar{\tilde{T}}')]\,|\tilde{T}\bar{\tilde{T}}'|^{n_1+n_2-p}\prod_{i=1}^{p} t_{ii}^{2p-2i+1}\,d\tilde{T}\,d\tilde{L}_I\Big/\left|\frac{\partial(\tilde{L}\bar{\tilde{L}}')}{\partial(\tilde{L}_D)}\right|_{\tilde{L}_I}.$$

Transform $\tilde{S} = \bar{\tilde{T}}'\tilde{T}$, $J(\tilde{T}:\tilde{S}) = 2^{-p}\prod_{i=1}^{p} t_{ii}^{-2p+2i-1}$. Then (3.9) is

$$(3.10)\quad \frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)}\int_{\tilde{L}\bar{\tilde{L}}'=\tilde{I}}\int_{\tilde{S}>\tilde{0}}[n_1p-\mathrm{tr}(\tilde{S}\tilde{L}\tilde{D}_{c_k(1+c_k)^{-1}}\bar{\tilde{L}}')]\exp[-\mathrm{tr}\tilde{S}]\,|\tilde{S}|^{n_1+n_2-p}\,d\tilde{S}\;d\tilde{L}_I\Big/\left|\frac{\partial(\tilde{L}\bar{\tilde{L}}')}{\partial(\tilde{L}_D)}\right|_{\tilde{L}_I}.$$

Apply the following equation of Khatri [8]:

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} \exp[-\mathrm{tr}\underset{\sim}{S}]|\underset{\sim}{S}|^{r-p}\tilde{C}_\kappa(\underset{\sim}{A}\underset{\sim}{S})d\underset{\sim}{S} = \tilde{\Gamma}_p(r,\kappa)\tilde{C}_\kappa(A),$$

where $\tilde{\Gamma}_p(r,\kappa) = \tilde{\Gamma}_p(r)[r]_\kappa$, then (3.10) becomes

$$\frac{2^p\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{\underset{\sim}{L}\underset{\sim}{L}'=\underset{\sim}{I}} [n_1p-(n_1+n_2)\mathrm{tr}(\underset{\sim}{L}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\underset{\sim}{L}')]d\underset{\sim}{L}_I \Big/ \left|\frac{\partial(\underset{\sim}{L}\underset{\sim}{L}')}{\partial(\underset{\sim}{L}_D)}\right|_{\underset{\sim}{L}_I}$$

$$= \frac{\pi^{p(p-1)}\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)} [n_1p-(n_1+n_2)\sum_{j=1}^{p} c_j/(1+c_j)].$$

Therefore, we get

$$\left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1 = \ldots = \lambda_p = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda = 1}$$

$$= C(p)\Big[\int_{a\le\omega(c_1,\ldots,c_p)\le b} n_1p \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)} \prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j -$$

$$\int_{a\le\omega(c_1,\ldots,c_p)\le b} (n_1+n_2)[\sum_{j=1}^{p} c_j/(1+c_j)] \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)} \prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j\Big].$$

Now equating the above to zero, we have Theorem 2.

Thus, the acceptance region based on criterion $\omega(c_1,\ldots,c_p)$ with local unbiasedness property (ℓup) and α level of significance can be written as

$$R: \; a(p,n_1,n_2) \le \omega(c_1,\ldots,c_p) \le b(p,n_1,n_2),$$

where a and b are so chosen as to satisfy:

$$\text{(i)}\quad C(p)\int_{a\le\omega(c_1,\ldots,c_p)\le b}\prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)}\prod_{j>j'}(c_j-c_{j'})^2\prod_{j=1}^{p}dc_j = 1-\alpha$$

and

$$\text{(ii)}\quad C(p)\Big[\int_{a\le\omega(c_1,\ldots,c_p)\le b} n_1 p\prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)}\prod_{j>j'}(c_j-c_{j'})^2\prod_{j=1}^{p}dc_j -$$

$$\int_{a\le\omega(c_1,\ldots,c_p)\le b}(n_1+n_2)\Big[\sum_{j=1}^{p} c_j/(1+c_j)\Big]\prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)}\prod_{j>j'}(c_j-c_{j'})^2\prod_{j=1}^{p}dc_j\Big] = 0,$$

where $C(p) = \dfrac{\pi^{p(p-1)}\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)}$.

4. The acceptance regions based on the five criteria with ℓup for $p = 2$

In this section, we will consider the acceptance regions of tests 1) to 5) in that order.

1) Roy's largest root, $L_2^{(2)} = c_2/(1+c_2)$. By using Theorem 2 in the previous section, we know that

$$\text{(4.1)}\qquad \left.\frac{\partial P(a \le L_2^{(2)} \le b\,|\,\lambda_1 = \lambda_2 = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$= C(2)\Big[\int_{a\le L_2^{(2)}\le b} g_2(c_1,c_2;n_1,n_2)dc_1dc_2 - \int_{a\le L_2^{(2)}\le b} h_2(c_1,c_2;n_1,n_2)dc_1dc_2\Big],$$

where $g_2(c_1,c_2;n_1,n_2) = 2n_1(c_1c_2)^{n_1-2}[(1+c_1)(1+c_2)]^{-(n_1+n_2)}(c_1-c_2)^2$ and

$$h_2(c_1,c_2;n_1,n_2) = (n_1+n_2)[c_1/(1+c_1)+c_2/(1+c_2)](c_1c_2)^{n_1-2}$$
$$\times [(1+c_1)(1+c_2)]^{-(n_1+n_2)}(c_1-c_2)^2.$$

Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then (4.1) becomes

$$(4.2)\qquad C(2)[\int_a^b \int_0^{\ell_2} g_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2 - \int_a^b \int_0^{\ell_2} h_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2],$$

where $g_3(\ell_1,\ell_2;n_1,n_2)=2n_1(\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2$ and

$$h_3(\ell_1,\ell_2;n_1,n_2)=(n_1+n_2)(\ell_1+\ell_2)(\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2.$$

Further, note that by making the same transformation,

$$(4.3)\qquad P(a \le L_2^{(2)} \le b|\lambda_1=\lambda_2=1)$$
$$= C(2)\int_a^b \int_0^{\ell_2} (\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2.$$

Khatri [5] has shown that

$$\int_0^x \int_0^{\ell_2} (\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$$
$$= B_x(n_1-1,n_2-1)B_x(n_1+1,n_2-1)-B_x(n_1,n_2-1)B_x(n_1,n_2-1),$$

where $B_x(r,s) = \int_0^x t^{r-1}(1-t)^{s-1}dt.$

Now $\int_0^x \int_0^{\ell_2} (\ell_1+\ell_2)(\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$

$$=\int_0^x \int_0^{\ell_2} [1-(1-\ell_1)(1-\ell_2)+\ell_1\ell_2](\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$$

$$= B_x(n_1-1,n_2-1)B_x(n_1+1,n_2-1)-B_x(n_1,n_2-1)B_x(n_1,n_2-1)$$
$$- B_x(n_1-1,n_2)B_x(n_1+1,n_2)+B_x(n_1,n_2)B_x(n_1,n_2)$$
$$+ B_x(n_1,n_2-1)B_x(n_1+2,n_2-1)-B_x(n_1+1,n_2-1)B_x(n_1+1,n_2-1).$$

Thus, we have proved the following:

Theorem 3. Let $T_1(x) = B_x(n_1-1,n_2-1)B_x(n_1+1,n_2-1)-B_x(n_1,n_2-1)B_x(n_1,n_2-1)$ and $T_2(x)=T_1(x)-B_x(n_1-1,n_2)B_x(n_1+1,n_2)+B_x(n_1,n_2)B_x(n_1,n_2)+B_x(n_1,n_2-1)$ $B_x(n_1+2,n_2-1)-B_x(n_1+1,n_2-1)B_x(n_1+1,n_2-1)$. Then the acceptance region based on Roy's largest root, $L_2^{(2)} = c_2/(1+c_2)$ with ℓup and α level is given by $a \leq L_2^{(2)} \leq b$ where a and b are so chosen as to satisfy:

(i) $C(2)[T_1(b)-T_1(a)] = 1-\alpha$ and

(ii) $C(2)\{2n_1[T_1(b)-T_1(a)]-(n_1+n_2)[T_2(b)-T_2(a)]\} = 0.$

2) Hotelling's trace, $U^{(2)} = c_1+c_2$. From the previous section,

(4.4) $P(a \leq U^{(2)} \leq b|\lambda_1 = \lambda_2 = \lambda)$

$$= \pi^{-2(n_1+n_2)} \int_{a\leq c_1+c_2\leq b} \lambda^{-2n_1} \exp[-\mathrm{tr}(\lambda^{-1}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1'+\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')]d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Now transform $\lambda^{-\frac{1}{2}}\underset{\sim}{X}_1 = \underset{\sim}{Y}_1$ and $\underset{\sim}{X}_2 = \underset{\sim}{Y}_2$. $J(\underset{\sim}{Y}_1:\underset{\sim}{X}_1) = \lambda^{-2n_1}$ and let $0 < d_1 \leq d_2 < \infty$ be the characteristic roots of $|\underset{\sim}{Y}_1\bar{\underset{\sim}{Y}}_1'-d\underset{\sim}{Y}_2\bar{\underset{\sim}{Y}}_2'| = 0$. Then (4.4) becomes

(4.5) $$C(2)\int_{a\leq\lambda d_1+\lambda d_2\leq b}(d_1d_2)^{n_1-2}[(1+d_1)(1+d_2)]^{-(n_1+n_2)}(d_1-d_2)^2 dd_1 dd_2.$$

Let $u = d_1 + d_2$ and $g = d_1d_2$. We get

$$P(a \leq U^{(2)} \leq b|\lambda_1 = \lambda_2 = \lambda)$$

$$= C(2)\int_{a\lambda^{-1}}^{b\lambda^{-1}} \int_0^{\frac{1}{4}u^2} g^{n_1-2}(1+u+g)^{-(n_1+n_2)}(u^2-4g)^{\frac{1}{2}}dgdu.$$

Thus

$$\left.\frac{\partial P(a \leq U^{(2)} \leq b|\lambda_1 = \lambda_2 = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$=C(2)[b\int_0^{\frac{1}{4}b^2} g^{n_1-2}(1+b+g)^{-(n_1+n_2)}(b^2-4g)^{\frac{1}{2}}dg-a\int_0^{\frac{1}{4}a^2} g^{n_1-2}$$

$$(1+a+g)^{-(n_1+n_2)}(a^2-4g)^{\frac{1}{2}}dg]$$

$$= bT_2(b)-aT_2(a),$$

where $T_2(u)$, the density function of $U^{(2)}$, has been found by Pillai and Jouris [12] as follows:

$$(4.6)\quad T_2(u) = C(2)\sum_{r=0}^{\infty}(-1)^r \frac{\binom{-(n_1+n_2)}{r}(n_1-2)!u^{2r+2n_1-1}}{4^{r+n_1-1}(1+u/2)^{2r+2n_1+2n_2}}$$

$$\times \frac{1}{(r+3/2)(r+5/2)\ldots[r+(2(n_1-3)+5)/2]}.$$

Furthermore,

$$P(a \leq U^{(2)} \leq b|\lambda_1 = \lambda_2 = 1)$$

$$= C(2)\int_a^b \int_0^{\frac{1}{4}u^2} g^{n_1-2}(1+u+g)^{-(n_1+n_2)}(u^2-4g)^{\frac{1}{2}}\, dgdu$$

$$= T_1(b)-T_1(a),$$

where $T_1(u)$ is defined as follows:

$$(4.7)\quad T_1(u)=C(2)\sum_{r=0}^{\infty}(-1)^r\binom{-(n_1+n_2)}{r}(n_1-2)!\ 4B_{\frac{u}{2+u}}(2r+2n_1,2n_2)$$

$$\times\frac{1}{(r+3/2)(r+5/2)\dots[r+(2n_1-1)/2]}.$$

Therefore, we have proved the following:

Theorem 4. The acceptance region based on $U^{(2)} = c_1+c_2$ with ℓup and α level is given by $a \le U^{(2)} \le b$ where a and b are chosen as to satisfy:

(i) $T_1(b)-T_1(a) = 1-\alpha$ and (ii) $bT_2(b)-aT_2(a) = 0$, where $T_1(u)$ and $T_2(u)$ are defined in (4.7) and (4.6) respectively.

3) Pillai's trace, $V^{(2)} = [c_1/(1+c_1)]+[c_2/(1+c_2)]$. From Theorem 2, we obtain $\left.\frac{\partial P(a\le V^{(2)}\le b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$ by replacing $L_2^{(2)}$ in the limits of the integrals in (4.1) by $V^{(2)}$. Now transform $\ell_1=c_1/(1+c_1)$ and $\ell_2=c_2/(1+c_2)$. Then

$$(4.8)\qquad \left.\frac{\partial P(a\le V^{(2)}\le b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$=C(2)\Big[\int_{a\le\ell_1+\ell_2\le b} g_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2-\int_{a\le\ell_1+\ell_2\le b} h_3(\ell_1,\ell_2;$$

$$n_1,n_2)d\ell_1 d\ell_2\Big].$$

Let $v = \ell_1+\ell_2$ and $g = \ell_1\ell_2$. Then (4.8) becomes

$$C(2)[2n_1\int_a^b\int_0^{\frac{1}{4}v^2} g^{n_1-2}(1-v+g)^{n_2-2}(v^2-4g)^{\frac{1}{2}}dg\,dv$$

$$-(n_1+n_2)\int_a^b \int_0^{\frac{1}{4}v^2} vg^{n_1-2}(1-v+g)^{n_2-2}(v^2-4g)^{\frac{1}{2}}dgdv].$$

And $P(a \leq V^{(2)} \leq b|\lambda_1=\lambda_2=1)$

$$= C(2)\int_a^b \int_0^{\frac{1}{4}v^2} g^{n_1-2}(1-v+g)^{n_2-2}(v^2-4g)^{\frac{1}{2}}dgdv.$$

Therefore, we have the following theorem:

Theorem 5. Let

$$T_1(x)=\begin{cases} \sum_{r=0}^{n_2-2} \dfrac{\binom{n_2-2}{r}(n_1+r-2)!}{2^{n_1+r-1}\,3.5\ldots(2(n_1+r)-1)} B_x(2n_1+2r,n_2-r-1) \text{ if } 0\leq x\leq 1, \\[2ex] \sum_{r=0}^{n_2-2} \dfrac{\binom{n_2-2}{r}(n_1+r-2)!}{2^{n_1+r-1}\,3.5\ldots(2(n_1+r)-1)} B_1(2n_1+2r,\ n_2-r-1) + \\[2ex] 2\sum_{r=0}^{n_1-2}(-1)^{n_1-r-2}\dfrac{\binom{n_1-2}{r}2^{n_2+r}(n_2+r-2)!}{3.5\ldots(2(n_2+r)-1)}\sum_{r=0}^{n_1-r-2}(-2)^j\binom{n_1-r-2}{j} \\[2ex] \qquad [B_{\frac{x}{2}}(j+1,2(n_2+r))-B_{\frac{1}{2}}(j+1,2(n_2+r))] \text{ if } 1\leq x\leq 2, \end{cases}$$

and

$$T_2(x)=\begin{cases} \sum_{r=0}^{n_2-2} \dfrac{\binom{n_2-2}{r}(n_1+r-2)!}{2^{n_1+r-1}\,3.5\ldots(2(n_1+r)-1)} B_x(2n_1+2r+1,n_2-r-1) \text{ if } 0\leq x\leq 1, \\[2ex] \sum_{r=0}^{n_2-2} \dfrac{\binom{n_2-2}{r}(n_1+r-2)!}{2^{n_1+r-1}\,3.5\ldots(2(n_1+r)-1)} B_1(2n_1+2r+1,n_2-r-1) + \\[2ex] 4\sum_{r=0}^{n_1-2}(-1)^{n_1-r-2}\dfrac{\binom{n_1-2}{r}2^{n_2+r}(n_2+r-2)!}{3.5\ldots(2(n_2+r)-1)}\sum_{j=0}^{n_1-r-2}(-2)^j\binom{n_1-r-2}{j} \end{cases}$$

$$\times\ [B_{\frac{x}{2}}(j+2,2(n_2+r))-B_{\frac{1}{2}}(j+2,2(n_2+r))], \text{ if } 1 \leq x \leq 2.$$

Then the acceptance region based on $V^{(2)}=[c_1/(1+c_1)]+[c_2/(1+c_2)]$ with ℓup and α level is given by $a \leq V^{(2)} \leq b$ where a and b are so chosen as to satisfy

(i) $C(2)[T_1(b)-T_1(a)] = 1-\alpha$ and

(ii) $C(2)\{2n_1[T_1(b)-T_1(a)]-(n_1+n_2)[T_2(b)-T_2(a)]\} = 0.$

4) Wilks' criterion $W^{(2)} = [(1+c_1)(1+c_2)]^{-1}$. Again from Theorem 2, we obtain $\left.\frac{\partial P(a \leq W^{(2)} \leq b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$ by replacing $L_2^{(2)}$ in the limits of the integrals in (4.1) by $W^{(2)}$. Now transform $\ell_1=c_1/(1+c_1)$ and $\ell_2=c_2/(1+c_2)$.

Then $\left.\frac{\partial P(a \leq W^{(2)} \leq b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$

$$= C(2)[\int_{a\leq(1-\ell_1)(1-\ell_2)\leq b} g_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2 - \int_{a\leq(1-\ell_1)(1-\ell_2)\leq b} h_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2].$$

The density function of $W^{(2)}$ has been found by Pillai and Jouris [12] as follows:

$$f(w) = \frac{\tilde{\Gamma}_2(n_1+n_2)}{\tilde{\Gamma}_2(n_2)\Gamma(2n_1)} \sum_{k=0}^{\infty} \frac{(n_1)_k(n_1-1)_k}{(2n_1)_k k!} w^{n_2-2}(1-w)^{2n_1+k-1}.$$

Thus, the distribution function of $W^{(2)}$ is

$$-2\sum_{j=0}^{\infty}\frac{(n_1+j-2)!}{4xj!(j-\frac{1}{2})\ldots.(n_1+j-\frac{1}{2})}\sum_k{}'(-1)^k\binom{-2j+1}{k}B_{w^{\frac{1}{2}}}(2(n_2+k-1),2(n_1+2j)-1)$$

where $\sum_k{}' = \sum_{k=0}^{1}$ if $j = 0$ and $\sum_k{}' = \sum_{k=0}^{\infty}$ if $j > 0$. Therefore, we have the following:

Theorem 6. Let

$$T_1(w) = -2\sum_{j=0}^{\infty}\frac{(n_1+j-2)!}{4xj!(j-\frac{1}{2})\ldots.(n_1+j-\frac{1}{2})}\sum_k{}'(-1)^k\binom{-2j+1}{k}B_{w^{\frac{1}{2}}}(2(n_2+k-1),\ 2(n_1+2j)-1),$$

$$T_2(w) = -2\sum_{j=0}^{\infty}\frac{(n_1+j-2)!}{4xj!(j-\frac{1}{2})\ldots.(n_1+j-\frac{1}{2})}\sum_k{}'(-1)^k\binom{-2j+1}{k}B_{w^{\frac{1}{2}}}(2(n_2+k),\ 2(n_1+2j)-1),$$

and

$$T_3(w) = -2\sum_{j=0}^{\infty}\frac{(n_1+j-1)!}{4xj!(j-\frac{1}{2})\ldots(n_1+j+\frac{1}{2})}\sum_k{}'(-1)^k\binom{-2j+1}{k}B_{w^{\frac{1}{2}}}(2(n_2+k-1),\ 2(n_1+2j)+1).$$

Then the acceptance region based on $W^{(2)} = [(1+c_1)(1+c_2)]^{-1}$ with ℓup and α level is given by $a \le W^{(2)} \le b$ where a and b are so chosen as to satisfy

(i) $C(2)[T_1(b)-T_1(a)] = 1-\alpha$ and

(ii) $C(2)[(n_1-n_2)\{T_1(b)-T_1(a)\}+(n_1+n_2)\{T_2(b)-T_2(a)\}-(n_1+n_2)\{T_3(b)-T_3(a)\}] = 0.$

5) Roy's largest-smallest roots, $LS^{(2)} = c_2/(1+c_2),\ c_1/(1+c_1)$.

From Theorem 2 in the previous section, we have

$$(4.11)\quad \left.\frac{\partial P(a \le LS^{(2)} \le b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$= C(2)[\int_{a\le c_1/(1+c_1)\le c_2/(1+c_2)\le b} g_2(c_1,c_2;n_1,n_2)dc_1dc_2 - \int_{a\le c_1/(1+c_1)\le c_2/(1+c_2)\le b} h_2(c_1,c_2;n_1,n_2)dc_1dc_2].$$

Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then (4.11) becomes

$$C(2)[\int_a^b \int_a^{\ell_2} g_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2 - \int_a^b \int_a^{\ell_2} h_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2].$$

Further, note that by making the same transformation,

$$P(a \le LS^{(2)} \le b|\lambda_1=\lambda_2=1)$$

$$= C(2)\int_a^b \int_a^{\ell_2} (\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2.$$

Use the technique as in Khatri [5], we have

$$\int_x^y \int_x^{\ell_2} (\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$$

$$= B_{x,y}(n_1-1,n_2-1)B_{x,y}(n_1+1,n_2-1) - B_{x,y}(n_1,n_2-1)B_{x,y}(n_1,n_2-1),$$

where $B_{x,y}(r,s) = \int_x^y t^{r-1}(1-t)^{s-1}dt$.

Now $\int_x^y \int_x^{\ell_2} (\ell_1+\ell_2)(\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$

$$= \int_x^y \int_x^{\ell_2} [1-(1-\ell_1)(1-\ell_2)+\ell_1\ell_2](\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 \, d\ell_1 d\ell_2$$

$$= B_{x,y}(n_1-1,n_2-1)B_{x,y}(n_1+1,n_2-1)-B_{x,y}(n_1,n_2-1)B_{x,y}(n_1,n_2-1)$$
$$- B_{x,y}(n_1-1,n_2)B_{x,y}(n_1+1,n_2)+B_{x,y}(n_1,n_2)B_{x,y}(n_1,n_2)$$
$$+ B_{x,y}(n_1,n_2-1)B_{x,y}(n_1+2,n_2-1)-B_{x,y}(n_1+1,n_2-1)B_{x,y}(n_1+1,n_2-1).$$

Thus, we have proved the following:

Theorem 7. Let $T_1(x,y) = B_{x,y}(n_1-1,n_2-1)B_{x,y}(n_1+1,n_2-1)-B_{x,y}(n_1,n_2-1)$ $B_{x,y}(n_1,n_2-1)$ and $T_2(x,y) = T_1(x,y)-B_{x,y}(n_1-1,n_2)B_{x,y}(n_1+1,n_2)+$ $B_{x,y}(n_1,n_2)B_{x,y}(n_1,n_2) +B_{x,y}(n_1,n_2-1)B_{x,y}(n_1+2,n_2-1)-B^2_{x,y}(n_1+1,n_2-1)$. Then the acceptance region based on Roy's largest-smallest roots, $LS^{(2)}=c_2/(1+c_2)$, $c_1/(1+c_1)$ with ℓup and α level is given by $a \le LS^{(2)} \le b$ where a and b are so chosen as to satisfy

(i) $C(2)\ T_1(a,b) = 1-\alpha$

(ii) $C(2)[2n_1T_1(a,b)-(n_1+n_2)T_2(a,b)] = 0.$

5. $P(a \le [c_1/(1+c_1)] \le [c_2/(1+c_2)] \le b)$ in the non-null case

The non-null distribution of $c_1,\ldots,c_p$ was obtained by Khatri [7] in the form

$$(5.1)\quad C(p)|\underset{\sim}{\Lambda}|^{-n_1}|\underset{\sim}{C}|^{n_1-p}|\underset{\sim}{I}+\underset{\sim}{C}|^{-(n_1+n_2)}\ {}_1\tilde{F}_0(n_1+n_2;\underset{\sim}{I}-\underset{\sim}{\Lambda}^{-1},\underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{C})^{-1})$$
$$\prod_{i>j}(c_i-c_j)^2,$$
$$0 < c_1 \le\ldots\le c_p < \infty,$$

where the complex hypergeometric function of a matrix argument is defined by James [4]:

$${}_s\tilde{F}_t(a_1,\dots,a_s;b_1,\dots,b_t;\underset{\sim}{A},\underset{\sim}{B}) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots\cdots[a_s]_\kappa}{[b_1]_\kappa\cdots\cdots[b_t]_\kappa}\,\frac{\tilde{C}_\kappa(\underset{\sim}{A})\tilde{C}_\kappa(\underset{\sim}{B})}{k!\,\tilde{C}_\kappa(I)},$$

where $a_1,\dots,a_s,b_1,\dots,b_t$ are real or complex constants and the coefficient $[a]_\kappa$ is defined by

$$[a]_\kappa = \prod_{i=1}^{p}(a-i+1)_{k_i}$$

where $(a)_k = a(a+1)\dots(a+k-1)$ and κ of k is a partition of k, $\kappa = (k_1,\dots,k_p)$, $k_1 \geq \dots \geq k_p \geq 0$ such that $k_1+\dots+k_p = k$ and the complex zonal polynomials $\tilde{C}_\kappa(\underset{\sim}{A})$ are expressible in terms of elementary symmetric functions of the characteristic roots of the hermitian matrix $\underset{\sim}{A}$. Now putting $p = 2$, in (5.1), the joint density of c_1, c_2 is given by

$$(5.2)\quad C(2)(\lambda_1\lambda_2)^{-n_1}(c_1c_2)^{n_1-2}[(1+c_1)(1+c_2)]^{-(n_1+n_2)}(c_2-c_1)^2$$

$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[n_1+n_2]_\kappa}{k!}\,\frac{\tilde{C}_\kappa\begin{pmatrix}1-1/\lambda_1 & 0\\ 0 & 1-1/\lambda_2\end{pmatrix}\tilde{C}_\kappa\begin{pmatrix}c_1/(1+c_1) & 0\\ 0 & c_2/(1+c_2)\end{pmatrix}}{\tilde{C}_\kappa(\underset{\sim}{I})}.$$

Using the relation $\tilde{C}_\kappa\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} = \sum_{r+2s=k}\tilde{b}_\kappa(r,s)(a+b)^r(ab)^s$, where the $\tilde{b}_\kappa(r,s)$'s were given by Pillai and Hsu [**10**] up to $k = 6$, (5.2) becomes

$$(5.3)\quad C(2)(\lambda_1\lambda_2)^{-n_1}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[n_1+n_2]_\kappa}{k!}\,\frac{\tilde{C}_\kappa\begin{pmatrix}1-1/\lambda_1 & 0\\ 0 & 1-1/\lambda_2\end{pmatrix}}{\tilde{C}_\kappa(\underset{\sim}{I})}$$

$$\times\sum_{r+2s=k}\tilde{b}_\kappa(r,s)[c/(1+c_1)+c_2/(1+c_2)]^r(c_1c_2)^{n_1+s-2}$$

$$\times[(1+c_1)(1+c_2)]^{-(n_1+n_2+s)}(c_2-c_1)^2.$$

Now transform $\ell_1 = c_1/(1+c_1)$, $\ell_2 = c_2/(1+c_2)$, then the joint distribution of ℓ_1, ℓ_2 is

$$C(2)(\lambda_1\lambda_2)^{-n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n_1+n_2]_\kappa}{k!} \frac{\tilde{C}_\kappa\begin{pmatrix} 1-1/\lambda_1 & 0 \\ 0 & 1-1/\lambda_2 \end{pmatrix}}{\tilde{C}_\kappa(I)}$$

$$\times \sum_{r+2s=k} \tilde{b}_\kappa(r,s)(\ell_1+\ell_2)^r(\ell_1\ell_2)^{n_1+s-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_2-\ell_1)^2.$$

Now let

$$h_{r,s}(a,b) = \int_a^b \int_a^{\ell_2} (\ell_1+\ell_2)^r(\ell_1\ell_2)^{n_1+s-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_2-\ell_1)^2 d\ell_1 d\ell_2$$

$$= \sum_{i=0}^{r} \binom{r}{i} \sum_{j=0}^{n_2-2} (-1)^j \binom{n_2-2}{j}$$

$$\times\{[B_{a,b}(2n_1+2s+r+j,n_2-1)-a^{i+j+n_1+s-1}B_{a,b}(n_1+s+r-i+1,n_2-1)]/(i+j+n_1+s-1)$$

$$-2[B_{a,b}(2n_1+2s+r+j,n_2-1)-a^{i+j+n_1+s}B_{a,b}(n_1+s+r-i,n_2-1)]/(i+j+n_1+s)$$

$$+[B_{a,b}(2n_1+2s+r+j,n_2-1)-a^{i+j+n_1+s+1}B_{a,b}(n_1+s+r-i-1,n_2-1)]/(i+j+n_1+s+1)\},$$

where $B_{x,y}(r,s) = \int_x^y t^{r-1}(1-t)^{s-1}dt$.

Then we have, in the complex non-null case,

$$P(a \le [c_1/(1+c_1)] \le [c_2/(1+c_2)] \le b]$$

$$=P(a \le \ell_1 \le \ell_2 \le b)$$

$$= C(2)(\lambda_1\lambda_2)^{-n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n_1+n_2]_\kappa}{k!} \frac{\tilde{C}_\kappa\begin{pmatrix} 1-1/\lambda_1 & 0 \\ 0 & 1-1/\lambda_2 \end{pmatrix}}{\tilde{C}_\kappa(\underset{\sim}{I})}$$

$$\times \sum_{r+2s=k} \tilde{b}_\kappa(r,s) h_{r,s}(a,b).$$

6. Numerical study of power

The results in the previous sections were used to obtain five percent points for the tests of H_0: $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$ based on criteria 1) to 4) in the unbiased as well as equal tail areas cases and criterion 5) in the unbiased case for $p = 2$, values of $n_1 = 2$, 3, 4, 7 and $n_2 = 7$, 17, 32, 62, and are given in Table 1.

The next step was to compute the powers of the various tests using the percentage points evaluated and the non-null distributions. For tests 1) to 4), non-null distributions were available in Pillai and Hsu [10] and for test 5), it has been obtained in Section 5. Before computing the power for a specific value of (λ_1, λ_2), the total probability in that case over the whole range of the respective statistic for all the terms included in the formula was calculated and the number of decimal places included in the tables was determined depending on the number of places of accuracy obtained in the total probability, at least as many decimal places as in the tables. Powers for tests 1) to 5) in the unbiased as well as equal tail areas cases for $p = 2$, for values of $n_1 = 2,3,4,7$, $n = 7,17,32,62$, and various (λ_1, λ_2) are presented in Table 2.

A few findings seem to emerge from tabulations of powers in Table 2, and they are in general agreement with those discussed before in the real case and are stated below for convenience.

1. $\lambda_1 \geq 1$, $\lambda_2 \geq 1$. It may be seen from Table 2 that equal tail areas tests based on 1) to 4) generally seem to perform better than corresponding unbiased ones except when very close to H_0 in which case bias is observed in some instances, mostly when n_1 is close to n_2.
2. $\lambda_1 < 1$, $\lambda_2 > 1$ or $\lambda_1 > 1$, $\lambda_2 < 1$. For tests 1) to 4), unbiased test is better than equal tails except when $\lambda_1 + \lambda_2 > 2$. When $\lambda_1+\lambda_2 \leq 2$, bias is observed.
3. $\lambda_1 < 1$, $\lambda_2 < 1$. For tests 1) to 4), unbiased test seems to be better than equal tails. There exists some bias when close to H_0.
4. $\lambda_1 \geq 1$, $\lambda_2 \geq 1$. In regard to comparative performance of the criteria, findings in the equal tail areas case are as in the one-sided case for 1) to 4) described by Pillai and Hsu [10], whose findings are similar to those in the real case by Pillai and Jayachandran [11], in the unbiased case when λ_1 and λ_2 are far apart but both greater than unity, in terms of power, $U^{(2)} > W^{(2)} > V^{(2)} > L_2^{(2)} > LS^{(2)}$, but with only one large positive deviation, $L_2^{(2)} > U^{(2)}$ $W^{(2)} > V^{(2)} > LS^{(2)}$. But if λ_1 and λ_2 are close, then $V^{(2)} > W^{(2)} > U^{(2)} > L_2^{(2)} > LS^{(2)}$.
5. $\lambda_1 < 1$, $\lambda_2 > 1$ or $\lambda_1 > 1$, $\lambda_2 < 1$. In the unbiased case, $U^{(2)} > W^{(2)} > L_2^{(2)} > V^{(2)} > LS^{(2)}$ when $\lambda_1 + \lambda_2 < 2$, $LS^{(2)} > L_2^{(2)} > U^{(2)}$ $W^{(2)} > V^{(2)}$ when $\lambda_1 + \lambda_2 = 2$, $L_2^{(2)} > U^{(2)} > W^{(2)} > LS^{(2)} > V^{(2)}$ when $\lambda_1 + \lambda_2 > 2$. In the equal tail areas case, $L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$.

6. $\lambda_1 < 1$, $\lambda_2 < 1$. In the unbiased case, $V^{(2)} > W^{(2)} > U^{(2)} > L_2^{(2)} > LS^{(2)}$, and in the equal tail areas case, $L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$.

7. $L_2^{(2)}$ seems to be least biased, then $U^{(2)}$, then $W^{(2)}$ and lastly $V^{(2)}$.

8. If a single test has to be recommended on an overall basis over the whole parameter space, Roy's largest root seems to be the proper candidate. In the two-sided case as well as when both λ_1 and λ_2 are less than unity, among tests 1) to 4), largest root performs best in the equal tail areas case. Since the largest root is the least biased, even equal tail areas could be adequate. However, for the two-sided case, the unbiased largest root test compares favorably with $LS^{(2)}$ when $\lambda_1+\lambda_2=2$ and is even the best when $\lambda_1+\lambda_2 > 2$.

TABLE 2

Powers of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and $LS^{(2)}$ in the unbiased and equal tail areas cases for testing $\lambda_1 = 1$, $\lambda_2 = 1$ against different simple two-sided alternative hypotheses in the complex case, $\alpha = .05$

$n_1 = 2$, $n_2 = 7$

		With local unbiased property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050007	.050009	.050014	.050011
1	1.1	.050671	.050690	.050619	.050686	.050343	.051346	.051523	.051931	.051779
1.05	1.05	.050537	.050576	.050629	.050610	.050214	.051211	.051410	.051955	.051709
1	1.5	.0656	.0659	.0640	.0657	.0595	.0688	.0698	.0700	.0708
1.25	1.25	.0621	.0630	.0641	.0638	.0557	.0653	.0670	.0705	.0691
1	2	.1177	.1183	.1123	.1175	.1021	.1231	.1250	.1225	.1262
1.333	1.5	.0853	.0875	.0898	.0893	.0714	.0903	.0937	.0997	.0976
1	3	.288	.289	.279	.287	.264	.295	.297	.291	.298
2.5	1.5	.276	.280	.282	.283	.247	.283	.289	.297	.295
2	2	.272	.278	.284	.283	.242	.280	.287	.299	.295
1.00001	0.9	.050747	.050773	.050717	.050773	.050373	.050047	.049908	.049333	.049633
1.00001	0.8	.052792	.052916	.052687	.052919	.051121	.051365	.051150	.049844	.050591
1.01	0.99	.050006	.050005	.050000	.050004	.050006	.050006	.050005	.050000	.050003
1.1	0.9	.050586	.050518	.050044	.050369	.050561	.050587	.050507	.049978	.050340
1.1	0.8	.051863	.051777	.050799	.051491	.051407	.051165	.050888	.049278	.050292
1.05	0.95	.050146	.050129	.050010	.050092	.050140	.050147	.050127	.049994	.050085
1.2	0.99	.0524	.0524	.0521	.0524	.0513	.0537	.0540	.0545	.0544
1.2	0.8	.0522	.0519	.0500	.0513	.0521	.0522	.0519	.0498	.0512
2	0.9	.110	.109	.100	.107	.097	.115	.116	.109	.115
2	0.7	.098	.096	.079	.091	.090	.102	.101	.086	.097
3	0.9	.27	.27	.26	.27	.25	.28	.28	.27	.28
0.99999	0.9	.050748	.050773	.050717	.050773	.050373	.050047	.049908	.049333	.049633
0.9	0.999	.050756	.050783	.050731	.050784	.050374	.050049	.049909	.049333	.049633
0.9	0.9	.052444	.052610	.052839	.052749	.050870	.051012	.050849	.050057	.050443
0.9	0.8	.0547	.0551	.0555	.0554	.0507	.0525	.0524	.0513	.0519
0.9	0.76	.0525	.0531	.0534	.0535	.0506	.0504	.0502	.0488	.0497
0.85	0.9	.0538	.0540	.0544	.0543	.0512	.0520	.0518	.0509	.0513
0.85	0.8	.0551	.0557	.0565	.0562	.0522	.0525	.0526	.0516	.0522
0.81	0.9	.0547	.0550	.0554	.0553	.0510	.0526	.0524	.0513	.0519
0.8	0.8	.0517	.0527	.0539	.0536	.0529	.0499	.0493	.0486	.0491

TABLE 8 (continued)

$n_1 = 2,\ n_2 = 17$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050021	.050024	.050025	.050024
1	1.1	.050973	.050983	.050931	.050969	.050479	.053057	.053329	.053386	.053388
1.05	1.05	.050766	.050822	.050836	.050835	.050310	.052839	.053170	.053300	.053259
1	1.5	.0724	.0724	.0710	.0720	.0630	.0826	.0838	.0829	.0836
1.25	1.25	.0674	.0688	.0691	.0690	.0583	.0775	.0803	.0812	.0810
1	2	.1361	.1358	.1319	.1346	.1127	.1538	.1554	.1523	.1547
1.333	1.5	.0983	.1014	.1020	.1020	.0782	.1144	.1199	.1212	.1210
1	3	.318	.317	.310	.315	.282	.339	.341	.335	.339
2.5	1.5	.306	.312	.312	.313	.265	.330	.340	.341	.341
2	2	.304	.312	.314	.314	.260	.328	.340	.343	.343
1.00001	0.9	.051052	.051071	.051027	.051055	.050493	.048959	.048701	.048532	.048606
1.00001	0.8	.054106	.054207	.054032	.054150	.051639	.049917	.049448	.049005	.049225
1.01	0.99	.050009	.050007	.050005	.050006	.050007	.050009	.050007	.050004	.050006
1.1	0.9	.050876	.050707	.050459	.050606	.050709	.050915	.050698	.050416	.050581
1.1	0.8	.052779	.052501	.051983	.052293	.051828	.050770	.050116	.049399	.049796
1.05	0.95	.050219	.050176	.050114	.050151	.050177	.050229	.050174	.050104	.050145
1.2	0.99	.0535	.0535	.0533	.0534	.0518	.0575	.0580	.0579	.0580
1.2	0.8	.0534	.0527	.0517	.0523	.0527	.0535	.0527	.0516	.0522
2	0.9	.127	.125	.120	.123	.107	.143	.143	.138	.141
2	0.7	.114	.109	.100	.105	.100	.128	.122	.114	.119
3	0.9	.30	.30	.29	.29	.27	.32	.32	.31	.32
0.99999	0.9	.051052	.051071	.051028	.051055	.050493	.048959	.048701	.048532	.048605
0.9	0.999	.051064	.051085	.051043	.051070	.050495	.048950	.048692	.048523	.048596
0.9	0.9	.053392	.053619	.053677	.053655	.051204	.049145	.048851	.048677	.048737
0.9	0.8	.0572	.0577	.0578	.0578	.0516	.0508	.0506	.0503	.0504
0.9	0.76	.0566	.0574	.0573	.0574	.0514	.0497	.0496	.0492	.0494
0.85	0.9	.0553	.0557	.0557	.0557	.0517	.0500	.0497	.0495	.0496
0.85	0.8	.0587	.0595	.0597	.0597	.0536	.0513	.0513	.0511	.0512
0.81	0.9	.0569	.0574	.0575	.0575	.0518	.0508	.0505	.0502	.0503
0.8	0.8	.0574	.0588	.0591	.0591	.0544	.0499	.0499	.0497	.0498

TABLE 8 (continued)

$n_1 = 2, \quad n_2 = 32$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050027	.050030	.050030	.050030
1	1.1	.051109	.051107	.051073	.051097	.050542	.053803	.054071	.054070	.054083
1.05	1.05	.050867	.050927	.050932	.050936	.050354	.053544	.053891	.053933	.053922
1	1.5	.0755	.0752	.0743	.0749	.0648	.0888	.0897	.0889	.0894
1.25	1.25	.0698	.0713	.0714	.0714	.0596	.0830	.0860	.0863	.0862
1	2	.1447	.1436	.1412	.1426	.1181	.1677	.1685	.1663	.1677
1.333	1.5	.1042	.1076	.1077	.1078	.0817	.1252	.1310	.1314	.1314
1	3	.331	.329	.325	.328	.291	.359	.359	.356	.358
2.5	1.5	.320	.327	.326	.327	.274	.351	.361	.361	.361
2	2	.318	.327	.328	.328	.270	.350	.363	.364	.363
1.00001	0.9	.051183	.051192	.051164	.051177	.050545	.048511	.048229	.048158	.048188
1.00001	0.8	.054664	.054725	.054613	.054672	.051860	.049350	.048810	.048600	.048700
1.01	0.99	.050010	.050008	.050006	.050007	.050008	.050011	.050008	.050006	.050007
1.1	0.9	.051006	.050780	.050638	.050717	.050773	.051074	.050780	.050616	.050706
1.1	0.8	.053173	.052783	.052484	.052647	.052003	.050645	.049823	.049435	.049640
1.05	0.95	.050251	.050195	.050159	.050179	.050193	.050269	.050195	.050154	.050176
1.2	0.99	.0540	.0540	.0538	.0539	.0521	.0592	.0596	.0595	.0596
1.2	0.8	.0539	.0530	.0525	.0528	.0530	.0542	.0530	.0524	.0527
2	0.9	.135	.132	.129	.130	.112	.156	.154	.151	.153
2	0.7	.122	.114	.109	.112	.105	.139	.132	.126	.129
3	0.9	.31	.31	.30	.31	.28	.34	.34	.33	.34
0.99999	0.9	.051184	.051193	.051165	.051177	.050545	.048510	.048229	.048158	.048188
0.9	0.999	.051197	.051209	.051181	.051193	.050548	.048497	.048215	.048145	.048175
0.9	0.9	.053798	.054032	.054052	.054037	.051349	.048392	.048095	.048041	.048055
0.9	0.8	.0583	.0588	.0588	.0588	.0520	.0502	.0500	.0498	.0499
0.9	0.76	.0584	.0591	.0590	.0591	.0516	.0496	.0495	.0493	.0494
0.85	0.9	.0560	.0563	.0563	.0563	.0520	.0492	.0489	.0488	.0489
0.85	0.8	.0602	.0611	.0611	.0611	.0542	.0509	.0509	.0508	.0508
0.81	0.9	.0579	.0584	.0584	.0584	.0522	.0501	.0498	.0497	.0497
0.8	0.8	.0599	.0613	.0614	.0614	.0556	.0504	.0502	.0502	.0502

TABLE 8 (continued)

$n_1 = 2, \; n_2 = 62$

λ_1	λ_2	With local unbiasedness property					With equal tail areas			
		$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050030	.050033	.050033	.050033
1	1.1	.051194	.051182	.051162	.051177	.050583	.054262	.054511	.054500	.054509
1.05	1.05	.050931	.050991	.050992	.050995	.050382	.053977	.054318	.054330	.054328
1	1.5	.0775	.0769	.0764	.0767	.0660	.0927	.0932	.0928	.0930
1.25	1.25	.0714	.0728	.0729	.0729	.0605	.0865	.0894	.0895	.0895
1	2	.1501	.1483	.1470	.1477	.1217	.1764	.1763	.1750	.1757
1.333	1.5	.1080	.1114	.1114	.1114	.0839	.1319	.1377	.1378	.1378
1	3	.340	.337	.335	.336	.298	.372	.371	.369	.370
2.5	1.5	.329	.335	.335	.335	.280	.364	.374	.374	.374
2	2	.327	.336	.337	.337	.276	.364	.376	.376	.376
1.00001	0.9	.051265	.051265	.051248	.051254	.050577	.048242	.047956	.047926	.047939
1.00001	0.8	.055003	.055030	.054966	.054994	.051995	.049018	.048447	.048348	.048395
1.01	0.99	.050011	.050008	.050007	.050008	.050008	.050012	.050008	.050007	.050008
1.1	0.9	.051087	.050824	.050746	.050787	.050813	.051176	.050830	.050741	.050788
1.1	0.8	.053415	.052946	.052785	.052867	.052110	.050581	.049657	.049454	.049558
1.05	0.95	.050272	.050206	.050186	.050197	.050203	.050294	.050207	.050185	.050197
1.2	0.99	.0544	.0542	.0542	.0542	.0522	.0602	.0606	.0605	.0606
1.2	0.8	.0542	.0532	.0529	.0531	.0532	.0546	.0532	.0529	.0531
2	0.9	.140	.136	.134	.135	.115	.164	.161	.160	.161
2	0.7	.127	.117	.115	.116	.108	.147	.137	.134	.136
3	0.9	.32	.32	.31	.31	.28	.35	.35	.35	.35
0.99999	0.9	.051265	.051265	.051249	.051254	.050577	.048241	.047956	.047926	.047939
0.9	0.999	.051279	.051282	.051256	.051271	.050580	.048225	.047940	.047910	.047923
0.9	0.9	.054046	.054279	.054284	.054275	.051438	.047944	.047665	.047650	.047652
0.9	0.8	.0589	.0594	.0594	.0594	.0523	.0498	.0496	.0496	.0496
0.9	0.76	.0594	.0601	.0601	.0601	.0518	.0496	.0494	.0493	.0494
0.85	0.9	.0564	.0567	.0567	.0567	.0521	.0487	.0485	.0484	.0484
0.85	0.8	.0611	.0620	.0620	.0620	.0546	.0507	.0507	.0506	.0506
0.81	0.9	.0584	.0589	.0589	.0589	.0524	.0497	.0494	.0494	.0494
0.8	0.8	.0613	.0628	.0628	.0628	.0564	.0505	.0504	.0504	.0504

TABLE 8 (continued)

$n_1 = 3, \quad n_2 = 7$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050000	.050002	.050010	.050007
1	1.1	.050846	.050904	.050824	.050900	.050603	.050831	.051086	.051823	.051554
1.05	1.05	.050694	.050784	.050896	.050853	.050385	.050678	.050966	.051913	.051512
1	1.5	.0736	.0744	.0720	.0742	.0691	.0736	.0752	.0764	.0771
1.25	1.25	.0676	.0694	.0720	.0711	.0619	.0675	.0703	.0767	.0742
1	2	.1702	.1717	.1655	.1709	.1603	.1701	.1730	.1719	.1754
1.333	1.5	.1126	.1166	.1215	.1200	.1015	.1125	.1178	.1286	.1246
1	3	.447	.448	.441	.447	.436	.447	.450	.447	.451
2.5	1.5	.453	.458	.463	.462	.439	.453	.459	.470	.467
2	2	.458	.464	.472	.469	.443	.458	.465	.479	.474
1.00001	0.9	.050935	.051000	.050963	.051018	.050675	.050951	.050808	.049891	.050328
1.00001	0.8	.051531	.051874	.051631	.051934	.050319	.051563	.051488	.049447	.050542
1.01	0.99	.050007	.050005	.050000	.050003	.050010	.050006	.050005	.049998	.050002
1.1	0.9	.050648	.050542	.050029	.050270	.050982	.050648	.050539	.049755	.050247
1.1	0.8	.051486	.051398	.050463	.050890	.052039	.051501	.051204	.048750	.05016
1.05	0.95	.050162	.050135	.050008	.050067	.050244	.050162	.050135	.049938	.050061
1.2	0.99	.0530	.0532	.0527	.0531	.0523	.0530	.0535	.0546	.0543
1.2	0.8	.0522	.0518	.0500	.0507	.0536	.0522	.0518	.0486	.0506
2	0.9	.154	.154	.142	.151	.146	.154	.155	.148	.155
2	0.7	.128	.125	.102	.116	.127	.128	.125	.105	.119
3	0.9	.42	.42	.41	.42	.41	.42	.42	.41	.42
0.99999	0.9	.050935	.051000	.050964	.051019	.050675	.050951	.050808	.049891	.050328
0.9	0.999	.050948	.051015	.050984	.051036	.050678	.050963	.050821	.049901	.050339
0.9	0.9	.053038	.053405	.053917	.053724	.051451	.053070	.053012	.051766	.052325
0.9	0.8									
0.9	0.76									
0.85	0.9	.0536	.0542	.0550	.0547	.0510	.0538	.0537	.0523	.0530
0.85	0.8									
0.81	0.9	.0505	.0505	.0515	.0512	.0505	.0499	.0499	.0485	.0493

TABLE 8 (continued)

$n_1 = 3, \quad n_2 = 17$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050016	.050020	.050022	.050022
1	1.1	.051319	.051358	.051264	.051332	.050875	.052872	.053335	.053457	.053457
1.05	1.05	.051054	.051185	.051221	.051214	.050571	.052594	.053164	.053429	.053347
1	1.5	.0836	.0839	.0814	.0831	.0755	.0911	.0933	.0916	.0931
1.25	1.25	.0755	.0785	.0793	.0792	.0666	.0829	.0882	.0901	.0895
1	2	.1934	.1935	.1872	.1915	.1758	.2050	.2080	.2029	.2069
1.333	1.5	.1300	.1364	.1377	.1376	.1124	.1411	.1508	.1538	.1532
1	3	.474	.473	.466	.471	.454	.484	.487	.480	.485
2.5	1.5	.478	.486	.487	.487	.456	.490	.501	.503	.503
2	2	.483	.493	.496	.495	.460	.495	.508	.512	.512
1.00001	0.9	.051410	.051478	.051407	.051455	.050925	.049846	.049467	.049150	.049284
1.00001	0.8	.053888	.054221	.053848	.054110	.051623	.050829	.050254	.049371	.049818
1.01	0.99	.050011	.050008	.050003	.050006	.050013	.050011	.050008	.050002	.050005
1.1	0.9	.051077	.050760	.050285	.050558	.051303	.051130	.050751	.050216	.050523
1.1	0.8	.052946	.052472	.051469	.052057	.053034	.051518	.050474	.049105	.049849
1.05	0.95	.050270	.050190	.050071	.050139	.050324	.050283	.050188	.050054	.050131
1.2	0.99	.0548	.0549	.0544	.0547	.0533	.0578	.0586	.0585	.0587
1.2	0.8	.0540	.0527	.0508	.0519	.0550	.0542	.0527	.0505	.0518
2	0.9	.176	.173	.164	.170	.162	.187	.186	.178	.184
2	0.7	.150	.140	.123	.133	.144	.160	.151	.134	.144
3	0.9	.45	.44	.43	.44	.43	.46	.46	.45	.45
0.99999	0.9	.051411	.051478	.051407	.051456	.050925	.049846	.049467	.049150	.049284
0.9	0.999	.051428	.051500	.051432	.051479	.050930	.049848	.049468	.049153	.049286
0.9	0.9	.054587	.055133	.055279	.055230	.052141	.051384	.051072	.050759	.050864
0.9	0.8	.0518	.0535	.0536	.0537	.0518	.0479	.0478	.0473	.0476
0.9	0.76									
0.85	0.9	.0563	.0572	.0574	.0573	.0522	.0523	.0521	.0518	.0519
0.85	0.8									
0.81	0.9	.0540	.0554	.0556	.0556	.0518	.0498	.0498	.0494	.0496

TABLE 8 (continued)

$n_1 = 3, \quad n_2 = 32$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050023	.050028	.050029	.050028
1	1.1	.051551	.051565	.051498	.051546	.051008	.053856	.054327	.054327	.054353
1.05	1.05	.051228	.051371	.051383	.051388	.050662	.053508	.054133	.054222	.054200
1	1.5	.0887	.0883	.0866	.0877	.0789	.1000	.1016	.1001	.1011
1.25	1.25	.0795	.0829	.0831	.0831	.0691	.0906	.0964	.0971	.0969
1	2	.2053	.2039	.1995	.2022	.1842	.2229	.2243	.2204	.2229
1.333	1.5	.1388	.1458	.1462	.1462	.1181	.1554	.1662	.1672	.1670
1	3	.487	.485	.480	.483	.464	.504	.504	.500	.503
2.5	1.5	.491	.500	.500	.500	.465	.508	.521	.521	.521
2	2	.496	.507	.508	.508	.469	.514	.529	.530	.530
1.00001	0.9	.051633	.051683	.051630	.051655	.051038	.049350	.048910	.048770	.048827
1.00001	0.8	.054988	.055234	.054969	.055114	.052227	.050557	.049802	.049347	.049567
1.01	0.99	.050013	.050008	.050006	.050007	.050014	.050014	.050008	.050005	.050007
1.1	0.9	.051290	.050846	.050554	.050713	.051449	.051389	.050843	.050509	.050691
1.1	0.8	.053638	.052912	.052292	.052628	.053475	.051596	.050176	.049385	.049798
1.05	0.95	.050323	.050211	.050138	.050178	.050361	.050348	.050211	.050127	.050172
1.2	0.99	.0557	.0556	.0553	.0555	.0539	.0601	.0609	.0606	.0608
1.2	0.8	.0548	.0531	.0519	.0525	.0556	.0553	.0531	.0517	.0525
2	0.9	.188	.182	.176	.180	.170	.204	.201	.195	.199
2	0.7	.162	.147	.136	.142	.152	.177	.162	.151	.157
3	0.9	.46	.45	.45	.45	.44	.48	.47	.47	.47
0.99999	0.9	.051634	.051684	.051631	.051655	.051038	.049350	.048910	.048770	.048827
0.9	0.999	.051654	.051709	.051658	.051681	.051044	.049346	.048908	.048769	.048825
0.9	0.9	.055304	.055883	.055933	.055901	.052458	.050644	.050314	.050213	.050234
0.9	0.8	.0544	.0560	.0560	.0560	.0519	.0489	.0483	.0481	.0482
0.9	0.76									
0.85	0.9	.0575	.0585	.0585	.0585	.0527	.0517	.0515	.0514	.0514
0.85	0.8									
0.81	0.9	.0562	.0576	.0576	.0576	.0519	.0501	.0500	.0498	.0498

TABLE 8 (continued)

$n_1 = 3, \quad n_2 = 62$

λ_1	λ_2	With local unbiasedness property					With equal tail areas			
		$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050028	.050032	.050033	.050033
1	1.1	.051703	.051695	.051653	.051685	.051096	.054492	.054931	.054907	.054929
1.05	1.05	.051342	.051487	.051491	.051499	.050723	.054097	.054722	.054749	.054744
1	1.5	.0921	.0912	.0901	.0907	.0812	.1058	.1067	.1057	.1063
1.25	1.25	.0822	.0856	.0857	.0857	.0707	.0956	.1016	.1017	.1017
1	2	.2132	.2104	.2079	.2093	.1899	.2346	.2345	.2320	.2335
1.333	1.5	.1446	.1517	.1518	.1519	.1220	.1649	.1758	.1760	.1760
1	3	.497	.493	.490	.492	.471	.516	.515	.512	.514
2.5	1.5	.500	.509	.508	.509	.471	.521	.533	.533	.533
2	2	.505	.516	.517	.517	.475	.526	.541	.542	.542
1.00001	0.9	.051776	.051808	.051776	.051784	.051110	.049040	.048583	.048522	.048546
1.00001	0.8	.055687	.055850	.055690	.055758	.052612	.050401	.049554	.049327	.049432
1.01	0.99	.050014	.050009	.050007	.050008	.050015	.050016	.050009	.050007	.050008
1.1	0.9	.051429	.050895	.050732	.050819	.051542	.051564	.050898	.050710	.050809
1.1	0.8	.054080	.053173	.052825	.053002	.053751	.051664	.050002	.049575	.049794
1.05	0.95	.050358	.050224	.050183	.050205	.050384	.050392	.050224	.050177	.050202
1.2	0.99	.0563	.0561	.0559	.0560	.0542	.0617	.0623	.0621	.0622
1.2	0.8	.0554	.0533	.0526	.0530	.0560	.0560	.0533	.0526	.0530
2	0.9	.196	.188	.184	.186	.175	.216	.210	.207	.209
2	0.7	.170	.152	.145	.149	.158	.188	.169	.163	.167
3	0.9	.47	.46	.46	.46	.45	.49	.48	.48	.48
0.99999	0.9	.051777	.051809	.051776	.051784	.051110	.049040	.048583	.048521	.048546
0.9	0.999	.C51798	.051836	.051804	.051812	.051116	.049034	.048578	.048517	.048541
0.9	0.9	.055760	.056341	.056356	.056328	.052658	.050190	.049884	.049855	.049852
0.9	0.8	.0560	.0576	.0575	.0576	.0522	.0488	.0487	.0486	.0486
0.9	0.76									
0.85	0.9	.0583	.0592	.0592	.0592	.0531	.0514	.0512	.0512	.0512
0.85	0.8									
0.81	0.9	.0575	.0590	.0589	.0589	.0527	.0502	.0501	.0500	.0500

TABLE 8 (continued)

$n_1 = 4, \quad n_2 = 7$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050001	.050000	.050000	.050000	.049995	.049997	.050007	.050003
1	1.1	.050977	.051105	.051010	.051079	.050813	.050451	.050754	.051719	.051376
1.05	1.05	.050812	.050980	.051121	.051056	.050524	.050289	.050627	.051844	.051356
1	1.5	.0856	.0871	.0847	.0867	.0824	.0833	.0856	.0876	.0880
1.25	1.25	.0763	.0791	.0826	.0813	.0714	.0739	.0775	.0859	.0827
1	2	.2443	.2468	.2415	.2458	.2382	.2412	.2448	.2453	.2475
1.333	1.5	.1595	.1646	.1708	.1687	.1511	.1564	.1624	.1753	.1706
1	3	.605	.607	.603	.606	.601	.603	.606	.605	.607
2.5	1.5	.633	.637	.642	.641	.627	.631	.636	.644	.642
2	2	.647	.651	.657	.655	.640	.645	.650	.659	.656
1.00001	0.9	.051057	.051142	.051172	.051210	.050913	.051609	.051515	.050404	.050894
1.00001	0.8									
1.01	0.99	.050007	.050006	.050000	.050002	.050013	.050007	.050006	.049996	.050002
1.1	0.9	.050676	.050559	.050018	.050184	.051316	.060665	.050564	.049648	.050172
1.1	0.8						.050128	.049955	.046989	.0485[illegible]8
1.05	0.95	.050169	.050140	.050006	.050046	.050327	.050167	.050141	.049911	.050043
1.2	0.99	.0536	.0540	.0534	.0538	.0531	.0526	.0533	.0547	.0544
1.2	0.8									
2	0.9	.214	.215	.205	.212	.211	.211	.213	.208	.213
2	0.7	.167	.164	.139	.153	.171	.164	.163	.141	.155
3	0.9	.57	.57	.56	.57	.56	.57	.57	.56	.57
0.99999	0.9	.051057	.051142	.051173	.051210	.050913	.051610	.051516	.050404	.050895
0.9	0.999	.051071	.051159	.051198	.051232	.050916	.051630	.051537	.050422	.050913
0.9	0.9	.052702	.053245	.054101	.053812	.051149	.053845	.054009	.052571	.053174

TABLE 8 (continued)

$n_1 = 4, \quad n_2 = 17$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050011	.050017	.050020	.050019
1	1.1	.051612	.051691	.051567	.051656	.051222	.052722	.053332	.053518	.053513
1.05	1.05	.051300	.051512	.051573	.051558	.050801	.052398	.053153	.053543	.053424
1	1.5	.0980	.0987	.0954	.0977	.0910	.1032	.1062	.1041	.1061
1.25	1.25	.0863	.0908	.0921	.0918	.0778	.0914	.0986	.1015	.1007
1	2	.2680	.2688	.2619	.2665	.2553	.2750	.2788	.2735	.2778
1.333	1.5	.1786	.1869	.1890	.1887	.1638	.1856	.1977	.2018	.2009
1	3	.625	.625	.619	.623	.615	.629	.632	.627	.631
2.5	1.5	.650	.656	.657	.658	.639	.655	.663	.666	.665
2	2	.663	.670	.672	.672	.651	.667	.677	.680	.679
1.00001	0.9	.051696	.051828	.051744	.051805	.051301	.050576	.050149	.049717	.049894
1.00001	0.8									
1.01	0.99	.050012	.050008	.050001	.050005	.050018	.050013	.050008	.050000	.050005
1.1	0.9	.051225	.050787	.050126	.050493	.051825	.051276	.050779	.050044	.050454
1.1	0.8	.051750	.051118	.050114	.050486	.052882	.050798	.049504	.047530	.048585
1.05	0.95	.050307	.050197	.050031	.050123	.050453	.050320	.050195	.050011	.050114
1.2	0.99	.0560	.0561	.0555	.0559	.0547	.0581	.0592	.0591	.0594
1.2	0.8	.0540	.0522	.0509	.0510	.0566	.0542	.0522	.0492	.0509
2	0.9	.239	.235	.224	.231	.228	.246	.245	.235	.242
2	0.7	.193	.180	.158	.170	.193	.200	.188	.166	.179
3	0.9	.59	.59	.58	.58	.58	.59	.59	.59	.59
0.99999	0.9	.051696	.051828	.051745	.051805	.051301	.050576	.050149	.049717	.049894
0.9	0.999	.051718	.051856	.051777	.051836	.051307	.050586	.050161	.049730	.049906
0.9	0.9	.054924	.055860	.056117	.056039	.052264	.052627	.052480	.052081	.052213
0.9	0.8									
0.9	0.76									
0.85	0.9	.0514	.0532	.0536	.0535	.0516	.0496	.0492	.0488	.0489

TABLE 8 (continued)

$n_1 = 4, \quad n_2 = 32$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050020	.050026	.050027	.050027
1	1.1	.051948	.051993	.051894	.051962	.051435	.053914	.054554	.054557	.054595
1.05	1.05	.051553	.051790	.051812	.051814	.050946	.053491	.054351	.054489	.054454
1	1.5	.1049	.1047	.1022	.1037	.0959	.1142	.1165	.1144	.1158
1.25	1.25	.0917	.0969	.0974	.0973	.0814	.1009	.1092	.1102	.1100
1	2	.2813	.2802	.2750	.2781	.2651	.2940	.2962	.2914	.2945
1.333	1.5	.1890	.1985	.1992	.1992	.1710	.2016	.2155	.2169	.2167
1	3	.635	.634	.630	.633	.623	.644	.645	.641	.644
2.5	1.5	.659	.666	.666	.667	.645	.668	.677	.678	.678
2	2	.671	.680	.681	.681	.657	.679	.691	.692	.692
1.00001	0.9	.052018	.052129	.052058	.052092	.051486	.050073	.049548	.049349	.049428
1.00001	0.8	.050073	.050528	.050131	.050276	.050165	.046600	.045821	.044958	.045402
1.01	0.99	.050015	.050009	.050005	.050007	.050021	.050016	.050009	.050004	.050006
1.1	0.9	.051521	.050881	.050450	.050681	.052069	.051635	.050877	.050389	.050651
1.1	0.8	.052853	.051799	.050815	.051347	.053732	.051248	.049336	.048113	.048751
1.05	0.95	.050381	.050220	.050112	.050170	.050514	.050410	.050219	.050097	.050163
1.2	0.99	.0573	.0572	.0567	.0570	.0556	.0611	.0621	.0617	.0620
1.2	0.8	.0553	.0527	.0509	.0518	.0577	.0558	.0527	.0507	.0518
2	0.9	.252	.246	.238	.242	.239	.265	.261	.253	.258
2	0.7	.208	.188	.173	.181	.205	.220	.201	.185	.194
3	0.9	.60	.60	.59	.59	.59	.61	.61	.60	.61
0.99999	0.9	.052018	.052129	.052059	.052093	.051486	.050073	.049548	.049349	.049428
0.9	0.999	.052044	.052163	.052094	.052127	.051494	.050079	.049556	.049358	.049436
0.9	0.9	.056029	.057046	.057138	.057089	.052806	.052057	.051883	.051750	.051774
0.9	0.8									
0.9	0.76									
0.85	0.9	.0536	.0556	.0557	.0556	.0518	.0499	.0494	.0493	.0493

Table 8 (continued)

$n_1 = 4,\ n_2 = 62$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050025	.050032	.050032	.050032
1	1.1	.052178	.052193	.052123	.052172	.051589	.054719	.055319	.055282	.055317
1.05	1.05	.051725	.051975	.051976	.051987	.051054	.054225	.055100	.055142	.055135
1	1.5	.1097	.1086	.1070	.1079	.0994	.1219	.1232	.1216	.1225
1.25	1.25	.0955	.1009	.1010	.1010	.0839	.1074	.1160	.1163	.1162
1	2	.2906	.2877	.2844	.2863	.2721	.3072	.3073	.3042	.3060
1.333	1.5	.1962	.2061	.2062	.2063	.1761	.2127	.2270	.2273	.2273
1	3	.643	.640	.638	.639	.629	.654	.654	.651	.653
2.5	1.5	.666	.673	.673	.673	.650	.676	.686	.686	.686
2	2	.677	.686	.687	.687	.662	.688	.700	.700	.700
1.00001	0.9	.052231	.052311	.052271	.052280	.051597	.049749	.049189	.049099	.049131
1.00001	0.8	.051492	.051743	.051380	.051554	.051098	.047082	.046072	.045603	.045828
1.01	0.99	.050017	.050009	.050007	.050008	.050022	.050019	.050009	.050006	.050008
1.1	0.9	.051725	.050934	.050685	.050817	.052230	.051889	.050935	.050650	.050801
1.1	0.8	.053590	.052212	.051645	.051938	.054280	.051576	.049242	.048557	.048909
1.05	0.95	.050432	.050234	.050171	.050204	.050554	.050474	.050234	.050163	.050200
1.2	0.99	.0582	.0579	.0576	.0578	.0562	.0631	.0639	.0636	.0638
1.2	0.8	.0562	.0529	.0519	.0524	.0584	.0568	.0530	.0518	.0524
2	0.9	.262	.252	.248	.250	.246	.278	.271	.266	.269
2	0.7	.219	.193	.184	.189	.213	.234	.209	.200	.204
3	0.9	.61	.60	.60	.60	.60	.62	.62	.61	.61
0.99999	0.9	.052232	.052312	.052271	.052281	.051598	.049749	.049189	.049099	.049131
0.9	0.999	.052260	.052348	.052309	.052318	.051606	.049751	.049194	.049106	.049137
0.9	0.9	.056758	.057778	.057819	.057769	.053146	.051699	.051555	.051516	.051501
0.9	0.8									
0.9	0.76									
0.85	0.9	.0551	.0570	.0571	.0570	.0526	.0498	.0497	.0496	.0496

TABLE 8 (continued)

$n_1 = 7,\ n_2 = 7$

λ_1	λ_2	With local unbiasedness property $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.049985	.049988	.050000	.049995
1	1.1	.051263	.052055	.051485	.051512	.051293	.049766	.050421	.051485	.051034
1.05	1.05	.051062	.052013	.051633	.051519	.050847	.049579	.050161	.051633	.051036
1	1.5	.1636	.1703	.1649	.1659	.1631	.1582	.1641	.1649	.1642
1.25	1.25	.1467	.1533	.1553	.1535	.1440	.1412	.1465	.1553	.1517
1	2	.5394	.5453	.5401	.5413	.5387	.5353	.5407	.5401	.5401
1.333	1.5	.4163	.4231	.4255	.4237	.4135	.4115	.4170	.4255	.4221
1	3	.905	.906	.905	.905	.905	.904	.905	.905	.905
2.5	1.5	.937	.938	.939	.938	.937	.937	.938	.939	.938
2	2	.947	.948	.948	.948	.946	.946	.947	.948	.948
1.00001	0.9	.050797	.050448	.051204	.051172	.050970	.052377	.052468	.051204	.051688
1.00001	0.8									
1.01	0.99	.050007	.050009	.050001	.050000	.050020	.050006	.050010	.049996	.050000
1.1	0.9	.050679	.050882	.050008	.050012	.052020	.050623	.050955	.049567	.050037
1.1	0.8									
1.05	0.95	.050174	.050221	.050003	.050006	.050502	.050160	.050239	.049893	.050012
1.2	0.99	.0561	.0584	.0564	.0566	.0562	.0533	.0551	.0564	.0558
1.2	0.8									
2	0.9	.467	.473	.463	.466	.467	.462	.469	.463	.464
2	0.7	.334	.344	.312	.321	.344	.328	.340	.312	.320
3	0.9	.87	.88	.87	.87	.87	.87	.88	.87	.87
0.99999	0.9	.050797	.050448	.051204	.051172	.050970	.052378	.052468	.051204	.051689
0.9	0.999	.050795	.050444	.051220	.051183	.050955	.052392	.052485	.051220	.051705

TABLE 8 (continued)

$n_1 = 7, \quad n_2 = 17$

λ_1	λ_2	With local unbiasedness property $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	With equal tail areas $L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050000	.050000	.050000	.050000	.050000	.050001	.050008	.050014	.050012
1	1.1	.052319	.052308	.052377	.052497	.052095	.052435	.053346	.053709	.053682
1.05	1.05	.051888	.052102	.052479	.052441	.051382	.052002	.053141	.053833	.053635
1	1.5	.1783	.1792	.1766	.1790	.1746	.1788	.1831	.1814	.1834
1.25	1.25	.1590	.1649	.1681	.1676	.1528	.1594	.1690	.1735	.1723
1	2	.5536	.5544	.5514	.5538	.5499	.5540	.5573	.5550	.5572
1.333	1.5	.4300	.4367	.4399	.4394	.4235	.4304	.4404	.4448	.4437
1	3	.909	.909	.908	.909	.908	.909	.910	.909	.909
2.5	1.5	.940	.940	.941	.941	.939	.940	.941	.941	.941
2	2	.949	.949	.950	.950	.948	.949	.950	.950	.950
1.00001	0.9	.051898	.052495	.052188	.052255	.051824	.051781	.051423	.050780	.051021
1.00001	0.8									
1.01	0.99	.050015	.050008	.050000	.050003	.050031	.050015	.050008	.049997	.050003
1.1	0.9	.051497	.050808	.050084	.050301	.053111	.051506	.050804	.049699	.050264
1.1	0.8									
1.05	0.95	.050378	.050206	.050018	.050078	.050772	.050380	.050204	.049927	.050069
1.2	0.99	.0599	.599	.0594	.0600	.0592	.0601	.0619	.0618	.0622
1.2	0.8									
2	0.9	.484	.481	.474	.479	.482	.484	.484	.478	.482
2	0.7	.359	.344	.322	.334	.368	.360	.347	.325	.337
3	0.9	.88	.88	.88	.88	.88	.88	.88	.88	.88
0.99999	0.9	.051898	.052495	.052189	.052256	.051824	.051781	.051423	.050781	.051021
0.9	0.999	.051912	.052524	.052222	.052287	.051815	.051794	.051441	.050801	.051040

TABLE 8 (continued)

$n_1 = 7, \quad n_2 = 32$

λ_1	λ_2	With local unbiasedness property					With equal tail areas			
		$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050012	.050021	.050001	.050001	.050000	.050012	.050021	.050023	.050022
1	1.1	.054133	.055168	.053042	.053132	.052512	.054133	.055168	.055187	.055248
1.05	1.05	.053514	.054936	.053044	.053013	.051638	.053514	.054936	.055209	.055139
1	1.5	.1926	.1964	.1855	.1875	.1818	.1926	.1964	.1935	.1955
1.25	1.25	.1712	.1833	.1764	.1761	.1580	.1712	.1833	.1851	.1847
1	2	.5667	.5692	.5600	.5620	.5571	.5667	.5692	.5661	.5682
1.333	1.5	.4429	.4554	.4491	.4489	.4297	.4429	.4554	.4571	.4568
1	3	.912	.913	.910	.911	.910	.912	.913	.912	.912
2.5	1.5	.941	.943	.942	.942	.940	.941	.943	.943	.943
2	2	.950	.952	.951	.951	.949	.950	.952	.952	.952
1.00001	0.9	.051430	.050873	.052749	.052830	.052352	.051430	.050873	.050541	.050671
1.00001	0.8									
1.01	0.99	.050022	.050009	.050002	.050005	.050037	.050022	.050009	.050001	.050005
1.1	0.9	.052140	.050913	.050145	.050540	.053714	.052140	.050913	.050062	.050500
1.1	0.8									
1.05	0.95	.050539	.050231	.050039	.050138	.050921	.050539	.050231	.050018	.050128
1.2	0.99	.0646	.0663	.0617	.0622	.0610	.0646	.0663	.0657	.0662
1.2	0.8									
2	0.9	.500	.497	.484	.488	.491	.500	.497	.490	.494
2	0.7	.383	.359	.334	.344	.384	.383	.359	.340	.350
3	0.9	.88	.88	.88	.88	.88	.88	.88	.88	.88
0.99999	0.9	.051431	.050874	.052750	.052830	.052352	.051431	.050874	.050541	.050672
0.9	0.999	.051442	.050893	.052792	.052872	.052347	.051442	.050893	.050563	.050692

TABLE 8 (continued)

$n_1 = 7,\ n_2 = 62$

		With local unbiasedness property					With equal tail areas			
λ_1	λ_2	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	$LS^{(2)}$	$L_2^{(2)}$	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$
1	1.001	.050019	.050029	.050029	.050029	.050000	.050019	.050028	.050029	.050029
1	1.1	.055413	.056409	.056335	.056404	.053743	.055413	.056409	.056335	.056404
1.05	1.05	.054644	.056160	.056250	.056235	.052766	.054644	.056160	.056249	.056235
1	1.5	.2034	.2057	.2033	.2047	.1825	.2034	.2057	.2033	.2047
1.25	1.25	.1802	.1934	.1940	.1939	.1571	.1802	.1934	.1940	.1939
1	2	.5766	.5777	.5752	.5766	.5588	.5766	.5777	.5752	.5766
1.333	1.5	.4524	.4660	.4666	.4665	.4298	.4524	.4660	.4665	.4665
1	3	.915	.915	.914	.914	.910	.915	.915	.914	.914
2.5	1.5	.943	.945	.945	.945	.940	.943	.945	.945	.945
2	2	.951	.953	.953	.953	.949	.951	.953	.953	.953
1.00001	0.9	.051184	.050534	.050372	.050425	.053683	.051184	.050534	.050372	.050425
1.00001	0.8									
1.01	0.99	.050027	.050010	.050004	.050007	.050041	.050027	.050010	.050004	.050007
1.1	0.9	.052647	.050977	.050433	.050717	.054102	.052646	.050977	.050433	.050717
1.1	0.8									
1.05	0.95	.050666	.050247	.050111	.050182	.051017	.050666	.050247	.050111	.050182
1.2	0.99	.0680	.0694	.0688	.0691	.0602	.0680	.0694	.0688	.0691
1.2	0.8									
2	0.9	.512	.506	.501	.504	.493	.512	.506	.501	.504
2	0.7	.401	.367	.355	.361	.391	.401	.367	.355	.361
3	0.9	.89	.89	.88	.89	.88	.89	.89	.88	.89
0.99999	0.9	.051184	.050535	.050372	.050426	.053683	.051184	.050534	.050372	.050426
0.9	0.999	.051194	.050555	.050394	.050447	.053691	.051194	.050555	.050394	.050447

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List of Symbols

Greek letters:

Σ	u.c. sigma	(e.g. p.1, line 19)
λ	l.c. lambda	(e.g. p.1, line 21)
α	l.c. alpha	(e.g. p.3, line 1)
β	l.c. beta	(e.g. p.5, line 9)
π	l.c. pi	(e.g. p.6, line 16)
ω	l.c. omega	(e.g. p.7, line 16)
Π	u.c. pi	(e.g. p.8, line 13)
Γ	u.c. gamma	(e.g. p.9, line 15)
κ	l.c. kappa	(e.g. p.13, line 2)
Λ	u.c. lambda	(e.g. p.23, line 17)

Non-italic symbols:

exp	(e.g. p.6, line 16)
tr	(e.g. p.6, line 16)
det	(e.g. p.7, line 6)

Short title:

Two-Sided Tests of Equality of Hermitian Covariance Matrices.