Almost Sure Linearity with a Rate for Linear Rank Statistics with a Bounded Score Function

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Linear rank statistics are shown to be almost surely linear with a rate under the assumption that the score function has a bounded second derivative, but with otherwise very mild conditions on the statistic and the underlying distribution function as compared to previous results of this type. This paper is a continuation of the study of asymptotic linearity of linear rank statistics that began with Jurečková (1969). Several previous results on almost sure linearily of linear rank statistics are obtained with more precision and generality as corollaries of the main theorem of this paper.

Section 0. Introduction.

Let $X_{1n},\ldots,X_{nn}$ be independent with distribution function $F(\cdot+d_{\ell n}\Theta_o)$ $\ell=1,2,\ldots n$ respectively. For each Θ let $R^{\Theta}_{\ell n}$ = the rank of $X_{\ell N}+d_{\ell n}\Theta$ among $X_{1N}+d_{1N}\Theta,\ldots,X_{nn}+d_{nn}\Theta$. Let $T_n(\Theta)$ be a statistic which is a function of the ranks $R^{\Theta}_{\ell n}$ $\ell=1,\ldots,n$.

Jurečková (1969) proved under very general conditions, including F having a finite Fisher information number, that if $T^*_n(\Theta)$ is a linear rank statistic appropriately normalized, $T^*_n(\Theta)$ is linear in probability. That is, for all positive c_1

$$\sup_{\Theta\in I_n(\Theta_o)} \left| T^*_n(\Theta_o) - T^*_n(\Theta) - A_n(\Theta_o-\Theta) \right| = o_p(1/n^{\frac{1}{2}}),$$

where $I_n(\theta_o) = [-c_1/\sqrt{n} + \theta_o, \theta_o + c_1/\sqrt{n}]$,

and A_n is a constant depending on T_n and F. See Section 5 for the exact form of A_n and the conditions on T_n.

We will investigate conditions on F and T^*_n such that a stronger kind of linearity holds, which we will call almost sure linearity with a rate. Meaning that if α_n and ψ_n $n=1,2,\ldots$ are appropriately chosen positive constants, then for all $c_1 > 0$ and $\lambda > 0$, there exists a $c > 0$ such that

$$P\left(\sup_{\Theta\in I_n(\Theta_o)} \left| T^*_n(\Theta_o)-T^*_n(\Theta)-A_n(\Theta_o-\Theta)\right| > c\psi_n\right) < n^{-\lambda},$$

where $I_n(\Theta_o) = [-c_1\alpha_n+\Theta_o, \Theta_o + c_1\alpha_n]$. This will be written:

$$\sup_{\Theta\in I_n(\Theta_o)} \left| T^*_n(\Theta_o)-T^*_n(\Theta)-A_n(\Theta_o-\Theta)\right| = SSO(\psi_n).$$

Previous results of this type for rank statistics are as follows: Wilcoxon one sample statistic, Lemma 4.2 Geerstema (1970); One sample rank statistics Theorem 4.3 Sen and Ghosh (1971); Kendall's Tau, Theorem 3.1 Ghosh and Sen (1971); Spearman's rank test, Theorem 2.3 Ghosh (1972); Wilcoxon two sample statistic, Inagaki (1973); Linear rank statistics under assumptions stated in Section 5, Theorem 3.2 Ghosh and Sen (1972).

In all the above it is assumed that the underlying distribution F has a density f which has a bounded derivative f'. In Theorem 2.1, Section 2, we will show that under very general conditions almost sure linearity with a rate holds for linear rank statistics; not only when f' is bounded, but also when

$$\int_{-\infty}^{\infty} |f'(u)|du < \infty.$$

This includes distributions which have a finite Fisher information number, Jurečková's condition on F for linearity of T_n in probability. In Section 3, Theorem 2.1 will be applied to several common rank statistics and the rate obtained will be compared to previously published results for such statistics when available. Finally in Section 5, Theorem 2.1 will be compared to a similar theorem obtained by Ghosh and Sen (1972).

Section 1. Basic Notation and Assumptions

We will establish here notation and assumptions that will be used throughout. Let $X_{\ell n}$ $\ell=1,\ldots,n$ be independent random variables with distribution function $F(\cdot + d_{\ell n}\Theta_o)$ $\ell=1,\ldots,n$, respectively. Note $X_{\ell n} + d_{\ell n}\Theta_o$ has distribution function F. We will write $F_{\ell n}(x) = F(x + d_{\ell n}\Theta_o)$ $\ell=1,\ldots,n$. By the rank of $X_{kn} + \Theta d_{kn}$ among $X_{1n} + \Theta d_{1n},\ldots,X_{nn} + \Theta d_{nn}$ written R_{kn}^{Θ}, we will mean

(1.1) $$R_{kn}^{\Theta} = \sum_{\ell=1}^{n} I(X_{kn} - X_{\ell n} + (d_{kn} - d_{\ell n})\Theta),$$

where $$I(u) = \begin{cases} 1 & u \geq 0 \\ 0 & u < 0 \end{cases}.$$

Denote $\tilde{F}_{kn}(\Theta) = R_{kn}^{\Theta}/n$. Observe that

$$E(\tilde{F}_{kn}(\Theta)|X_{kn}) = \frac{1}{n}\sum_{\ell\neq k}^{n} F_{\ell n}(X_{kn} + (d_{kn} - d_{\ell n})\Theta) + \frac{1}{n}.$$

Let c_{kn} k=1,...,n and d_{kn} k=1,...,n be constants,

(1.2) $$\bar{c}_n = \sum_{\ell=1}^{n} c_{\ell n}/n, \bar{d}_n = \sum_{\ell=1}^{n} d_{\ell n}/n,$$

(1.3) $$c^*_{kn} = c_{kn}/(\sum_{\ell=1}^{n} c^2_{\ell n})^{\frac{1}{2}},$$

(1.4) $$D_{kn} = (\sum_{\ell=1}^{n} (d_{kn} - d_{\ell n})^2/n)^{\frac{1}{2}},$$

(1.5) $$\sigma^2_{nd} = \sum_{\ell=1}^{n} (d_{\ell n} - \bar{d}_n)^2/n, \text{ and}$$

(1.6) $$\Gamma_n = \sum_{\ell=1}^{n} (c_{\ell n} - \bar{c}_n)(d_{\ell n} - \bar{d}_n)/(n \sum_{\ell=1}^{n} c^2_{\ell n})^{\frac{1}{2}}$$

$$= n^{-\frac{1}{2}} \sum_{\ell=1}^{n} c^*_{\ell n}(d_{\ell n} - \bar{d}_n).$$

Note that we will often drop the subscript n in such terms as $\bar{d}_n$, $\bar{c}_n$, $F_{\ell n}$, etc.

a(u) will be a nonconstant nondecreasing function on [0,1] such that $|a'(u)| < A'$ and $|a''(u)| < A''$ for all $u \in (0,1)$. Set $\bar{a} = \int_0^1 a(u)du$ and $\sigma^2_a = \int_0^1 (a(u)-\bar{a})^2 du$.

$T_n(\Theta)$ and $T^*_n(\Theta)$ will be the linear rank statistics,

(1.7) $$T_n(\Theta) = \sum_{\ell=1}^{n} c_{\ell n} a(R^{\Theta}_{\ell n}/n)$$

and

(1.8) $$T^*_n(\Theta) = n^{-\frac{1}{2}} \sum_{\ell=1}^{n} c^*_{\ell n} a(R^{\Theta}_{\ell n}/n).$$

Let $G_{kn}(\Theta_o,\Theta) = \tilde{F}_{kn}(\Theta_o) - \tilde{F}_{kn}(\Theta) - E(\tilde{F}_{kn}(\Theta_o) - \tilde{F}_{kn}(\Theta) | X_{kn})$ **(1.9)**

and

(1.10) $$G_n(\Theta_o,\Theta) = n^{-\frac{1}{2}} \sum_{k=1}^{n} |c^*_{kn}| G_{kn}(\Theta_o,\Theta).$$

Definition 1.1.

A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ will be said to be of strong stochastic order $\phi(n)$, if for every $\lambda>0$ there exists a $c > 0$ such that $P(|X_n|>c\phi(n)) < n^{-\lambda}$ for all n sufficiently large. This will be written $X_n = SSO(\phi(n))$.

Some easily verifiable properties of SSO are

(1.a) If $X_n = SSO(\phi(n))$, then for any two sequences of non-negative constants $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, $a_nX_n+b_n = SSO(a_n\phi(n)+b_n)$.

(1.b) If $X_n^{(i)} = SSO(\phi_i(n))$ i=1...k, then

$$X_n^{(1)} + \dots + X_n^{(k)} = SSO(\phi_1(n) + \dots + \phi_k(n)) = SSO(\phi_1(n) \vee \dots \vee \phi_k(n)).$$

$(a \vee b = \max\{a,b\})$

(1.c) If $X_n = SSO(\phi_1(n))$ and $Y_n = SSO(\phi_2(n))$, then

$$X_nY_n = SSO(\phi_1(n)\phi_2(n)).$$

(1.d) If $X_n = SSO(\phi_1(n))$ and $\psi(n)$ is such that $\lim_{n\to\infty} \psi(n)/\phi(n) > 0$, then $X_n = SSO(\psi(n))$.

(1.e) If $X_n = SSO(\phi(n))$ then $X_n \overset{a.s.}{=} O(\phi(n))$.

(1.f) If $\{X_n\}_{n=1}^{\infty}$ is nonrandom and $X_n = O(\phi(n))$, then $X_n = SSO(\phi(n))$.

Section 2. Almost Sure Linearily with a Rate for Linear Rank Statistics with a Bounded Score Function.

Theorem 2.1.

Let $\{d_{\ell n}\}_{\ell=1}^{n}$ n=1,2,... be such that for some positive constants $D_2 > D_1 > 0$, $D_2 > \overline{\lim_{n\to\infty}}\, \sigma_{nd}^2 \geq \underline{\lim_{n\to\infty}}\, \sigma_{nd}^2 > D_1$.

Assume that F has a density f which is absolutely continuous with derivative f' such that either

(2.1.i) $\int_{-\infty}^{\infty} |f'(u)|du < \infty$

or

(2.1.ii) $|f'| < M'$ for some $M' > 0$.

Also assume the conditions on a(u) described in Section 1.

Then for any $c_1 > 0$

(2.2) $$\sup_{\theta \in I_n(\theta_o)} |T_n^*(\theta_o)-T_n^*(\theta)-(\theta_o-\theta)\Gamma_n \int_{-\infty}^{\infty} a'(F(u))f^2(u)du| = SSO(\psi(n))$$

where $I_n(\theta_o) = [-c_1\sqrt{\log\log n}/\sqrt{n} + \theta_o, \theta_o + c_1\sqrt{\log\log n}/\sqrt{n}]$

and $\psi(n) =$

$$\sqrt[4]{\log n \log n}\sqrt{\log n}/n^{3/4} \vee (\sum_{k=1}^{n} (c_{kn}^*(d_{kn}-\bar{d}))^2)^{\frac{1}{2}}\sqrt{\log\log n}\ \log n/n$$

(Proof delayed until Section 4).

Remark 2.1.

Theorem 2.1 will be proven with $\int_{-\infty}^{\infty} a'(F(u))f^2(u)du$ replaced by

$\int_{-\infty}^{\infty} a'(F(u) \frac{(n-1)}{n} + \frac{1}{n})f^2(u)du$.

Remark 2.2.

(2.1.i) or (2.1.ii) imply that there exists an $M > 0$ such that $|f| < M$.

Remark 2.3.

If f has a finite Fisher information number then (2.1.i) is satisfied.

Remark 2.4.

$\underline{\lim}_{n\to\infty} \sigma_{nd}^2 > D_1 > 0$ iff $\underline{\lim}_{n\to\infty} \min_{1\leq k\leq n} D_{kn} > D$ for some $D > 0$.

Remark 2.5.

In Section 3 examples will be given where each of the terms in the expression for $\psi(n)$ is the dominating term.

Section 3. Examples.

We will show how Theorem 2.1 applies for various conditions on the $c_{\ell n}$'s and the $d_{\ell n}$'s.

Example 3.1.

Suppose in addition to the conditions in Theorem 2.1 there exists a $c > 0$ such that for all n sufficiently large $\max_{1 \le k \le n} |c^*_{kn}| < c/n^{\frac{1}{2}}$. Then simple calculations show that

$(2.2) = SSO\,(\sqrt[4]{\log\log n}\,\sqrt{\log n}\,/n^{3/4})$.

In particular:

Example 3.1.1. The Wilcoxon Two Sample Statistic

Let $X_{1n},\ldots,X_{n_m n},X_{n_m+1n},\ldots,X_{nn}$ be independent such that $X_{kn} \overset{d}{\sim} F(\cdot+\Theta_o)$ for $k=1,\ldots,n_m$ and $X_{kn} \overset{d}{\sim} F(\cdot)$ for $k=n_m+1,\ldots,n$.

Let $a(u) = u$, $d_{\ell n} = c_{\ell n} = \begin{cases} 1 & \ell=1,\ldots,n_m \\ 0 & \ell=n_{m+1},\ldots,n \end{cases}$,

so that $T_n(\Theta) = \sum_{k=1}^{n_m} \frac{R^{\Theta}_{Rn}}{n}$.

Let $\lambda_n = n_m/n$. Now $\sum_{\ell=1}^{n} (d_{\ell n}-\bar{d})^2 = n\lambda_n(1-\lambda_n)$, $\sum_{\ell=1}^{n} c^2_{\ell n} = n_m$ and $\Gamma_n = \lambda_n^{\frac{1}{2}}(1-\lambda_n)$.

Assume that there exist $0 < \lambda_1 < \lambda_2 < 1$, such that (3.1.1.i) $\lambda_1 \le \lambda_n \le \lambda_2$ for all n sufficiently large.

It is easily checked that with (3.1.1.i) the Wilcoxon Two Sample Test is a special case of Example 3.1.

Hence (3.1.1.ii) ≡

$$\sup_{\Theta \in I_n(\Theta_o)} |T^*_n(\Theta_o) - T^*_n(\Theta) - (\Theta_o-\Theta)\lambda_n^{\frac{1}{2}}(1-\lambda_n) \int_{-\infty}^{\infty} f^2(u)du| =$$

$$SSO(\sqrt[4]{\log\log n}\,\sqrt{\log n}\,/n^{3/4}).$$

Inagaki (1973) shows that $(3.1.1.ii) \overset{a.s.}{=} O(\sqrt[4]{\log\log n}\,\sqrt{\log n}\,/n^{3/4})$, and only under condition (2.1.ii).

Example 3.1.2. Spearman's Rho

Let $X_{1n},\ldots,X_{nn}$ be independent such that $X_{kn} \overset{d}{\sim} F(\cdot + d_{kn}\Theta_o)$ $k=1,\ldots,n$. Assume that the d_{kn}'s are distinct. Let $c_{\ell n} = R_{\ell n}$ = rank of $d_{\ell n}$ among $d_{1n},\ldots,d_{nn}$. Hence $\sum_{\ell=1}^{n} c_{\ell n}^2 = n(n+1)(2n+1)/6$ and $\Gamma_n = \sum_{\ell=1}^{n} \ell(d_{\ell n}^{(\ell)}-\bar{d}) = \sqrt{6}\,/(n(n+1)(2n+1))^{\frac{1}{2}}$, where $d_{1n}^{(1)} <\ldots< d_{nn}^{(n)}$ are the reordered values of d_{1n}, $\ldots,d_{nn}$. Now $T_n(\Theta) = \sum_{\ell=1}^{n} \frac{R_{\ell n}\, R_{\ell n}^{\Theta}}{n}$ (letting $a(u) = u$).

Assume that there exist $D_1 > 0$ and $D_2 > 0$ such that for all n sufficiently large $D_2 > \sigma_{nd}^2 > D_1$. Note that $\max_{1\le k\le n} |c_{kn}^*| \le \sqrt{6}/n^{\frac{1}{2}}$.

Hence

$$\sup_{\Theta\in I_n(\Theta_o)} \left|T_n^*(\Theta_o)-T_n^*(\Theta) - (\Theta_o-\Theta)\sum_{k=1}^{n} \frac{k(d_{kn}^{(k)} - \bar{d})}{(n(n+1)(2n+1))^{\frac{1}{2}}}\sqrt{6}\int_{-\infty}^{\infty} f^2(u)du\right|$$

$$= SSO(\sqrt[4]{\log\log n}\,\sqrt{\log n}\,/n^{3/4}).$$

Ghosh (1972) proves an SSO rate type result for T_n, but in his case the $d_{\ell n}$'s are random variables.

Remark 3.1.

Analogous to Kiefer (1967), for Example 3.1 the best almost sure rate is probably

$$(2.2) \overset{a.s.}{=} O(\,(\log\log n)^{3/4}/n^{3/4}).$$

Example 3.2. Jurečková (1971)

Assume that $T_n(\Theta)$ satisfies all the conditions of Theorem 2.1. In addition assume that there exists a $c > 0$ such that for all n sufficiently large $\max_{1\le\ell\le n} |d_{\ell n}-\bar{d}| < c\, n^{\frac{1}{4}-\delta}$ where $0 < \delta < \frac{1}{4}$.

Then (2.2) = $SSO(\sqrt{\log\log n}\ \log n/n^{3/4-\delta})$

A similar result without a rate and in general more restrictive assumptions on $\hat{T}_n$ by Jurečková (1971) is as follows:

If $a(u)$ is a nondecreasing function with a bounded first derivative, $(c_{\ell n}-c_{kn})(d_{\ell n}-d_{kn}) \geq 0$ for $1 \leq k,\ \ell \leq n$, $\sigma^2_{nd} < D_2$, $\max_{1\leq \ell \leq n} |c^*_{\ell n}| = o(1)$, $\sum_{\ell=1}^{n} c^*_{\ell n} = 0$, the $d_{\ell n}$'s satisfy the above condition and $f(u)$ has the property that there exists an $\varepsilon > 0$ such that $\sup_{|t|<\varepsilon} \int_{-\infty}^{\infty} \frac{(f'(u-t))^2}{f(u)} du < \infty$ for all u and $f(u) > 0$. Then (2.2) $\overset{a.s.}{=}$ $o(1/\sqrt{n})$ for all $c_1 > 0$ where $I_n(\Theta_o) = [c_1/\sqrt{n} + \Theta_o, \Theta_o + c_1/\sqrt{n}]$.

In the remaining examples only conditions on the $c_{\ell n}$'s or $d_{\ell n}$'s will be stated and then the corresponding SSO rate.

<u>Example 3.3.</u>

Assume $\max_{1\leq \ell\leq n} |c^*_{\ell n}| = \frac{\delta(n)}{\log n}$ where $\delta(n) \underset{n\to\infty}{\to} 0$.

Then

(2.2) = $SSO(\delta(n)\sqrt{\log\log n}\ /\sqrt{n})$.

In particular if $c_{\ell n} = d_{\ell n}-\bar{d}$ $1 \leq \ell \leq n$,

$$\sup_{\Theta\in I_n(\Theta_o)} \left| T^*_n(\Theta_o)-T^*_n(\Theta) - (\Theta_o-\Theta)\sigma_{nd} \int_{-\infty}^{\infty} a'(F(u))f^2(u)du \right| =$$

$SSO(\delta(n)\sqrt{\log\log n}\ /\sqrt{n})$.

<u>Example 3.4.</u>

Assume $\sum_{\ell=1}^{n} (d_{\ell n}-\bar{d})^4 /n \leq D$ for some $D > 0$. Then (2.2) = $SSO(\sqrt{\log\log n}\ \log n/n^{3/4})$.

Section 4. Proof of Theorem 2.1

Preliminaries

Frequent use will be made of Bernstein's inequality. The version we will use is as follows:

Let $X_1,\ldots,X_n$ be independent random variables such that $|X_\ell - EX_\ell| < B$ for $1 \le \ell \le n$ and some constant $B > 0$. Then for all $t > 0$,

$P(|S_n - ES_n| > t) < 2\exp(-t^2/(2\sigma^2 + \frac{2}{3}Bt))$,

where $S_n = X_1 + \ldots + X_n$ and $\sigma^2 = \text{Var } S_n$. (See Bennet (1962)).

Let $\alpha_n = \sqrt{\log\log n}/\sqrt{n}$, $\gamma_n = \alpha_n^{\frac{1}{2}}\sqrt{\log n}/\sqrt{n}$ and $\eta_n = \alpha_n \log n/\sqrt{n}$.

Pick constants γ, δ, ν, α, β and ε such that

(4.a) $\quad \underline{\lim}_{n\to\infty}\, n\gamma_n^2/(\alpha_n \log n) > \tau > 0$

(4.b) $\quad 0 \le \overline{\lim}_{n\to\infty}\, \gamma_n/\alpha_n < \delta < 0$

(4.c) $\quad 0 \le \overline{\lim}_{n\to\infty}\, \log(\alpha_n/\gamma_n)/\log n < \nu < \infty$

(4.d) $\quad \overline{\lim}_{n\to\infty}\, \alpha_n/(\eta_n n^{\frac{1}{2}}) < \alpha < \infty$

(4.e) $\quad \underline{\lim}_{n\to\infty}\, n^{\frac{1}{2}}\eta_n/(\alpha_n \log n) > \beta > 0$

(4.f) $\quad 0 \le \overline{\lim}_{n\to\infty}\, \log(\alpha_n/\eta_n)/\log n < \varepsilon < \infty$

The reason for the generality of these conditions will be explained in the remarks after the proof. Now set

$$\psi_1(n) = n^{-\frac{1}{2}}\gamma_n \sum_{k=1}^{n} |c_{kn}^*| D_{kn}$$

$$\psi_2(n) = \gamma_n^2/\alpha_n$$

$$\psi_3(n) = n^{-\frac{1}{2}} \sum_{k=1}^{n} |c_{kn}^*| D_{kn}\alpha_n^{\frac{1}{2}}\gamma_n$$

$$\psi_4(n) = n^{-\frac{1}{2}}\gamma_n^2 \sum_{k=1}^{n} |c_{kn}^*| D_{kn}^2$$

$$\psi_5(n) = \eta_n\left(\sum_{k=1}^{n} c_{kn}^{*2} D_{kn}^2\right)^{\frac{1}{2}}$$

and

$$\psi_6(n) = n^{-\frac{1}{2}} \sum_{k=1}^{n} |c_{kn}^*| D_{kn}^2 \alpha_n^2.$$

Routine calculations show that if a random variable is $SSO(\psi_1(n) \vee \ldots \vee \psi_6(n))$, it is $SSO(\psi(n))$.

The proof of Theorem 2.1 will now be given in the following steps:

<u>Outline of Proof</u>

We can write (2.2)

$$\leq \sup_{\Theta \in I_n(\Theta_o)} \Big| \sum_{k=1}^{n} \frac{c_{kn}^*}{n^{\frac{1}{2}}} \Big(a(\tilde{F}_{kn}(\Theta_o)) - a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn}))$$

(4.1.i)

$$-[a(\tilde{F}_{kn}(\Theta)) - a(E(\tilde{F}_{kn}(\Theta)|X_{kn}))]\Big)\Big|$$

$$+ \sup_{\Theta \in I_n(\Theta_o)} \Big| \sum_{k=1}^{n} \frac{c_{kn}^*}{n^{\frac{1}{2}}} \Big(a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn})) - a(E(\tilde{F}_{kn}(\Theta)|X_{kn}))$$

(4.1.ii)

$$- E[a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn})) - a(E(\tilde{F}_{kn}(\Theta)|X_{kn}))]\Big)\Big|$$

$$+ \sup_{\Theta \in I_n(\Theta_o)} \Big| \sum_{k=1}^{n} \frac{c_{kn}^*}{n^{\frac{1}{2}}} \Big(E[a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn})) - a(E(\tilde{F}_{kn}(\Theta)|X_{kn}))]$$

(4.1.iii)

$$- (\Theta_o - \Theta)(d_{kn} - \bar{d}) \int_{-\infty}^{\infty} a'\left(\frac{(n-1)}{n} F(u) + \frac{1}{n}\right) f^2(u)\,du \Big)\Big|$$

By Proposition 4.2, (4.1.i) = $SSO(\psi_1(n) \vee \psi_2(n) \vee \psi_3(n) \vee \psi_4(n))$, by Proposition 4.3, (4.1.ii) = $SSO(\psi_5(n))$, and by Proposition 4.4, (4.1.iii) = $SSO(\psi_6(n))$. Hence, Theorem 2.1 will follow immediately once Propositions 4.2, 4.3 and 4.4 have been established.

Proposition 4.2.

Let $\{d_{\ell n}\}_{\ell=1}^{n}$ $n \geq 1$ be such that $\min_{1\leq k\leq n} D_{kn} > D > 0$ for all n sufficiently large and some $D > 0$.

Also let α_n, γ_n $n \geq 1$ be positive constants such that (4.a), (4.b) and (4.c) are satisfied, then for all $c_1 > 0$,

$$(4.1.i) = SSO(\psi_1(n) \vee \psi_2(n) \vee \psi_3(n) \vee \psi_4(n)),$$

where $I_n(\Theta_o) = [\Theta_o - c_1\alpha_n, \Theta_o + c_1\alpha_n]$.

Proof.

Before we prove Proposition 4.2 we need some lemmas.

Lemma 4.2.1.

For all Θ

$$|a(\tilde{F}_{kn}(\Theta_o)) - a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn})) - [a(\tilde{F}_{kn}(\Theta)) - a(E(\tilde{F}_{kn}(\Theta)|X_{kn})]|$$

$$\leq A'|G_{kn}(\Theta_o,\Theta)| + \frac{A''}{2} | \tilde{F}_{kn}(\Theta_o) - E(F_{kn}(\Theta_o)|X_{kn})|^2$$

$$+ \frac{A''}{2}|\tilde{F}_{kn}(\Theta) - E(\tilde{F}_{kn}(\Theta)|X_{kn})|^2$$

$$+ A''|E(\tilde{F}_{kn}(\Theta_o)|X_{kn}) - E(\tilde{F}_{kn}(\Theta)|X_{kn})||\tilde{F}_{kn}(\Theta_o) - E(\tilde{F}_{kn}(\Theta_o)|X_{kn})|.$$

Proof.

Repeated use of Taylor's theorem and the mean value theorem gives for all $x, y, z, w \in (0,1)$

$$|a(x)-a(y)-(a(w)-a(z))|$$

$$\leq A'|x-y-(w-z)| + \frac{A''}{2}|x-y|^2 + \frac{A''}{2}|w-z|^2 + A''|y-z||x-y|.$$

Now let $x = \tilde{F}_{kn}(\Theta_o)$, $y = E(\tilde{F}_{kn}(\Theta_o)\ X_{kn})$, $w = \tilde{F}_{kn}(\Theta)$ and $z = E(F_{kn}(\Theta)|X_{kn})$. □

Lemma 4.2.2.

For all $\lambda > 0$ there exists $c_2 > 0$ such that

$$(4.2.2.i) \equiv P\Big(\bigcup_{k=1}^{n}\{\sup_{\theta\in I_n(\theta_o)} |G_{kn}(\theta_o,\theta)| > c_2 D_{kn}\gamma_n\}\Big) < n^{-\lambda}$$

and

$$(4.2.2.ii) \qquad \sup_{\theta\in I_n(\theta_o)} |G_n(\theta_o,\theta)| = SSO(\psi_1(n)).$$

Proof.

Pick $\lambda > 0$.

Claim 4.2.2.1.

For $\beta_1>0$, $\beta_2>0$ and $1\geq k\geq n$,

let $S(\beta_1,\beta_2,k,n) = \{S_n:\ S_n = \sum_{\ell=1}^{n} Y_\ell,\ Y_1,\ldots,Y_n$

are independent, Var $S_n \leq \beta_1 D_{kn}\alpha_n/n$, and $|Y_\ell| < \beta_2/n$ $\ell=1,\ldots,n\}$. Then for all $\beta_1>0$, $\beta_2>0$ and $\lambda^*>0$, there exists a $c > 0$ such that for all n sufficiently large,

$$\max_{1\leq k\leq n}\ \sup_{S_n\in S(\beta_1,\beta_2,k,n)} P(|S_n-ES_n| > c\gamma_n D_{kn}) < n^{-(\lambda^*+1)}.$$

Proof. (Of Claim 4.2.2.1).

Pick $\lambda^*>0$, $\beta_1>0$, $\beta_2>0$. Now pick N_o large enough so that for all $n\geq N_o$ $n\gamma_n^2/(\alpha_n \log n) > \tau$, $\gamma_n/\alpha_n < \delta$ and $\min_{1\leq k\leq n} D_{kn} > D$. Choose any $n\geq N_o$, k such that $1\leq k\leq n$ and $S_n\in S(\beta_1,\beta_2,k,n)$, then since $|Y_\ell-EY_\ell| < 2\beta_2/n$, by Bernstein's inequality,

$$P(|S_n-ES_n| > c\gamma_n D_{kn})$$

$$\leq 2\exp(-c^2\gamma_n^2 D_{kn}^2/(2\beta_1 D_{kn}\frac{\alpha_n}{n} + \frac{4}{3n}\beta_2 c\gamma_n D_{kn}))$$

$$= 2\exp(-c^2 n\gamma_n^2 D_{kn}/(2\beta_1\alpha_n + \frac{4}{3}c\beta_2\gamma_n)).$$

Now, since for all $n \geq N_o$, $\min_{1 \leq k \leq n} D_{kn} > D$, $\frac{n\gamma_n^2}{\alpha_n \log n} > \tau > 0$ and $\frac{\gamma_n}{\alpha_n} < \delta < \infty$, the above

is $\leq 2\exp(-c^2 D\tau \log n/(2\beta_1 + \frac{4\beta_2}{3}c\delta)) = 2n^{-c^2 D\tau/(2\beta_1 + \frac{4}{3}\beta_2 c\delta)} \equiv \rho_n$.

Note that ρ_n does not depend on k or the particular S_n chosen. By making c sufficiently large $\rho_n < n^{-(\lambda^*+1)}$.□

Now some observations:

Let $W_{k\ell n}(\Theta_o,\Theta) =$

$\frac{1}{n}\{I(X_{kn}-X_{\ell n} + (d_{kn}-d_{\ell n})\Theta_o) - I(X_{kn}-X_{\ell n} + (d_{kn}-d_{\ell n})\Theta)\}$.

Let $a = \min\{X_{kn} + (d_{kn}-d_{\ell n})\Theta_o,\ X_{kn} + (d_{kn}-d_{\ell n})\Theta\}$,

$b = \max\{X_{kn} + (d_{kn}-d_{\ell n})\Theta_o,\ X_{kn} + (d_{kn}-d_{\ell n})\Theta\}$.

For $X_{\ell n} \in [a,b)$, $W_{k\ell n}(\Theta_o,\Theta) = \text{sign}((d_{kn}-d_{\ell n})(\Theta_o-\Theta))/n$ and for $X_{\ell n} \notin [a,b)$, $W_{k\ell n}(\Theta_o,\Theta) = 0$. So that for X_{kn} fixed,

$$W_{k\ell n}(\Theta_o,\Theta) = \begin{cases} \text{sign}((d_{kn}-d_{\ell n})(\Theta_o-\Theta))/n & \text{w.p. } P_{k\ell n}(\Theta_o,\Theta) \\ 0 & \text{w.p. } 1-P_{k\ell n}(\Theta_o,\Theta), \end{cases}$$

where $P_{kn}(\Theta_o,\Theta) = F_{\ell n}(b) - F_{\ell n}(a)$.

Note that $P_{k\ell n}(\Theta_o,\Theta) \leq M' |d_{kn}-d_{\ell n}|\ |\Theta_o-\Theta|$ and $\text{Var}(W_{k\ell n}(\Theta_o,\Theta)|X_{kn})$

$$= \frac{1}{n^2} P_{k\ell n}(\Theta_o,\Theta)(1-P_{k\ell n}(\Theta_o,\Theta)) \leq \frac{M}{n^2} |d_{kn}-d_{\ell n}||\Theta_o-\Theta|.$$

<u>Claim 4.2.2.2.</u>

Let $S_n = \sum_{\ell=1}^{n} W_{k\ell n}(\Theta_o,\Theta_\ell)$. If all the Θ_ℓ's are in $I_n(\Theta_o)$, then

$(S_n|X_{kn}) = \sum_{\ell=1}^{n} (W_{k\ell n}(\Theta_o,\Theta_\ell)|X_{kn})$ is in $S(\beta_1,\beta_2,k,n)$ where $\beta_1 = M\,c_1$ and $\beta_2 = 1$.

Proof. (Of Claim 4.2.2.2).

We easily see that $(S_n|X_{kn})$, that is, S_n with X_{kn} fixed, is a sum of independent random variables $(W_{k\ell n}(\Theta_o,\Theta_\ell)|X_{kn})$ $\ell=1\ldots n$, such that $|(W_{k\ell n}(\Theta_o,\Theta_\ell)|X_{kn})| \leq \frac{1}{n}$ and

$$\mathrm{Var}(S_n|X_{kn}) = \frac{1}{n^2}\sum_{\ell=1}^{n} P_{k\ell n}(\Theta_o,\Theta_\ell)(1-P_{k\ell n}(\Theta_o,\Theta_\ell)) \leq \frac{M}{n}\sum_{\ell=1}^{n}\frac{|d_{kn}-d_{\ell n}|}{n}c_1\alpha_n,$$

which by Schwarz's inequality is $\leq \frac{M}{n}c_1\alpha_n D_{kn}$. Hence, $(S_n|X_{kn})\in S(\beta_1,\beta_2,k,n)$.□

Now to complete the proof of Lemma 4.2.2.

Let $\eta_r(\Theta_o) = \Theta_o + c_1 r\alpha_n b_n^{-1}$, where $b_n^{-1} = \gamma_n/\alpha_n$ and r are integers such that $|r| \leq b_n+1$.

Let $B_{kn}(\eta_r(\Theta_o)) = \frac{1}{n}\sum_{\ell=1}^{n} I(X_{kn}-X_{\ell n}+(d_{kn}-d_{\ell n})(\Theta_o+c_1b_n^{-1}\alpha_n(r+i_{k\ell})))$,

$$\text{where } i_{k\ell} = \begin{cases} 1 \text{ if } d_{kn}-d_{\ell n} \geq 0 \\ 0 \text{ if } d_{kn}-d_{\ell n} < 0 \end{cases}$$

and $b_{kn}(\eta_r(\Theta_o)) = \frac{1}{n}\sum_{\ell=1}^{n} I(X_{kn}-X_{\ell n}+(d_{kn}-d_{\ell n})(\Theta_o+c_1b_n^{-1}\alpha_n(r+i_{\ell k})))$.

Pick $\Theta\in I_n(\Theta_o)$, then there exists an integer r such that

$$-1-b_n\leq r\leq b_n \text{ and } \Theta_o+\alpha_n b_n^{-1}c_1 r\leq\Theta<\alpha_n b_n^{-1}c_1(r+1)+\Theta_o.$$

Note that for such an r depending on Θ,

$$b_{kn}(\eta_r(\Theta_o)) \leq \tilde{F}_{kn}(\Theta) \leq B_{kn}(\eta_r(\Theta_o)) \text{ for } k=1,\ldots,n.$$

So that for all k such that $1\leq k\leq n$,

$$b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)|X_{kn})$$

$$\leq G_{kn}(\Theta_o,\Theta) \leq B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)|X_{kn}).$$

Now write $b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)|X_{kn})$

$$= b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)|X_{kn})$$

$$+E(b_{kn}(\eta_r(\Theta_o))-B_{kn}(\eta_r(\Theta_o))|X_{kn}),$$

and $B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)|X_{kn})$

$$= B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)|X_{kn})$$

$$+ E(B_{kn}(\eta_r(\Theta_o))-b_{kn}(\eta_r(\Theta_o))|X_{kn}).$$

So that for all k such that $1\leq k\leq n$.

$$\sup_{\Theta\varepsilon I_n(\Theta_o)} | G_{kn}(\Theta_o,\Theta) | \leq$$

$$\max \{ \sup_{|r|\leq b_n+1} | b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(b_{kn}(\eta_r(\Theta_o))-\tilde{F}_n(\Theta_o)|X_{kn})|;$$

$$\sup_{|r|\leq b_n+1} | B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)|X_{kn})|\}$$

$$+ \sup_{|r|\leq b_n+1} | E(B_{kn}(\eta_r(\Theta_o))-b_{kn}(\eta_r(\Theta_o))|X_{kn})|.$$

Note that for all r such that $|r|\leq b_n+1$

$$|E(B_{kn}(\eta_r(\Theta_o))-b_{kn}(\eta_r(\Theta_o))|X_{kn})| \leq Mc_1 \sum_{\ell=1}^{n} \frac{|d_{kn}-d_{\ell n}|}{n} \alpha_n b_n^{-1},$$

which by Schwarz's inequality is $\leq Mc_1\gamma_n D_{kn}$.

Hence,

$$\sup_{|r|\leq b_n+1} | E(B_{kn}(\eta_r(\Theta_o))-b_{kn}(\eta_r(\Theta_o))|X_{kn})|\leq Mc_1\gamma_n D_{kn}. \tag{4.2.2.iii}$$

Note that $b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)$

$$= \frac{1}{n}\sum_{\ell=1}^{n} (I(X_{kn}-X_{\ell n}+(d_{kn}-d_{\ell n})Z_\ell)-I(X_{kn}-X_{\ell n}+(d_{kn}-d_{\ell n})\Theta_o))$$

$$= -\sum_{\ell=1}^{n} W_{k\ell n}(\Theta_o,Z_\ell), \text{ where } Z_\ell = \Theta_o + c_1 b_n^{-1}\alpha_n(r+i_{\ell k})$$

and similarly $B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o) = -\sum_{\ell=1}^{n} W_{k\ell n}(\Theta_o,W_\ell)$,

where $W_\ell = \Theta_o + c_1 b_n^{-1}\alpha_n(r+i_{k\ell})$.

Now observe that $|\Theta_o-Z_\ell| \le c_1\alpha_n$ $\ell=1,\ldots,n$ and $|\Theta_o-W_\ell| \le c_1\alpha_n$ $\ell=1,\ldots,n$.

Hence, by Claim 4.2.2.2, $(b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)|X_{kn})$ and $(B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)|X_{kn})$ are in $S(\beta_1,\beta_2,k,n)$ where $\beta_1=M\ c_1$ and $\beta_2=1$.

Therefore, by Claim 4.2.2.1 for all $\lambda^*>0$, there exists a $c'>0$ and an $N_o>0$ such that for all $n \ge N_o$

$$\max_{1\le k\le n}\ \sup_{|r|\le b_n+1} P\Big(\big|B_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(B_{kn}(\eta_r(\Theta_o)) -F_{kn}(\Theta_o)|X_{kn})\big|>c'D_{kn}\gamma_n|X_{kn}\Big)< n^{-(\lambda^*+1)}$$

and

$$\max_{1\le k\le n}\ \sup_{|r|\le b_{n+1}} P\Big(\big|b_{kn}(\eta_r(\Theta_o))-\tilde{F}_{kn}(\Theta_o)-E(b_{kn}(\eta_r(\Theta_o)) -F_{kn}(\Theta_o)|X_{kn})\big|>c'D_{kn}\gamma_n|X_{kn}\Big)<n^{-(\lambda^*+1)}.$$

Since c' and N_o do not depend on X_{kn}, we have in combination with (4.2.2.iii) that (4.2.2.i) $\le 2(2b_n+3)n^{-\lambda^*}$ for some $c_2>0$.

Also, with probability $\ge 1 - 2(2b_n+3)n^{-\lambda^*}$,

$$\sup_{\Theta\in I_n(\Theta_o)} |G_n(\Theta_o,\Theta)| = \sup_{\Theta\in I_n(\Theta_o)} \Big|\sum_{k=1}^{n} \frac{|c^*_{kn}|}{n^{\frac{1}{2}}} G_{kn}(\Theta_o,\Theta)\Big|$$

$$\le \sum_{k=1}^{n} \frac{|c^*_{kn}|}{n^{\frac{1}{2}}} \sup_{\Theta\in I_n(\Theta_o)} |G_{kn}(\Theta_o,\Theta)|$$

$$\le \sum_{k=1}^{n} \frac{|c^*_{kn}|}{n^{\frac{1}{2}}}\, c\, D_{kn}\gamma_n = c_2\psi_1(n)$$

for some $c_2>0$.

Note that $\log(2(2b_n+3))/\log n-\lambda^*$

$$= \log(2(2(\alpha_n/\gamma_n)+3))/\log n-\lambda^*$$

which by (4.c) is less than $\nu-\lambda^*$ for all n sufficiently large.

Now pick λ^* so that $\nu-\lambda^* < -\lambda$. This completes the proof of Lemma 4.2.2.□

Lemma 4.2.3.

$$\max_{1\le k\le n} | \tilde{F}_{kn}(\Theta_o)-E(\tilde{F}_{kn}(\Theta_o)|X_{kn})| = SSO(\gamma_n/\alpha_n^{\frac{1}{2}})$$

Proof.

Pick $\lambda > 0$. Note first that $E(\tilde{F}_{kn}(\Theta_o)|X_{kn})$

$$= \sum_{\ell=1}^{n} \frac{F_{\ell n}(X_{kn}+(d_{kn}-d_{\ell n})\Theta_o)}{n} - \frac{F_{kn}(X_{kn})}{n} + \frac{1}{n}$$

$$= \sum_{\ell=1}^{n} \frac{F(X_{kn}+(d_{kn}-d_{\ell n})\Theta_o+d_{\ell n}\Theta_o)}{n} - \frac{F_{kn}(X_{kn})}{n} + \frac{1}{n}$$

$$= F(X_{kn}+d_{kn}\Theta_o) - \frac{F_{kn}(X_{kn})}{n} + \frac{1}{n}$$

Now $\frac{|F_{kn}(X_{kn})|}{n} \le \frac{1}{n}$ implies that

$$|\tilde{F}_{kn}(\Theta_o)-E(\tilde{F}_{kn}(\Theta_o)|X_{kn})| \le |\tilde{F}_{kn}(\Theta_o)-F(X_{kn}+d_{kn}\Theta_o)| + \frac{2}{n} .$$

Therefore, $P(\max_{1\le k\le n} | \tilde{F}_{kn}(\Theta_o)-E(\tilde{F}_{kn}(\Theta_o)|X_{kn})| > c_3\, \gamma_n/\alpha_n^{\frac{1}{2}})$

$$\le P(\max_{1\le k\le n} | \tilde{F}_{kn}(\Theta_o)-F(X_{kn}+d_{kn}\Theta_o)| > c_3\gamma_n/\alpha_n^{\frac{1}{2}} - \frac{2}{n}) .$$

But for all n sufficiently large, we have by (4.a) that

$$\gamma_n/\alpha_n^{\frac{1}{2}} > (\tau \log n)^{\frac{1}{2}} / n^{\frac{1}{2}} > \frac{2}{n} .$$

Hence, for all n sufficiently large, the above is

$$\le P(\max_{1\le k\le n} | \tilde{F}_{kn}(\Theta_o)-F(X_{kn}+d_{kn}\Theta_o)| > (c_3-2)(\tau \log n)^{\frac{1}{2}}/n^{\frac{1}{2}}),$$

which by the Dvoretzky, Kiefer, Wolfowitz (1956) inequality is $\leq$ $C\exp(-2\,\tau\,\log n(c_3-2)^2) = Cn^{-2\tau(c_3-2)^2}$, where C is a constant independent of n, c_3 and τ. Now pick c_3 such that $\log C - 2\tau(c_3-2)^2 < -\lambda$. □

Lemma 4.2.4.

(4.2.4.i) $\equiv$

$$\sup_{\Theta\in I_n(\Theta_o)} \sum_{k=1}^{n} \frac{|c_{kn}^*|}{n} (|E(\tilde{F}_{kn}(\Theta_o)|X_{kn})-E(\tilde{F}_{kn}(\Theta)|X_{kn})||\tilde{F}_{kn}(\Theta_o)-E(\tilde{F}_{kn}(\Theta_o)|X_{kn})|)$$

$$= SSO(\psi_3(n)).$$

Proof.

First note that for $\Theta\in I_n(\Theta_o)$ $|E(\tilde{F}_{kn}(\Theta_o)|X_{kn})-E(\tilde{F}_{kn}(\Theta)|X_{kn})|$

$$= |\sum_{\ell-1}^{n} \left(\frac{F_{\ell n}(X_{kn}+(d_{kn}-d_{\ell n})\Theta_o)}{n} - \frac{F_{\ell n}(X_{kn}+(d_{kn}-d_{\ell n})\Theta)}{n}\right)|$$

$$\leq M \sum_{\ell=1}^{n} \frac{|d_{kn}-d_{\ell n}|}{n} |\Theta_o-\Theta| \leq Mc_1 D_{kn}\alpha_n .$$

Hence (4.2.4.i)

$$\leq Mc_1\alpha_n\, n^{-\frac{1}{2}} \sum_{k=1}^{n} |c_{kn}^*| D_{kn} \max_{1\leq k\leq n} |\tilde{F}_{kn}(\Theta_o)-E(\tilde{F}_{kn}(\Theta_o)|X_{kn})|$$

which by Lemma 4.2.3 is equal to

$$Mc_1\alpha_n\, n^{-\frac{1}{2}} \sum_{k=1}^{n} |c_{kn}^*| D_{kn} SSO(\gamma_n/\alpha_n^{\frac{1}{2}}) = SSO(\psi_3(n)). \square$$

Lemma 4.2.5.

$$\sup_{\Theta\in I_n(\Theta_o)} \sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} |\tilde{F}_{kn}(\Theta)-E(\tilde{F}_{kn}(\Theta)|X_{kn})|^2$$

$$= SSO(\psi_2(n)\vee\psi_4(n)).$$

Proof.

Pick $\lambda > 0$. By Minkowski's inequality,

$$\left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} (\tilde{F}_{kn}(\Theta_o)-E(\tilde{F}_{kn}(\Theta_o)|X_{kn})-\{\tilde{F}_{kn}(\Theta)-E(\tilde{F}_{kn}(\Theta)|X_{kn})\})^2\right)^{\frac{1}{2}}$$

$$+\left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} (\tilde{F}_{kn}(\Theta_o)-E(\tilde{F}_{kn}(\Theta_o)|X_{kn}))^2\right)^{\frac{1}{2}}$$

$$\geq \left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} (\tilde{F}_{kn}(\Theta)-E(\tilde{F}_{kn}(\Theta)|X_{kn}))^2\right)^{\frac{1}{2}}$$

Now, by Lemma 4.2.2, there exists a $c_2 > 0$ such that for all n sufficiently large,

$$P(\bigcup_{k=1}^{n} \{\sup_{\Theta\in I_n(\Theta_o)} |G_{kn}(\Theta_o,\Theta)|>c_2 D_{kn}\gamma_n\}) < \tfrac{1}{2} n^{-\lambda}.$$

By Lemma 4.2.3, there exists a $c_3 > 0$ such that for all n sufficiently large,

$$P(\max_{1\leq k\leq n} |\tilde{F}_{kn}(\Theta_o) - E(\tilde{F}_{kn}(\Theta_o)|X_{kn})| > c_3\gamma_n/\alpha_n^{\frac{1}{2}}) < \frac{1}{2} n^{-\lambda}$$

Observe that by the above

$$\sup_{\Theta\in I_n(\Theta_o)} \left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} (\tilde{F}_{kn}(\Theta)-E(\tilde{F}_{kn}(\Theta)|X_{kn}))^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} (\sup_{\Theta\in I_n(\Theta_o)} |G_{kn}(\Theta_o,\Theta)|)^2\right)^{\frac{1}{2}}$$

$$+\left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} (\max_{1\leq k\leq n} |\tilde{F}_n(\Theta_o)-E(\tilde{F}_{kn}(\Theta_o)|X_{kn})|)^2\right)^{\frac{1}{2}}$$

which with probability $\geq 1 - \frac{1}{2} n^{-\lambda} - \frac{1}{2} n^{-\lambda} = 1 - n^{-\lambda}$ is

$$\leq \left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} c_2^2 D_{kn}^2 \gamma_n^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} c_3^2\gamma_n^2/\alpha_n\right)^{\frac{1}{2}} .$$

Now, since $\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} \leq 1$, the above is

$$\leq \left(c_2 \left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} D_{kn}^2\right)^{\frac{1}{2}} + c_3/\alpha_n^{\frac{1}{2}}\right)\gamma_n.$$

Therefore, for all n sufficiently large with probability $< n^{-\lambda}$,

$$\sup_{\Theta \in I_n(\Theta_o)} \sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} |\tilde{F}_{kn}(\Theta) - E(\tilde{F}_{kn}(\Theta)|X_{kn})|^2$$

$$> \left(c_2\left(\sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} D_{kn}^2\right)^{\frac{1}{2}} + c_3/\alpha_n^{\frac{1}{2}}\right)^2 \gamma_n^2.$$

This completes the proof of Lemma 4.2.5.□

Now to finish the proof of Proposition 4.2. By Lemma 4.2.1 we can write

$$(4.1.i) \quad \leq A' \sup_{\Theta \in I_n(\Theta_o)} \sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} |G_{kn}(\Theta_o, \Theta)| \qquad (4.2.i)$$

$$+ \frac{A''}{2} \sup_{\Theta \in I_n(\Theta_o)} \sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} |\tilde{F}_{kn}(\Theta_o) - E(\tilde{F}_{kn}(\Theta_o)|X_{kn})|^2 \qquad (4.2.ii)$$

$$+ \frac{A''}{2} \sup_{\Theta \in I_n(\Theta_o)} \sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} |\tilde{F}_{kn}(\Theta) - E(\tilde{F}_{kn}(\Theta)|X_{kn})|^2 \qquad (4.2.iii)$$

$$+ A'' \sup_{\Theta \in I_n(\Theta_o)} \sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} (|E(\tilde{F}_{kn}(\Theta_o)|X_{kn}) - E(\tilde{F}_{kn}(\Theta)|X_{kn})| \cdot$$

$$|\tilde{F}_{kn}(\Theta_o) - E(\tilde{F}_{kn}(\Theta_o)|X_{kn})|) \qquad (4.2.iv)$$

Now by Lemma 4.2.2, (4.2.i) = $SSO(\psi_1(n))$,

Lemma 4.2.3, (4.2.ii) = $SSO(\psi_2(n))$,

Lemma 4.2.5, (4.2.iii) = $SSO(\psi_2(n) \vee \psi_4(n))$, and

Lemma 4.2.4, (4.2.iv) = $SSO(\psi_3(n))$.

By property (1.b) of SSQ,the proof of Proposition 4.2 is complete.

Proposition 4.3.

Let $H_{kn}(\Theta)$ =

$$= \frac{c^*_{kn}}{n^{\frac{1}{2}}} \Big(a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn})) - a(E(\tilde{F}_{kn}(\Theta)|X_{kn}))$$

$$- E[a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn})) - a\ (E(\tilde{F}_{kn}(\Theta)|X_{kn}))]\Big)$$

and

$$H_n(\Theta) = \sum_{k=1}^{n} H_{kn}(\Theta).$$

Let α_n, η_n $n \geq 1$ be any two sequences of positive constants such that (4.d), (4.e), and (4.f) hold. Then,

$$\sup_{\Theta \in I_n(\Theta_o)} |H_n(\Theta)| = SSO(\psi_5(n)).$$

Proof.

(Note that without loss of validity we will write $E(\tilde{F}_{kn}(\Theta)|X_{kn}) = \frac{1}{n} \sum_{\ell \neq k}^{n} F_{\ell n}(X_{kn} + (d_{kn} - d_{\ell n})\Theta)$, dropping the $\frac{1}{n}$ term).

Let $b_n = \alpha_n/\eta_n$. For integers r such that $|r| \leq b_n + 1$ set $\eta_{rn} = \Theta_o + c_1 r \alpha_n b_n^{-1}$.

$$\text{Let } i_{k\ell} = \begin{cases} 1 \text{ if } c^*_{kn}(d_{kn} - d_{\ell n}) \geq 0 \\ 0 \text{ if } c^*_{kn}(d_{kn} - d_{\ell n}) < 0 \end{cases}$$

and

$$j_{k\ell} = 1 - i_{k\ell}.$$

For $\Theta \in [\eta_{rn}, \eta_{r+1n}]$, by the nondecreasing property of a(u) and F, observe that

$$\frac{c^*_{kn}}{n^{\frac{1}{2}}}\left(a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn}))-a(\sum_{\ell\neq k}^{n} F_{\ell n}(X_{kn}+(d_{kn}-d_{\ell n})(\Theta_o + c_1\alpha_n b_n^{-1}(r+j_{\ell k})))/n)\right)$$

$$\geq \frac{c^*_{kn}}{n^{\frac{1}{2}}}\left(a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn}))-a(E(\tilde{F}_{kn}(\Theta)|X_{kn}))\right)$$

$$\geq \frac{c^*_{kn}}{n^{\frac{1}{2}}}\left(a(E(\tilde{F}_{kn}(\Theta_o)|X_{kn}))-a(\sum_{\ell\neq k}^{n} F_{\ell n}(X_{kn}+(d_{kn}-d_{\ell n})(\Theta_o + c_1\alpha_n b_n^{-1}(r+i_{\ell k})))/n)\right).$$

Now, let A_{kn} =

$$\frac{c^*_{kn}}{n^{\frac{1}{2}}}\left(a(E(F_{kn}(\Theta_o)\ X_{kn}))-a(\sum_{\ell\neq k}^{n} F_{\ell n}(X_{kn}+(d_{kn}-d_{\ell n})(\Theta_o + c_1\alpha_n b_n^{-1}(r+j_{\ell k})))/n)\right),$$

and let a_{knr} =

$$\frac{c^*_{kn}}{n^{\frac{1}{2}}}\ (a(E(F_{kn}(\Theta_o)|X_{kn}))-a(\sum_{\ell\neq k}^{n} F_{\ell n}(X_{kn}+(d_{kn}-d_{\ell n})(\Theta + c_1\alpha_n b_n^{-1}(r+i_{\ell k})))/n)\ )$$

and Δ_{knr} =

$$\frac{c^*_{kn}}{n^{\frac{1}{2}}}\ (Ea(\sum_{\ell\neq k}^{n} F_{\ell n}(X_{kn}+(d_{kn}-d_{\ell n})(\Theta_o+c_1\alpha_n b_n^{-1}(r+i_{k\ell})))/n)$$

$$-\ Ea\ (\sum_{\ell\neq k}^{n} F_{\ell n}(X_{kn}+(d_{kn}-d_{\ell n})(\Theta_o+c_1\alpha_n b_n^{-1}(r+j_{\ell k})))/n)).$$

By the above inequalities, we have

$$\sum_{k=1}^{n}(A_{knr}-EA_{knr}+\Delta_{knr}) \geq \sum_{k=1}^{n} H_{kn}(\Theta) \geq \sum_{k=1}^{n}(a_{kn}-Ea_{kn}-\Delta_{knr}).$$

Note that

$$|\Delta_{knr}| \leq \eta_n c_1 A'M \sum_{\ell=1}^{n} \frac{|d_{kn}-d_{\ell n}|}{n}\ \frac{|c^*_{kn}|}{n^{\frac{1}{2}}} \leq \eta_n c_1 A'M\ D_{kn}\ \frac{|c^*_{kn}|}{n^{\frac{1}{2}}}\ .$$

Now (4.3.1) $\left|\sum_{k=1}^{n} \Delta_{knr}\right| \leq c_1 A'M\eta_n \sum_{k=1}^{n} D_{kn}\ \frac{|c^*_{kn}|}{n^{\frac{1}{2}}}$

$$\leq c_1 A'M\eta_n(\sum_{k=1}^{n} D^2_{kn}c^{*2}_{kn})^{\frac{1}{2}} = c_1 A'M\psi_5(n).$$

(4.3.1), in combination with the following claim, will complete the proof of Proposition 4.3.

Claim 4.3.2.

For all $\lambda > 0$ there exists a $c^* > 0$ such that

(4.3.3) $\equiv$

$$\max_{1\le k\le n} P\left(\max\left\{\sup_{|r|\le b_n+1} \left|\sum_{k=1}^{n}(A_{knr}-EA_{knr})\right|, \sup_{|r|\le b_n+1}\left|\sum_{k=1}^{n}(a_{knr}-Ea_{knr})\right|\right\} > c^*\psi_5(n)\right) < n^{-\lambda}.$$

Proof.

Note that $\sum_{k=1}^{n} A_{knr}$ and $\sum_{k=1}^{n} a_{knr}$ are both sums of independent random variables. Now

$$\mathrm{Var}\left(\sum_{k=1}^{n} A_{knr}\right) = \sum_{k=1}^{n} \mathrm{Var}A_{knr} \le \sum_{k=1}^{n} \frac{c_{kn}^{*2}}{n} A'^2M^2 \left(\sum_{\ell=1}^{n} \frac{|d_{kn}-d_{\ell n}|}{n}\right)^2 c_1^2\alpha_n^2$$

$$\le A'^2M^2 c_1^2 \frac{\alpha_n^2}{n} \sum_{k=1}^{n} c_{kn}^{*2}D_{kn}^2.$$

Similarly, $\mathrm{Var}\left(\sum_{k=1}^{n} a_{knr}\right) \le A'^2M^2 c_1^2 \frac{\alpha_n^2}{n} \sum_{k=1}^{n} c_{kn}^{*2}D_{kn}^2$

Also $\max_{1\le k\le n} |A_{knr}| \le \max_{1\le k\le n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} D_{kn}c_1A'M\alpha_n \equiv B_n \frac{c_1A'M\alpha_n}{n^{\frac{1}{2}}}$

and $\max_{1\le k\le n} |a_{knr}| \le \max_{1\le k\le n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}} D_{kn}c_1A'M\alpha_n \equiv B_n \frac{c_1A'M\alpha_n}{n^{\frac{1}{2}}}$.

Now by Bernstein's inequality,

$$\left.\begin{array}{l} P(|A_{knr}-EA_{knr}| > c^*\psi_5(n)) \\ P(|a_{knr}-Ea_{knr}| > c^*\psi_5(n)) \end{array}\right\}$$

$$\leq 2\ \exp(-c^{*2}\psi_5^2(n)/(2A'^2M^2\ c_1^2\ \frac{\alpha_n^2}{n} \sum_{k=1}^{n} c_{kn}^{*2}D_{kn}^2 + \frac{4}{3}\ c_1\ \frac{c^* A'M}{n^{\frac{1}{2}}}\ B_n\alpha_n\psi_5(n)))$$

$$\leq 2\ \exp(-c^{*2}\eta_n^2/(2A'^2M^2\ \frac{c_1^2\alpha_n^2}{n} + \frac{4}{3}\ c_1c^*A'M\ \frac{\alpha_n\eta_n}{n^{\frac{1}{2}}}\ ))$$

$$= 2\ \exp(-c^{*2}n^{\frac{1}{2}}\ \frac{\eta_n}{\alpha_n}\ /(2A'^2M^2\ c_1^2\ \frac{\alpha_n}{\eta_n n^{\frac{1}{2}}} + \frac{4}{3}\ c_1c^*A'M)).$$

By (4.d) and (4.e) for all n sufficiently large, the above is $\leq$ $2\ \exp(-c^{*2}\beta \log n/(2A'^2M^2c_1^2\alpha + \frac{4}{3}\ c_1c^*A'M\ )) \equiv \rho_n$.

Now by Bonferroni's inequality,

(4.3.3) $\leq 2n(2\ b_n + 3)\ \rho_n$ for all n sufficiently large. By (4.f) for all n sufficiently large

$$(4.3.3) \qquad \leq 2\ n(2n^{\varepsilon} + 3)\ \rho_n \qquad (4.3.4)$$

If we choose c^* sufficiently large, we have (4.3.4) $< n^{-\lambda}$.□

Proposition 4.4.

Assume (2.1.i) or (2.1.ii). Then there exists a $c > 0$ depending only on $a(u)$ and f such that for all Θ,

$$\Big|\sum_{k=1}^{n} \frac{c_{kn}^*}{n^{\frac{1}{2}}}\ (Ea(E(\tilde{F}_{kn}(\Theta_o)|X_{kn}))-Ea(E(\tilde{F}_{kn}(\Theta)|X_{kn}))$$

$$-(\Theta_o-\Theta)(d_{kn}-\bar{d}) \int_{-\infty}^{\infty} a'(\frac{(n-1)}{n}\ F(u) + \frac{1}{n})f^2(u)du)\Big| \leq c \sum_{k=1}^{n} \frac{|c_{kn}^*|}{n^{\frac{1}{2}}}\ D_{kn}^2(\Theta_o-\Theta)^2.$$

Proof.

Note first that $Ea(E(\tilde{F}_{kn}(\Theta)|X_{kn}))$

$$= \int_{-\infty}^{\infty} a(\frac{1}{n} \sum_{\ell\neq k}^{n} F(u+(d_{kn}-d_{\ell n})(\Theta_o-\Theta)) + \frac{1}{n})f(u)du$$

and

$$\left| \int_{-\infty}^{\infty} \Big([a(\tfrac{(n-1)}{n} F(u) + \tfrac{1}{n}) - a(\tfrac{1}{n} \sum_{\ell \neq k}^{n} F(u+(d_{kn}-d_{\ell n})(\Theta_o-\Theta)) + \tfrac{1}{n})] f(u)$$

$$- (d_{kn}-\bar{d})(\Theta_o-\Theta) \; a'(\tfrac{(n-1)}{n} F(u) + \tfrac{1}{n}) \; f^2(u) \Big) du \right|$$

$$\leq \left| \int_{-\infty}^{\infty} a'(\tfrac{(n-1)}{n} F(u) + \tfrac{1}{n}) [\tfrac{1}{n} \sum_{\ell=1}^{n} \Big(F(u+(d_{kn}-d_{\ell n})(\Theta_o-\Theta))-F(u)\Big) f(u)$$

$$- (d_{kn}-\bar{d})(\Theta_o-\Theta) f^2(u)] du \right| \tag{4.4.i}$$

$$+ \frac{A''}{2} \int_{-\infty}^{\infty} \Big(\sum_{\ell=1}^{n} \frac{F(u+(d_{kn}-d_{\ell n})(\Theta_o-\Theta))-F(u)}{n} \Big)^2 f(u) du. \tag{4.4.ii}$$

The above follows from Taylor's theorem.

Now, (4.4.ii)

$$\leq \frac{A''}{2} M^2 \Big(\sum_{\ell=1}^{n} \frac{|d_{kn}-d_{\ell n}|}{n} \Big)^2 (\Theta_o-\Theta)^2 \leq \frac{A''M^2}{2} D_{kn}^2 \; (\Theta_o-\Theta)^2.$$

To bound (4.4.i) by a similar term, we need a lemma.

<u>Lemma 4.4.1</u>.

If (2.1.i) or (2.1.ii), there exists a $c^* > 0$ dependent only on f and a(u) such that for all Θ

$$\left| \int_{-\infty}^{\infty} a'(\tfrac{(n-1)}{n} F(u) + \tfrac{1}{n}) [\Big(F(u+(d_{kn}-d_{\ell n})(\Theta_o-\Theta))-F(u)\Big) f(u)$$

$$- (d_{kn}-d_{\ell n})(\Theta_o-\Theta) f^2(u)] du \right| \leq c^* (d_{kn}-d_{\ell n})^2 (\Theta_o-\Theta)^2. \tag{4.4.2}$$

Proof.

Assume (2.1.i) $\int_{-\infty}^{\infty} |f'(u)|du < \infty$, then

$$\int_{-\infty}^{\infty} a'(\frac{(n-1)}{n}F(u) + \frac{1}{n})[\Big(F(u+(d_{kn}-d_{\ell n})(\Theta_o-\Theta))-F(u)\Big)f(u)-(d_{kn}-d_{\ell n})(\Theta_o-\Theta)f^2(u)]du$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\Theta_o-\Theta} a'(\frac{(n-1)}{n}F(u) + \frac{1}{n})\Big(f(u+(d_{kn}-d_{\ell n})v)-f(u)\Big)f(u)dvdu(d_{kn}-d_{\ell n})$$

which by Fubini's Theorem

$$= \int_{0}^{\Theta_o-\Theta}\int_{-\infty}^{\infty} a'(\frac{(n-1)}{n}F(u) + \frac{1}{n})\Big(f(u+(d_{kn}-d_{\ell n})v)-f(u)\Big)f(u)dudv(d_{kn}-d_{\ell n}).$$

Now,

$$\int_{-\infty}^{\infty} a'(\frac{(n-1)}{n}F(u) + \frac{1}{n})\Big(f(u+(d_{kn}-d_{\ell n})v) - f(u)\Big)f(u)du$$

equals by integration by parts to

$$a'(\frac{(n-1)}{n}F(u) + \frac{1}{n})f(u)\Big(F(u+(d_{kn}-d_{\ell n})v)-F(u)\Big)\Big|_{-\infty}^{\infty}$$

$$-\int_{-\infty}^{\infty}[\Big(F(u+(d_{kn}-d_{\ell n})v)-F(u)\Big)\Big(a''(\frac{(n-1)}{n}F(u) + \frac{1}{n})f^2(u)\frac{(n-1)}{n}$$

$$+ a'(\frac{(n-1)}{n}F(u) + \frac{1}{n})f'(u)\Big)]du.$$

The absolute value of the first term is bounded by

$$A'M^2|d_{kn}-d_{\ell n}||v|$$

and the absolute value of the second term is bounded by

$$M|d_{kn}-d_{\ell n}|\ |v|\ \{\int_{-\infty}^{\infty} A''Mf(u)du + A'\int_{-\infty}^{\infty}|f'(u)|du\}.$$

Let $2c^* = A'M^2 + A''M^2 + A'M \int_{-\infty}^{\infty} |f'(u)|du$, then we have (4.4.2)

$$\leq \left|\int_0^{\Theta_0-\Theta} 2c^*|v|dv(d_{kn}-d_{\ell n})^2\right| = c^*(\Theta_o-\Theta)^2(d_{kn}-d_{\ell n})^2.$$

Now assume (2.1.ii) $|f'| < M'$. The conclusion follows from observing that

$$\left|\int_{-\infty}^{\infty} a'(\frac{(n-1)}{n} F(u) + \frac{1}{n})(f(u+(d_{kn}-d_{\ell n})v)-f(u))f(u)du\right|$$

$$\leq A'M'|v|\ |d_{kn}-d_{\ell n}|^2.\Box$$

To complete the proof of Proposition 4.4, note that (4.4.i)

$$\leq \sum_{\ell=1}^{n} \frac{1}{n} \left|\int_{-\infty}^{\infty} a'(\frac{(n-1)}{n} F(u) + \frac{1}{n})[(F(u+(d_{kn}-d_{\ell n})(\Theta_o-\Theta))\right.$$

$$\left.-F(u))f(u)-(d_{kn}-d_{\ell n})(\Theta_o-\Theta)f^2(u)]du\right|,$$

which by Lemma 4.4.1 is

$$\leq c^* \sum_{\ell=1}^{n} \frac{(d_{kn}-d_{\ell n})^2}{n} (\Theta_o-\Theta)^2 = (\Theta_o-\Theta)^2\ c^*D_{kn}^2.$$

Now let $c = \frac{A''M^2}{2} + c^*$.

This completes the proof of Proposition 4.4.□

Remark 4.1.

The only conditions on α_n, γ_n and η_n used in the proof of Theorem 2.1 are (4.a) through (4.f). So as long as these conditions are satisfied, Theorem 2.1 is valid for the more general interval $I_n(\Theta_o) = [-c_1\alpha_n + \Theta_o, \Theta_o + c_1\alpha_n]$ with $\psi(n) = \psi_1(n) \vee \dots \vee \psi_6(n)$.

Remark 4.2.

The proof of Theorem 2.1 can be modified to give an alternate proof of Jurečková's Theorem 3.1 (1969) in the case when a(u) is absolutely continuous inside (0,1). The essential ingredient is Lemma 5.1 Hájek (1968).

Section 5. Comparison of Theorem 2.1 with Theorem 3.2 Sen and Ghosh (1972)

Jurečková (1969) showed that if a(u) is nondecreasing on (0,1), $\int_0^1 a^2(u)du < \infty$, $\max_{1\leq \ell \leq n} |c^*_{\ell n}| = 0$, $\sum_{\ell=1}^n c_{\ell n} = 0$, $(c_{\ell n}-c_{kn})(d_{\ell n}-d_{kn}) \geq 0$ $1 \leq k,\ell \leq n$, $\overline{\lim}_{n\to\infty} \sigma^2_{nd} < \infty$ and f has a finite Fisher information number, then the linearily is probability statement for $T^*(\Theta)$ with $a(\frac{\ell}{n})$ replaced by $a(\frac{\ell}{n+1})$ given in Section 1 holds with $A_n = \Gamma_n \int_0^1 a(u)\varphi(u,f)du$, where $\varphi(u,f) = -f'(F^{-1}(u))/f(F^{-1}(u))$. The question naturally arises: When does a stronger kind of linearity hold? An SSO rate is needed in the construction of bounded length sequential confidence intervals for Θ_o, (See for example Sen and Ghosh (1972) or in the determination of the exact rate for which estimates for Θ_o based on $T_n(\Theta)$ converge to Θ_o. Inagaki (1975) considers the particular case of estimates of the shift based on the Wilcoxon Two Sample Statistic. The strongest published result on the SSO question is given by Sen and Ghosh (1972) for the case of regression rank statistics.

We quote Theorem 3.2 Ghosh and Sen (1972). Ghosh and Sen consider linear rank statistics of the form: (The following is in our notation).

Let $X_{1n},\ldots,X_{nn}$ be independent such that $X_{\ell n} \overset{d}{\sim} F(\cdot + d_{\ell n}\Theta_o)$ for $\ell=1,\ldots,n$.

Set $T_n(\Theta) = \sum_{\ell=1}^n (d_{\ell n}-\bar{d})a(R^{\Theta}_{\ell n}/(n+1))$, i.e. $c_{\ell n} = d_{\ell n}-\bar{d}$. (5.1)

Assume that there exists a $K_o > o$ such that for $u \in (o,1)$,

$$|a(u)| \leq -K_o \log[u(1-u)] \text{ and} \quad (5.2)$$
$$|a'(u)| \leq K_o(u(1-u))^{-1}.$$

(5.3) F has density f which is absolutely continuous with derivative f' such that

(5.3.1) there exist M>0 and M'>0 such that

$|f| < M$ and $|f'| < M'$, and

(5.3.2) $\lim_{x\to\pm\infty} f(x)a'(F(x))$ is finite

(5.4) $\max_{1\leq \ell\leq n} |d_{\ell n}-\bar{d}|/ (\sum_{\ell=1}^{n} (d_{\ell n}-\bar{d})^2)^{\frac{1}{2}} = 0(1/\sqrt{n})$

(5.5) $\lim_{n\to\infty} \sigma_{nd}^2 = \sigma_d^2 > 0$

Then for all $c_1 > 0$, integers $k\geq 1$ and all δ such that $0<\delta<\frac{1}{4}$,

$$(5.6)\equiv \sup_{\Theta\in I_n(\Theta_o)} | T_n^*(\Theta_o)-T_n^*(\Theta) - (\Theta_o-\Theta)\sigma_d \int_{-\infty}^{\infty} a'(F(u))f^2(u)du| = SSO((\log n)^{k+1}/n^{\frac{1}{2}+\delta}),$$

where $I_n(\Theta_o) = [-c_1 \frac{(\log n)^k}{n^{\frac{1}{2}}} + \Theta_o, \Theta_o + c_1 \frac{(\log n)^k}{n^{\frac{1}{2}}}]$.

Comparison with Theorem 2.1

Observe that if $|a'(u)| < A'$ and $|a''(u)| < A''$ (5.7)

then (5.3.2) is automatically satisfied.

(5.4) combined with (5.7) applied to Theorem 2.1 gives

$$(5.6) = SSO((\log\log n)^{\frac{1}{4}}(\log n)^{\frac{1}{2}}/n^{3/4}),$$

where $I_n(\Theta_o) = [-c_1\frac{(\log\log n)^{\frac{1}{2}}}{n^{\frac{1}{2}}} + \Theta_o,\Theta_o+c_1 \frac{(\log\log n)^{\frac{1}{2}}}{n^{\frac{1}{2}}}]$,

which is a finer rate than that obtainable from Theorem 3.2. Refer to Example 3.1. The smaller interval enables us to obtain law of the iterated logarithm rates of convergence of estimates of Θ_o based on $T_n(\Theta)$.

Theorem 3.1 Sen and Ghosh (1972) allows a moderately unbounded a(u) function, but requires a strict Noether condition on the $c_{\ell n}$'s. Theorem 2.1 in a sense reverses the restrictions. If we are granted (5.7), Theorem 2.1 becomes much more versatile that Theorem 3.2. The $c_{\ell n}$'s need not be of the form $c_{\ell n} = d_{\ell n}-\bar{d}$, the exchange of (5.2) for (5.7) permits condition (5.4) to be relaxed or dispensed with (refer to Example 3.4), and there is a finer tuning of the SSO term, since it is a function of the $c_{\ell n}$'s, $d_{\ell n}$'s and the particular α_n's, γ_n's and η_n's chosen. See Remark 4.1. Almost sure linearity with an SSO rate can now be considered for linear rank statistics for which Theorem 3.2 does not apply; for example, Spearman's Rho (Example 3.1.2). In addition, Theorem 2.1 extends the class of density functions for which an SSO rate holds. Not only can an SSO rate be obtained when the underlying distribution function has a density which satisfies (2.1.ii) but also the SSO rate can be obtained when only (2.1.i) is satisfied.

REFERENCES

Adichie, J. N. (1967): Estimates of regression coefficients based on rank tests. Ann. Math. Statist., 38:894-904.

Bahadur, R. R. (1966): A note on quantiles in larger samples. Ann. Math. Statist., 37:577-580.

Bennet, G. (1962): Probability inequalities for the sum of independent random variables. J. Amer. Statist. Assoc., 57:33-45.

Dvorestzky, A., Kiefer, J. and Wolfowitz (1956): Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator. Ann. Math. Statist., 27:642-669.

Geertsema, J. C. (1970): Sequential confidence intervals based on rank tests. Ann. Math. Statist., 41:1016-1026.

Ghosh, M. and Sen, P. K. (1971): Sequential confidence interval for the regression coefficient based on Kendall's Tau. Calcutta Statist. Assoc. Bull., 20:23-36.

Ghosh, M. and Sen, P. K. (1972): On bounded length confidence based on a class of rank statistics. Sankhya A, 34:33-52.

Ghosh, M. (1972): On some properties of a class of Spearman rank statistics with applications. Ann. Inst. Statist. Math. 24:57-68.

Ghosh, M. and Sen, P. K. (1973): On some sequential simultaneous confidence intervals procedures. Ann. Inst. Statist. Math., 25:123-133.

Hájek, J. and Šidák, Z. (1967): Theory of Rank Tests, Academic Press.

Hájek, J. (1968): Asymptotic normality of simple linear rank statistics under alternatives. Ann. Math. Statist., 19:293-325.

Hodges, J. L., Jr. and Lehmann, E. L. (1963): Estimates of location based on rank test. Ann. Math. Statist., 34:598-611.

Hoeffding, W. (1963): Probability inequalities for sums of bounded random variables. J. Amer. Statist. Assoc., 58:13-29.

Inagaki, N. (1973): The asymptotic representation of the Hodges-Lehmann estimate based on Wilcoxon two sample statistics. Ann. Inst. Statist. Math., 25:457-466.

Jurečková, J. (1969): Asymptotic linearity of a rank statistic in regression parameter. Ann. Math. Statist., 40:1889-1900.

Jurečková, J. (1971): Almost sure uniform asymptotic linearity of rank statistics in regression parameter. Trans. Sixth. Conf. Information Theor. Statist. Decis. F. 305-313.

Jurečková, J. (1971b): Nonparametric estimate of regression coefficients. Ann. Math. Statist., 42:1328-1338.

Jurečková, J. (1973): Central limit theorem for Wilcoxon rank statistics process. Ann. of Statist., 1:1046-1060.

Kiefer, J. On Bahadur's representation of sample quantiles. Ann. Math. Statist. 1967, 1323-1342.

Sen, P. K. and Ghosh, M. (1971): On bounded length sequential confidence intervals based on one sample rank order statistics. Ann. Math. Statist., 42:189-203.

Sen, P. K. and Ghosh, M. (1972): On strong convergence of regression rank statistics. Sankhya A, 34:335-348.

Steyn, H. S., Jr. and Geerstema, J. C. (1974): Nonparametric confidence sequences for the centre of symmetry of a symmetric distribution. S. Afr. Statist. J., 8:25-34.

Van Eeden, C. (1968): Nonparametric Estimation, Les Presses de l'Universite de Montreal.

Van Eeden, C. (1972): An analogue for signed statistics of Jurečková's asymptotic linearity theorem for rank statistics. Ann. Math. Statist., 43:791-802.