# POWER COMPARISONS OF TWO-SIDED TESTS OF EQUALITY OF TWO COVARIANCE MATRICES AND SOME DISTRIBUTION THEORY

by

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# CHAPTER I

# POWER COMPARISONS OF TWO-SIDED TESTS OF EQUALITY OF TWO COVARIANCE MATRICES BASED ON SIX CRITERIA

## 1. Introduction and Summary

Let $\underset{\sim}{X}_1$ $(p \times n_1)$ and $\underset{\sim}{X}_2(p \times n_2)$, $p \leq n_i$, $i = 1,2$, be independent matrix variates, columns of $\underset{\sim}{X}_1$ being independently distributed as $N(\underset{\sim}{0},\underset{\sim}{\Sigma}_1)$ and those of $\underset{\sim}{X}_2$ independently distributed as $N(\underset{\sim}{0},\underset{\sim}{\Sigma}_2)$. Let $0 < c_1 \leq \ldots \leq c_p < \infty$ be the characteristic roots of $|\underset{\sim}{X}_1\underset{\sim}{X}_1' - c\underset{\sim}{X}_2\underset{\sim}{X}_2'| = 0$ and $\lambda_1,\ldots,\lambda_p$, the characteristic roots of $|\underset{\sim}{\Sigma}_1 - \lambda\underset{\sim}{\Sigma}_2| = 0$. Power studies of tests of $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ or equivalently $\lambda_1 = \ldots = \lambda_p = 1$ against the alternative of a one-sided nature:

$$\lambda_i \geq 1, \quad \sum_{i=1}^{p} \lambda_i > p, \quad i = 1,\ldots,p,$$

were carried out by Pillai and Jayachandran [29] based on the following four criteria:

1) Roy's largest root, $c_p$, [Roy, 36] or $L_p^{(p)} = c_p/(1+c_p)$,

2) Hotelling's trace, $U^{(p)} = \sum_{i=1}^{p} c_i$, [Pillai,20],

3) Pillai's trace, $V^{(p)} = \sum_{i=1}^{p} [c_i/(1+c_i)]$, [Pillai,20],

4) Wilks' criterion, $W^{(p)} = \prod_{i=1}^{p} (1+c_i)^{-1}$, [Wilks, 43].

Exact power tabulations were made in the two-roots case for various $(\lambda_1,\lambda_2)$ and different degrees of freedom $n_1$ and $n_2$ (actually in terms of $m = (n_1-p-1)/2$ and $n = (n_2-p-1)/2$). Power comparisons were also made of tests of the above hypothesis against the one-sided alternative, based on each $c_i$ (equivalently $c_i/(1+c_i)$), $i = 1,\ldots,p$, in the two- and three-roots cases [Pillai and Al-Ani, 24]. In this chapter, a power comparison study has been attempted of tests of the hypothesis $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$ based on the above four criteria as well as

5) Roy's largest-smallest roots, $c_1$ ($L_1^{(p)} = c_1/(1+c_1)$) and $c_p$ ($L_p^{(p)} = c_p/(1+c_p)$), to be denoted by $LS^{(p)}$ defined in terms of $L_p^{(p)}$ and $L_1^{(p)}$, [Roy, 37],

6) Modified likelihood ratio (m$\ell$r), $Z^{(p)} = \prod_{i=1}^{p} [c_i^{\frac{1}{2}n_1}/(1+c_i)^{\frac{1}{2}(n_1+n_2)}]$, [Anderson, 1 ; Bartlett, 2].

Sugiura and Nagao [41] have shown the unbiasedness of the m$\ell$r test $Z^{(p)}$. The distribution of $Z^{(p)}$ for $p = 1$ and $2$ and $n_1 = n_2$ in the null case has been available in the literature for sometime [Anderson, 1]. Powers of $Z^{(2)}$ for $n_1 = n_2$ have been tabulated by Pillai and Young [34], after deriving the appropriate non-central distribution. Conditions for the unbiasedness of the largest-smallest roots test $LS^{(p)}$ could be obtained from the treatment of Roy [38]. Since the critical values a and b obtained through the "condition of local unbiasedness" are difficult to compute, Thompson has suggested an approximate approach [42] given by

$$(1.1)\qquad P(a \leq L_1^{(p)}) = 1 - \tfrac{1}{2}\alpha \text{ and } P(a \leq L_1^{(p)} \leq L_p^{(p)} \leq b) = 1 - \alpha.$$

(actually Thompson has suggested the approach for the one sample i.e. Wishart case). This will be called $LS_1^{(p)}$. A variation of $LS_1^{(p)}$, called $LS_2^{(p)}$, is also considered in this chapter, which is given by

$$(1.2)\qquad P(L_p^{(p)} \leq b) = 1 - \tfrac{1}{2}\alpha \text{ and } P(a \leq L_1^{(p)} \leq L_p^{(p)} \leq b) = 1-\alpha.$$

Further, Krishnaiah [40] has suggested the following alternate approach:

$$(1.3)\qquad P(1-b \leq L_1^{(p)} \leq \ldots \leq L_p^{(p)} \leq b) = 1-\alpha.$$

This will be called $LS_3^{(p)}$. Obviously, $LS_1^{(p)}$, $LS_2^{(p)}$ and $LS_3^{(p)}$ provide biased tests. Similarly the tests 1) to 4) are biased.

In this chapter, a theorem is proved first obtaining the condition of local unbiasedness for a class of tests of which 1) to 5) are special cases. Using the theorem, relations between the two critical values for each of the five tests are obtained as special cases for tests 1) to 5) for the two-roots case. Further, critical values for level $\alpha = .05$ (five percent points) for the five tests are computed for $p = 2$ and values of $m = 0,1,2,5$ and $n = 5, 10, 15, 20, 25, 30, 40, 60, 80, 100$, and are given in Table 1. Also, powers of the criteria 1) to 5) have been tabulated for various values of $(\lambda_1,\lambda_2)$, $m = 0,1,2,5$ and $n = 5, 15, 30, 60$, and these are presented in Table 4. In addition, power tabulations have also been carried out from the equal tail areas point of view, of tests 1) to 4) which are observed to be biased although the bias is not serious. These tabulations are also available in Table 4 for the same values of $(\lambda_1,\lambda_2)$, m and n as before facilitating comparisons with powers in the unbiased case. The critical values in this case are also given in Table 1.

For studying the m$\ell$r test $Z^{(p)}$ and comparing its powers with those of others, the non-central distribution of $Z^{(2)}$ is obtained using zonal polynomials up to the sixth degree for $n_2 = 2n_1$. Tabulations of powers of $Z^{(2)}$ are carried out obtaining the lower five percent points for $n_1 = 3, 5, 7, 13$ and for comparison those of test 1) to 5) in the unbiased case and 1) to 4) in the equal tail areas case. These are given in Table 5. The critical values are given in Table 2.

In order to compare the largest-smallest roots test with the three approximations as well as the largest root, a separate study has been made and the results on the powers of these five tests are given in Table 6 for selected $(\lambda_1,\lambda_2)$ and m = 0, 1, 2, 5 and n = 5, 15, 30 and the critical values in Table 3. The approximations $LS_1^{(2)}$, $LS_2^{(2)}$ and $LS_3^{(2)}$ are all biased and the largest root seems to fare better than all except for the largest-smallest roots for two-sided larger deviations.

A few findings seem to emerge from the numerical results of powers tabulated and in general it is observed that the largest root has some power advantage over the other criteria studied since it is less (very slightly) biased, and except for two-sided larger deviations, has better power generally than $Z^{(2)}$ and the largest-smallest roots.

These findings are presented in Section 7. In view of this, condition of local unbiasedness for $p = 3$ has been studied for the largest root and critical values obtained in the unbiased as well as equal tail areas cases which are presented in Table 1. The condition of local unbiasedness has been explicitly obtained in Section 4 for $U^{(3)}$ as well.

## 2. The condition of local unbiasedness

The acceptance regions based on criteria 1) to 5) with local unbiasedness property and $\alpha$ level of significance can be written in one form:

$$R: \quad a(p,n_1,n_2) \leq w(c_1,\ldots,c_p) \leq b(p,n_1,n_2),$$

where a and b are so chosen as to satisfy

$$(2.1) \qquad (i) \quad P(a \leq w(c_1,\ldots,c_p) \leq b|\lambda_1 = \ldots = \lambda_p = 1) = 1-\alpha,$$

$$(2.2) \qquad (ii) \quad \left.\frac{\partial P(a \leq w(c_1,\ldots,c_p) \leq b|\lambda_1,\ldots,\lambda_p)}{\partial \lambda_i}\right|_{\lambda_1=\ldots=\lambda_p=1} = 0, \; i=1,\ldots,p,$$

where $w(c_1,\ldots,c_p) = c_p/(1+c_p)$ for test 1), $\sum_{i=1}^{p} c_i$ for 2), $\sum_{i=1}^{p} [c_i/(1+c_i)]$ for 3), $\prod_{i=1}^{p} (1+c_i)^{-1}$ for 4) and $c_1/(1+c_1)$, $c_p/(1+c_p)$ for 5).

In this section, we will show that the p equations given in (ii) are really equivalent to one equation and are in turn equivalent to

$$(2.3) \qquad (ii') \quad \left.\frac{\partial P(a \leq w(c_1,\ldots,c_p) \leq b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial \lambda}\right|_{\lambda=1} = 0.$$

This enables us to compute a and b in a much simpler way for each of the five criteria. We call (ii) or equivalently (ii') "the condition of local unbiasedness".

Theorem 1. The p equations $\left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial\lambda_i}\right|_{\lambda_1=\ldots=\lambda_p=1}=0$,

$i = 1,\ldots,p$, are equivalent to one equation and are in turn equivalent to

$$\left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0.$$

Proof. The joint density of $\underset{\sim}{X}_1$, $\underset{\sim}{X}_2$ defined in the Introduction and Summary is given by

$$(2\pi)^{-\frac{1}{2}p(n_1+n_2)}|\underset{\sim}{\Sigma}_1|^{-\frac{1}{2}n_1}|\underset{\sim}{\Sigma}_2|^{-\frac{1}{2}n_2}\exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{X}_1\underset{\sim}{X}_1' + \underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{X}_2\underset{\sim}{X}_2')].$$

Without loss of generality, we may start directly from the following canonical form:

$$(2\pi)^{-\frac{1}{2}p(n_1+n_2)}\prod_{i=1}^{p}\lambda_i^{-\frac{1}{2}n_1}\exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{X}_1\underset{\sim}{X}_1' + \underset{\sim}{X}_2\underset{\sim}{X}_2')],$$

where $\underset{\sim}{D}_{\lambda_k^{-1}} = \mathrm{diag}(1/\lambda_k)$. Then

$$P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)$$

$$= (2\pi)^{-\frac{1}{2}p(n_1+n_2)}\int\limits_{a\le w(c_1,\ldots,c_p)\le b}\prod_{i=1}^{p}\lambda_i^{-\frac{1}{2}n_1}\exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{X}_1\underset{\sim}{X}_1'+\underset{\sim}{X}_2\underset{\sim}{X}_2')]d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Hence

$$(2.4)\qquad \frac{\partial P(a\le w(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial\lambda_i^{-1}}$$

$$=(2\pi)^{-\frac{1}{2}p(n_1+n_2)}\int\limits_{a\le w(c_1,\ldots,c_p)\le b}\prod_{j=1}^{p}\lambda_j^{-\frac{1}{2}n_1}[\tfrac{1}{2}n_1\lambda_i-\tfrac{1}{2}(\underset{\sim}{X}_1\underset{\sim}{X}_1')_{ii}]\exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{X}_1\underset{\sim}{X}_1' + \underset{\sim}{X}_2\underset{\sim}{X}_2')] \times d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Transform $\underset{\sim}{X}_1 = \underset{\sim}{U}\underset{\sim}{D}_{\sqrt{c_k}}\underset{\sim}{L}_1$, $\quad \underset{\sim}{X}_2 = \underset{\sim}{U}\underset{\sim}{L}_2$,

where $\underset{\sim}{U}$ is non-singular and $\underset{\sim}{L}_1\underset{\sim}{L}_1' = \underset{\sim}{L}_2\underset{\sim}{L}_2' = \underset{\sim}{I}$ and integrate out over $\underset{\sim}{L}_{1I}$ and $\underset{\sim}{L}_{2I}$, the independent elements of $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ respectively [Roy, 38]. Then (2.4) becomes

$$(2.5) \qquad c \int_{R^*} \prod_{j=1}^{p} \lambda_j^{-\frac{1}{2}n_1} [\tfrac{1}{2}n_1\lambda_i - \tfrac{1}{2}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')_{ii}] \exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}' + \underset{\sim}{U}\underset{\sim}{U}')]$$

$$\times |\underset{\sim}{U}|^{n_1+n_2-p} \, d\underset{\sim}{U} \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j,$$

where $R^*$ is $a \leq w(c_1,\ldots,c_p) \leq b$, $-\infty <$ all $u_{ij} < \infty$ and c is a positive and constant factor of proportionality. Thus we have

$$(2.6) \qquad \left.\frac{\partial P(a \leq w(c_1,\ldots,c_p) \leq b \mid \lambda_1,\ldots,\lambda_p)}{\partial \lambda_i^{-1}}\right|_{\lambda_1=\ldots=\lambda_p = 1}$$

$$= c \int_{R^*} [\tfrac{1}{2}n_1 - \tfrac{1}{2}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')_{ii}] \exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}' + \underset{\sim}{U}\underset{\sim}{U}')] |\underset{\sim}{U}|^{n_1+n_2-p} d\underset{\sim}{U}$$

$$\times \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j,$$

where $R^*$ is $a \leq w(c_1,\ldots,c_p) \leq b$, $-\infty <$ all $u_{ij} < \infty$. The only term in the above integrand which depends on i is $(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')_{ii} =$ $(u_{i1}^2 c_1 + u_{i2}^2 c_2 + \ldots + u_{ip}^2 c_p)$ and the integral is taken over the domain $R^*$. Therefore, $u_{i1},\ldots,u_{ip}$ are just dummy variables. Thus the integral is invariant under a change of the subscript i. So

$$\left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial \lambda_i^{-1}}\right|_{\lambda_1=\ldots=\lambda_p=1} \text{ is the same for } i=1,\ldots,p.$$

Hence the p equations are really equivalent to one equation. Now adding up the p formally different looking integrals like (2.6) over $i = 1,\ldots,p$, we have

$$(2.7) \quad c \int_{R^*} [\tfrac{1}{2}n_1 p - \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')]\exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}'+\underset{\sim}{U}\underset{\sim}{U}')]|\underset{\sim}{U}|^{n_1+n_2-p} d\underset{\sim}{U}$$

$$\times \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j,$$

where $R^*$ is $a \le w(c_1,\ldots,c_p) \le b$, $-\infty <$ all $u_{ij} < \infty$. We claim that this is the same as $\left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$.

In order to see this, consider

$$P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p=\lambda)$$

$$= (2\pi)^{-\frac{1}{2}p(n_1+n_2)} \int_{a \le w(c_1,\ldots,c_p) \le b} \lambda^{-\frac{1}{2}n_1 p} \exp[-\tfrac{1}{2}\mathrm{tr}(\lambda^{-1}\underset{\sim}{X}_1\underset{\sim}{X}_1'+\underset{\sim}{X}_2\underset{\sim}{X}_2')]d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Then we have

$$(2.8) \quad \left.\frac{\partial P(a \le w(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$=(2\pi)^{-\frac{1}{2}p(n_1+n_2)} \int_{a \le w(c_1,\ldots,c_p) \le b} [\tfrac{1}{2}n_1 p - \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{X}_1\underset{\sim}{X}_1')]\exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{X}_1\underset{\sim}{X}_1'+\underset{\sim}{X}_2\underset{\sim}{X}_2')]d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Now make the same transformation as before and integrate out over $L_{1I}$ and $L_{21}$. Then (2.8) becomes

$$c \int_{R^*}[\tfrac{1}{2}n_1 p-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')]\exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}'+\underset{\sim}{U}\underset{\sim}{U}')]|\underset{\sim}{U}|^{n_1+n_2-p}d\underset{\sim}{U}$$

$$\times \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'}(c_j-c_{j'})\prod_{j=1}^{p}dc_j,$$

where $R^*$ is $a \leq w(c_1,\ldots,c_p) \leq b$, $-\infty <$ all $u_{ij} < \infty$, which is the same as (2.7). Hence (ii) is equivalent to (ii').

Theorem 2. Condition (ii') can be written as

$$c(p,m,n)[\int_{a\leq w(c_1,\ldots,c_p)\leq b} \tfrac{1}{2}p(2m+p+1)\prod_{j=1}^{p}c_j^m(1+c_j)^{-(m+n+p+1)}\prod_{j>j'}(c_j-c_{j'})\prod_{j=1}^{p}dc_j -$$

$$(2.9)\quad \int_{a\leq w(c_1,\ldots,c_p)\leq b}(m+n+p+1)[\sum_{j=1}^{p}c_j/(1+c_j)]\prod_{j=1}^{p}c_j^m(1+c_j)^{-(m+n+p+1)}\prod_{j>j'}(c_j-c_{j'})\prod_{j=1}^{p}dc_j] = 0,$$

where $m = (n_1-p-1)/2$, $n = (n_2-p-1)/2$ and

$$(2.10)\quad c(p,m,n) = \pi^{\frac{1}{2}p}\prod_{i=1}^{p}\Gamma[\tfrac{1}{2}(2m+2n+p+i+2)]/\{\Gamma[\tfrac{1}{2}(2m+i+1)]\Gamma[\tfrac{1}{2}(2n+i+1)]\Gamma(\tfrac{1}{2}i)\}.$$

Proof. If $\underset{\sim}{U}$ in Theorem 1 is taken with a positive first row,

$$(2.11)\quad \left.\frac{\partial P(a\leq w(c_1,\ldots,c_p) \leq b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$= c \int_{R^*}[\tfrac{1}{2}n_1 p-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}')]\exp[-\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{U}\underset{\sim}{D}_{c_k}\underset{\sim}{U}'+\underset{\sim}{U}\underset{\sim}{U}')]|\underset{\sim}{U}|^{n_1+n_2-p}d\underset{\sim}{U}$$

$$\times \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} \prod_{j>j'} (c_j-c_{j'}) \prod_{j=1}^{p} dc_j,$$

where $c = \dfrac{2^p}{(2\pi)^{\frac{1}{2}p(n_1+n_2)}} \cdot \dfrac{\pi^{\frac{1}{2}pn_1-\frac{1}{4}p(p-1)}}{\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n_1-i+1)]} \cdot \dfrac{\pi^{\frac{1}{2}pn_2-\frac{1}{4}p(p-1)}}{\prod_{i=1}^{p}\Gamma[\frac{1}{2}(n_2-i+1)]}$

and R* is $a \le w(c_1,\ldots,c_p) \le b$, $\underset{\sim}{U}$ is nonsingular with a positive first row. Now transform $\underset{\sim}{U} = \underset{\sim}{B}\underset{\sim}{D}_{(1+c_k)^{-\frac{1}{2}}}$. Then (2.11) becomes

(2.12) $$c \int_{R^{**}} [\tfrac{1}{2}n_1p - \tfrac{1}{2}tr(\underset{\sim}{B}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\underset{\sim}{B}')]\exp[-\tfrac{1}{2}tr\underset{\sim}{B}\underset{\sim}{B}']|\underset{\sim}{B}|^{n_1+n_2-p} d\underset{\sim}{B}$$

$$\times \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)} (1+c_j)^{-\frac{1}{2}(n_1+n_2)} \prod_{j>j'} (c_j-c_{j'}) \prod_{j=1}^{p} dc_j,$$

where R** is $a \le w(c_1,\ldots,c_p) \le b$, $\underset{\sim}{B}$ is nonsingular with a positive first row. For integration with respect to $\underset{\sim}{B}$, use the result [see Roy,38, Appendix]

$$\int_{\underset{\sim}{B}} \exp[-\tfrac{1}{2}tr\underset{\sim}{B}\underset{\sim}{B}']|\underset{\sim}{B}|^q d\underset{\sim}{B} = 2^{\frac{1}{2}p(p+q)-p}\pi^{\frac{1}{2}p^2} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(q+p-i+1)]/\Gamma[\tfrac{1}{2}(p-i+1)].$$

We get

(2.13) $$c \int_{R^{**}} \tfrac{1}{2}n_1p \exp[-\tfrac{1}{2}tr\underset{\sim}{B}\underset{\sim}{B}']|\underset{\sim}{B}|^{n_1+n_2-p} d\underset{\sim}{B} \prod_{j=1}^{p} c_j^{\frac{1}{2}(n_1-p-1)}$$

$$(1+c_j)^{-\frac{1}{2}(n_1+n_2)} \prod_{j>j'} (c_j-c_{j'}) \prod_{j=1}^{p} dc_j$$

$$= c(p,m,n) \int_{a \le w(c_1,\ldots,c_p) \le b} \tfrac{1}{2}p(2m+p+1) \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j-c_{j'}) \prod_{j=1}^{p} dc_j,$$

where m, n and c(p,m,n) are given in (2.10). Now consider

$$(2.14)\quad \int_{B} \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{B}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\underset{\sim}{B}')\exp[\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}\underset{\sim}{B}']|\underset{\sim}{B}|^{n_1+n_2-p}d\underset{\sim}{B}.$$

Transform $\underset{\sim}{B} = \underset{\sim}{T}\underset{\sim}{L}$ where $\underset{\sim}{T}$ is lower triangular, $\underset{\sim}{L}\underset{\sim}{L}' = \underset{\sim}{I}(p)$ and the first row of $\underset{\sim}{L}$ is to be positive. Then (2.14) becomes

$$(2.15)\quad 2^{p}\int_{\underset{\sim}{L}\underset{\sim}{L}'=\underset{\sim}{I}}\int_{\underset{\sim}{T}} \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{T}\underset{\sim}{L}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\underset{\sim}{L}'\underset{\sim}{T}')\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{T}\underset{\sim}{T}']|\underset{\sim}{T}\underset{\sim}{T}'|^{\frac{1}{2}(n_1+n_2-p)}$$

$$\prod_{i=1}^{p} t_{ii}^{p-i}d\underset{\sim}{T}\; d\underset{\sim}{L}_I \Big/ \left|\frac{\partial \underset{\sim}{L}\underset{\sim}{L}'}{\partial \underset{\sim}{L}_D}\right|_{\underset{\sim}{L}_I}.$$

Furthermore, transform $\underset{\sim}{S} = \underset{\sim}{T}'\underset{\sim}{T}$, $J(\underset{\sim}{T}:\underset{\sim}{S}) = 2^{-p}\prod_{i=1}^{p} t_{ii}^{-p+i-1}$. Then (2.15) becomes

$$(2.16)\quad \int_{\underset{\sim}{L}\underset{\sim}{L}'=\underset{\sim}{I}}\int_{\underset{\sim}{S}>0} \tfrac{1}{2}\mathrm{tr}(\underset{\sim}{S}\underset{\sim}{L}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\underset{\sim}{L}')\exp[-\tfrac{1}{2}\underset{\sim}{S}]|\underset{\sim}{S}|^{\frac{1}{2}(n_1+n_2-p-1)}d\underset{\sim}{S}d\underset{\sim}{L}_I \Big/ \left|\frac{\partial \underset{\sim}{L}\underset{\sim}{L}'}{\partial \underset{\sim}{L}_D}\right|_{\underset{\sim}{L}_I}.$$

Apply equation (1) of Constantine [3], (2.16) becomes

$$\int_{\underset{\sim}{L}\underset{\sim}{L}'=\underset{\sim}{I}} \tfrac{1}{2}\Gamma_p(\tfrac{1}{2}(n_1+n_2),1)\mathrm{tr}(2\underset{\sim}{D}_{c_k(1+c_k)^{-1}})(\tfrac{1}{2})^{-\frac{1}{2}p(n_1+n_2)}d\underset{\sim}{L}_I \Big/ \left|\frac{\partial \underset{\sim}{L}\underset{\sim}{L}'}{\underset{\sim}{L}_D}\right|_{\underset{\sim}{L}_I}$$

$$=\{2^{-p}\pi^{\frac{1}{2}p^2-\frac{1}{4}p(p-1)}\Big/\prod_{i=1}^{p}\Gamma[\tfrac{1}{2}(p-i+1)]\}\tfrac{1}{2}\Gamma_p(\tfrac{1}{2}(n_1+n_2),1)\mathrm{tr}(2\underset{\sim}{D}_{c_k(1+c_k)^{-1}})\,2^{\frac{1}{2}p(n_1+n_2)}$$

$$=\{2^{\frac{1}{2}p(n_1+n_2)-p}\pi^{\frac{1}{2}p^2}\prod_{i=1}^{p}\Gamma[\tfrac{1}{2}(n_1+n_2-i+1)]/\Gamma[\tfrac{1}{2}(p-i+1)]\}\tfrac{1}{2}(n_1+n_2)[\sum_{j=1}^{p}c_j/(1+c_j)].$$

Thus we get

$$(2.17)\quad c\int_{R**}\tfrac{1}{2}\mathrm{tr}(\underset{\sim}{B}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\underset{\sim}{B}')\exp[-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{B}\underset{\sim}{B}']|\underset{\sim}{B}|^{n_1+n_2-p}d\underset{\sim}{B}$$

$$\times\prod_{j=1}^{p}c_j^{\frac{1}{2}(n_1-p-1)}(1+c_j)^{-\frac{1}{2}(n_1+n_2)}\prod_{j>j'}(c_j-c_{j'})\prod_{j=1}^{p}dc_j$$

$$= c(p,m,n) \int_{a \leq w(c_1,\ldots,c_p) \leq b} (m+n+p+1)\left[\sum_{j=1}^{p} c_j/(1+c_j)\right] \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)}$$

$$\prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j .$$

Therefore, by subtracting (2.17) from (2.13), we get

$$\left.\frac{\partial P(a \leq w(c_1,\ldots,c_p) \leq b | \lambda_1 = \ldots = \lambda_p = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$= c(p,m,n)\left[\int_{a \leq w(c_1,\ldots,c_p) \leq b} \tfrac{1}{2}p(2m+p+1) \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j - \right.$$

$$\left.\int_{a \leq w(c_1,\ldots,c_p) \leq b} (m+n+p+1)\left[\sum_{j=1}^{p} c_j/(1+c_j)\right] \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j - c_{j'}) \prod_{j=1}^{p} dc_j\right].$$

Now equating the above to zero, we get (2.9).

Thus, the acceptance region based on the criterion $w(c_1,\ldots,c_p)$ with local unbiasedness property ($\ell$up) and $\alpha$ level of significance can be written as

$$R: \quad a(p,m,n) \leq w(c_1,\ldots,c_p) \leq b(p,m,n),$$

where a and b are so chosen as to satisfy

$$\text{(i)} \quad c(p,m,n) \int_{a \leq w(c_1,\ldots,c_p) \leq b} \prod_{j=1}^{p} c_j^m (1+c_j)^{-(m+n+p+1)} \prod_{j>j'} (c_j - c_{j'})$$

$$\prod_{j=1}^{p} dc_j = 1-\alpha$$

and

$$\text{(ii')}\quad c(p,m,n)\Big[\int_{a\le w(c_1,\ldots,c_p)\le b} \tfrac{1}{2}p(2m+p+1)\prod_{j=1}^{p} c_j^m(1+c_j)^{-(m+n+p+1)}\prod_{j>j'}(c_j-c_{j'})\prod_{j=1}^{p} dc_j -$$

$$\int_{a\le w(c_1,\ldots,c_p)\le b}(m+n+p+1)\Big[\sum_{j=1}^{p} c_j/(1+c_j)\Big]\prod_{j=1}^{p} c_j^m(1+c_j)^{-(m+n+p+1)}$$

$$\times\prod_{j>j'}(c_j-c_{j'})\prod_{j=1}^{p} dc_j\Big] = 0.$$

3. <u>The acceptance regions based on the five criteria with $\ell$up for p=2</u>

In this section, we will consider the acceptance regions of tests 1) to 5) in that order.

1) <u>Roy's largest root, $L_2^{(2)} = c_2/(1+c_2)$.</u> By using Theorem 2 in the previous section, we know that

$$\text{(3.1)}\quad \left.\frac{\partial P(a\le L_2^{(2)} \le b\,|\,\lambda_1 = \lambda_2 = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$= c(2,m,n)\Big[\int_{a\le L_2^{(2)}\le b} g(c_1,c_2;m,n)dc_1dc_2 - \int_{a\le L_2^{(2)}\le b} h(c_1,c_2;m,n)\,dc_1dc_2\Big],$$

where $g(c_1,c_2;m,n)=(2m+3)(c_1c_2)^m[(1+c_1)(1+c_2)]^{-(m+n+3)}(c_2-c_1)$ and $h(c_1,c_2;m,n)=(m+n+3)[c_1/(1+c_1)+c_2/(1+c_2)](c_1c_2)^m[(1+c_1)(1+c_2)]^{-(m+n+3)}(c_2-c_1)$.

Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then (3.1) becomes

$$\text{(3.2)}\quad c(2,m,n)\Big[\int_a^b\int_0^{\ell_2} g_1(\ell_1,\ell_2;m,n)d\ell_1d\ell_2 - \int_a^b\int_0^{\ell_2} h_1(\ell_1,\ell_2;m,n)d\ell_1d\ell_2\Big],$$

where $g_1(\ell_1,\ell_2;m,n) = (2m+3)(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)$ and

$$h_1(\ell_1,\ell_2;m,n)=(m+n+3)(\ell_1+\ell_2)(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1).$$

Further, note that by making the same transformation,

(3.3) $P(a \leq L_2^{(2)} \leq b|\lambda_1 = \lambda_2 = 1)$

$$= c(2,m,n)\int_a^b \int_0^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2.$$

Now using Pillai's reduction formula, [Pillai,19,21],

(3.4) $\int_0^b \int_0^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2$

$$= [2\int_0^b \ell_2^{2m+1}(1-\ell_2)^{2n+1}d\ell_2 - b^{m+1}(1-b)^{n+1}\int_0^b \ell_1^m(1-\ell_1)^n d\ell_1]/(m+n+2),$$

and

(3.5) $\int_0^b \int_0^{\ell_2} (\ell_1+\ell_2)(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2$

$$= [2\int_0^b \ell_2^{2m+2}(1-\ell_2)^{2n+1}d\ell_2 - b^{m+2}(1-b)^{n+1}\int_0^b \ell_1^m(1-\ell_1)^n d\ell_1$$

$$+(m+2)\int_0^b \int_0^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2]/(m+n+3),$$

we have proved the following:

<u>Theorem 3</u>. Let $T_1(x) = 2B_x(2m+2,2n+2)-x^{m+1}(1-x)^{n+1}B_x(m+1,n+1)$ and $T_2(x)=2B_x(2m+3,2n+2)-x^{m+2}(1-x)^{n+1}B_x(m+1,n+1)$ where $B_x(r,s)=\int_0^x t^{r-1}(1-t)^{s-1}dt$.

Then the acceptance region based on Roy's largest root, $L_2^{(2)} = c_2/(1+c_2)$ with $\ell$up and $\alpha$ level is given by $a \leq L_2^{(2)} \leq b$ where a and b are so chosen as to satisfy

(i) $c(2,m,n)[T_1(b)-T_1(a)]/(m+n+2) = 1-\alpha$ and

(ii) $c(2,m,n)\{[T_1(b)-T_1(a)][(m+1)/(m+n+2)]-[T_2(b)-T_2(a)]\} = 0.$

2) Hotelling's trace, $U^{(2)} = c_1+c_2$. From the previous section

(3.6) $P(a \le U^{(2)} \le b|\lambda_1 = \lambda_2 = \lambda)$

$$= (2\pi)^{-(n_1+n_2)} \int_{a\le c_1+c_2\le b} \lambda^{-n_1} \exp[-\tfrac{1}{2}tr(\lambda^{-1}\underset{\sim}{X}_1\underset{\sim}{X}_1'+\underset{\sim}{X}_2\underset{\sim}{X}_2')]d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Now transform $\lambda^{-\frac{1}{2}}\underset{\sim}{X}_1 = \underset{\sim}{Y}_1$ and $\underset{\sim}{X}_2 = \underset{\sim}{Y}_2$. $J(\underset{\sim}{Y}_1:\underset{\sim}{X}_1) = \lambda^{-n_1}$ and let $0 < d_1 \le d_2 < \infty$ be the characteristic roots of $|\underset{\sim}{Y}_1\underset{\sim}{Y}_1'-d\underset{\sim}{Y}_2\underset{\sim}{Y}_2'| = 0$, then (3.6) becomes

(3.7) $$c(2,m,n)\int_{a\le\lambda d_1+\lambda d_2\le b}(d_1d_2)^m[(1+d_1)(1+d_2)]^{-(m+n+3)}(d_2-d_1)dd_1dd_2.$$

Let $u = d_1+d_2$ and $g = d_1d_2$. We get

$$c(2,m,n)\int_{a\lambda^{-1}}^{b\lambda^{-1}}\int_0^{\frac{1}{4}u^2} g^m/(1+u+g)^{m+n+3}\, dgdu.$$

Thus $\left.\dfrac{\partial P(a \le U^{(2)}\le b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$

(3.8) $$=c(2,m,n)[b\int_0^{\frac{1}{4}b^2} g^m/(1+b+g)^{m+n+3}dg-a\int_0^{\frac{1}{4}a^2} g^m/(1+a+g)^{m+n+3}dg].$$

Furthermore, transform $t = g/(1+b+g)$, (3.8) becomes

$$c(2,m,n)[b(1+b)^{-(n+2)}\int_0^{b^2/(b+2)^2} t^m(1-t)^{n+1}dt-a(1+a)^{-(n+2)}$$
$$\times\int_0^{a^2/(a+2)^2} t^m(1-t)^{n+1}dt].$$

And furthermore, $P(a \leq U^{(2)} \leq b \mid \lambda_1 = \lambda_2 = 1)$

$$(3.9) \qquad = c(2,m,n)\int_a^b \int_0^{\frac{1}{4}u^2} g^m/(1+u+g)^{m+n+3}dgdu.$$

Now using integration by parts, (3.9) becomes

$$c(2,m,n)\{[2\int_0^{b/(b+2)} t^{2m+1}(1-t)^{2n+2}dt-(1+b)^{-(n+1)}\int_0^{b^2/(b+2)^2} t^m(1-t)^{n+1}dt]$$

$$-[2\int_0^{a/(a+2)} t^{2m+1}(1-t)^{2n+2}dt-(1+a)^{-(n+1)}\int_0^{a^2/(a+2)^2} t^m(1-t)^{n+1}dt]\}/(n+1).$$

Therefore, we have proved the following:

Theorem 4. The acceptance region based on $U^{(2)} = c_1+c_2$ with $\ell$up and $\alpha$ level is given by $a \leq U^{(2)} \leq b$ where a and b are so chosen as to satisfy

(i) $c(2,m,n)\{[2B_{b/(b+2)}(2m+2,2n+3)-(1+b)^{-(n+1)}B_{b^2/(b+2)^2}(m+1,n+2)]$

$-[2B_{a/(a+2)}(2m+2,2n+3)-(1+a)^{-(n+1)}B_{a^2/(a+2)^2}(m+1,n+2)]\}/(n+1)=1-\alpha$ and

(ii) $c(2,m,n)\{b(1+b)^{-(n+2)}B_{b^2/(b+2)^2}(m+1,n+2)-a(1+a)^{-(n+2)}$

$\times B_{a^2/(a+2)^2}(m+1,n+2)\} = 0.$

3) Pillai's trace, $V^{(2)} = [c_1/(1+c_1)]+[c_2/(1+c_2)]$. From Theorem 2, we obtain $\left.\frac{\partial P(a \leq V^{(2)} \leq b \mid \lambda_1 = \lambda_2 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$ by replacing $L_2^{(2)}$ in the limits of the integrals in (3.1) by $V^{(2)}$. Now transform $\ell_1=c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then

$$(3.10)\quad \left.\frac{\partial P(a \le V^{(2)} \le b \mid \lambda_1 = \lambda_2 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$= c(2,m,n)\Big[\int_{a\le \ell_1+\ell_2\le b} g_1(\ell_1,\ell_2;m,n)\,d\ell_1 d\ell_2 - \int_{a\le \ell_1+\ell_2\le b} h_1(\ell_1,\ell_2;m,n)\,d\ell_1 d\ell_2\Big].$$

Let $v = \ell_1+\ell_2$ and $g = \ell_1\ell_2$. Then (3.10) becomes

$$c(2,m,n)\Big[(2m+3)\int_a^b \int_0^{\frac{1}{4}v^2} g^m(1-v+g)^n dg dv - (m+n+3)\int_a^b\int_0^{\frac{1}{4}v^2} v g^m(1-v+g)^n dg dv\Big].$$

and $P(a \le V^{(2)} \le b \mid \lambda_1=\lambda_2=1) = c(2,m,n)\int_0^b\int_0^{\frac{1}{4}v^2} g^m(1-v+g)^n dg dv.$

Therefore, we have the following Theorem:

<u>Theorem 5</u>. Let

$$T_1(x) = \begin{cases} [2/(m+1)]\sum_{\gamma=0}^{n}(-1)^\gamma\Big[\binom{n}{\gamma}\Big/\binom{m+\gamma+1}{\gamma}\Big]B_{\frac{1}{2}x}(2m+2\gamma+3,2n-2\gamma+1) \quad \text{if } 0 \le x \le 1, \\ [2/(m+1)]\sum_{\gamma=0}^{n}(-1)^\gamma\Big[\binom{n}{\gamma}\Big/\binom{m+\gamma+1}{\gamma}\Big]B_{\frac{1}{2}}(2m+2\gamma+3,2n-2\gamma+1) + \\ [2/(n+1)]\sum_{\gamma=0}^{m}(-1)^\gamma\Big[\binom{m}{\gamma}\Big/\binom{n+\gamma+1}{\gamma}\Big]\Big[B_{\frac{1}{2}x}(2m-2\gamma+1,2n+2\gamma+3) - B_{\frac{1}{2}}(2m-2\gamma+1,2n+2\gamma+3)\Big] \quad \text{if } 1 \le x \le 2, \end{cases}$$

and

$$T_2(x) = \begin{cases} [4/(m+1)]\sum_{\gamma=0}^{n}(-1)^\gamma\Big[\binom{n}{\gamma}\Big/\binom{m+\gamma+1}{\gamma}\Big]B_{\frac{1}{2}x}(2m+2\gamma+4,2n-2\gamma+1) \quad \text{if } 0 \le x \le 1, \\ [4/(m+1)]\sum_{\gamma=0}^{n}(-1)^\gamma\Big[\binom{n}{\gamma}\Big/\binom{m+\gamma+1}{\gamma}\Big]B_{\frac{1}{2}}(2m+2\gamma+4,2n-2\gamma+1) + \end{cases}$$

$$[4/(m+1)]\sum_{\gamma=0}^{m}(-1)^{\gamma}[\binom{m}{\gamma}/\binom{n+\gamma+1}{\gamma}][B_{\frac{1}{2}x}(2m-2\gamma+2,2n+2\gamma+3)-B_{\frac{1}{2}}(2m-2\gamma+2,$$
$$2n+2\gamma+3)] \quad \text{if } 1 \le x \le 2.$$

Then the acceptance region based on $V^{(2)} = [c_1/(1+c_1)] + [c_2/(1+c_2)]$ with $\ell$up and $\alpha$ level is given by $a \le V^{(2)} \le b$ where a and b are so chosen as to satisfy

(i) $c(2,m,n)[T_1(b)-T_1(a)] = 1-\alpha$ and

(ii) $c(2,m,n)\{(2m+3)[T_1(b)-T_1(a)]-(m+n+3)[T_2(b)-T_2(a)]\} = 0.$

4) Wilks' criterion, $W^{(2)} = [(1+c_1)(1+c_2)]^{-1}$. Again from Theorem 2, we obtain $\left.\frac{\partial P(a\le W^{(2)}\le b|\lambda_1 = \lambda_2 = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$ by replacing $L_2^{(2)}$ in the limits of the integrals in (3.1) by $W^{(2)}$. Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then

$$\left.\frac{\partial P(a\le W^{(2)}\le b|\lambda_1 = \lambda_2 = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1} \tag{3.11}$$

$$=c(2,m,n)[\int\int_{a\le(1-\ell_1)(1-\ell_2)\le b} g_1(\ell_1,\ell_2;m,n)d\ell_1 d\ell_2 -$$
$$\int\int_{a\le(1-\ell_1)(1-\ell_2)\le b} h_1(\ell_1,\ell_2;m,n)d\ell_1 d\ell_2].$$

Let $w = (1-\ell_1)(1-\ell_2)$ and $g = \ell_1\ell_2$. Then (3.11) becomes

$$c(2,m,n)[(2m+3)\int_a^b\int_0^{(1-w^{\frac{1}{2}})^2} g^m w^n dgdw-(m+n+3)\int_a^b\int_0^{(1-w^{\frac{1}{2}})^2} g^m w^n(1-w+g)dgdw],$$

and

$$P(a \le W^{(2)} \le b|\lambda_1 = \lambda_2 = 1) = c(2,m,n)\int_a^b\int_0^{(1-w^{\frac{1}{2}})^2} g^m w^n dgdw.$$

Therefore, we have the following theorem:

Theorem 6. The acceptance region based on $W^{(2)} = [(1+c_1)(1+c_2)]^{-1}$ with $\ell$up and $\alpha$ level is given by $a \le W^{(2)} \le b$ where a and b are so chosen as to satisfy

(i) $c(2,m,n)2[B_{b^{\frac{1}{2}}}(2n+2,2m+3)-B_{a^{\frac{1}{2}}}(2n+2,2m+3)]/(m+1) = 1-\alpha$ and

(ii) $c(2,m,n)2\{(m+n+3)[B_{b^{\frac{1}{2}}}(2n+4,2m+3)-B_{a^{\frac{1}{2}}}(2n+4,2m+3)]/(m+1)$

$-(n-m)[B_{b^{\frac{1}{2}}}(2n+2,2m+3)-B_{a^{\frac{1}{2}}}(2n+2,2m+3)]/(m+1)$

$-(m+n+3)[B_{b^{\frac{1}{2}}}(2n+2,2m+5)-B_{a^{\frac{1}{2}}}(2n+2,2m+5)]/(m+2)\} = 0.$

5) Roy's largest-smallest roots, $LS^{(2)} = c_2/(1+c_2), c_1/(1+c_1)$.

From Theorem 2, we have

$$(3.12) \quad \left.\frac{\partial P(a \le LS^{(2)} \le b|\lambda_1 = \lambda_2 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$= c(2,m,n)[\int_{a\le c_1/(1+c_1)\le c_2/(1+c_2)\le b} g(c_1,c_2;m,n)dc_1dc_2 - \int_{a\le c_1/(1+c_1)\le c_2/(1+c_2)\le b} h(c_1,c_2;m,n)dc_1dc_2].$$

Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then (3.12) becomes

$$c(2,m,n)][\int_a^b \int_a^{\ell_2} g_1(\ell_1,\ell_2;m,n)d\ell_1 d\ell_2 - \int_a^b \int_a^{\ell_2} h_1(\ell_1,\ell_2;m,n)d\ell_1 d\ell_2].$$

Further, $P(a \le LS^{(2)} \le b|\lambda_1 = \lambda_2 = 1)$

$$= c(2,m,n)\int_a^b \int_a^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2.$$

Now using the same technique as in Pillai [19,21,22],

$$\int_a^b \int_a^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2$$

$$= [2\int_a^b \ell_2^{2m+1}(1-\ell_2)^{2n+1}d\ell_2-\{b^{m+1}(1-b)^{n+1}+a^{m+1}(1-a)^{n+1}\}\int_a^b \ell_1^m(1-\ell_1)^n d\ell_1]/(m+n+2)$$

and $\int_a^b \int_a^{\ell_2} (\ell_1+\ell_2)(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2$

$$= [2\int_a^b \ell_2^{2m+2}(1-\ell_2)^{2n+1}d\ell_2-\{b^{m+2}(1-b)^{n+1}+a^{m+2}(1-a)^{n+1}\}\int_a^b \ell_1^m(1-\ell_1)^n d\ell_1$$

$$+ (m+2)\int_a^b \int_a^{\ell_2} (\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2]/(m+n+3).$$

Therefore, we have proved the following:

<u>Theorem 7</u>. The acceptance region based on Roy's largest-smallest roots $LS^{(2)} = c_2/(1+c_2),\ c_1/(1+c_1)$ with $\ell$up and $\alpha$ level is given by $a \le c_1/(1+c_1) \le c_2/(1+c_2) \le b$ where a and b are so chosen as to satisfy

(i) $c(2,m,n)[2B_{a,b}(2m+2,2n+2)-\{b^{m+1}(1-b)^{n+1}+a^{m+1}(1-a)^{n+1}\}B_{a,b}(m+1,n+1)]/(m+n+2)$

$= 1-\alpha$ and

(ii) $c(2,m,n)\{[2B_{a,b}(2m+2,2n+2)-\{b^{m+1}(1-b)^{n+1}+a^{m+1}(1-a)^{n+1}\}B_{a,b}(m+1,n+1)]$

$$[(m+1)/(m+n+2)]-[2B_{a,b}(2m+3,2n+2)-\{b^{m+2}(1-b)^{n+1}+a^{m+2}(1-a)^{n+1}\}$$

$$B_{a,b}(m+1,n+1)]\} = 0,$$

where $B_{x,y}(r,s) = \int_x^y t^{r-1}(1-t)^{s-1}dt.$

4. The acceptance regions based on two criteria with ℓup for p=3

In this section, we consider the acceptance regions of Roy's largest root and Hotelling's trace for p = 3. Largest root is taken up first.

1) Roy's largest root, $L_3^{(3)} = c_3/(1+c_3)$. From Theorem 2, we have

$$\left.\frac{\partial P(a\le L_3^{(3)}\le b|\lambda_1 = \lambda_2 = \lambda_3 = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$(4.1)\quad =c(3,m,n)[\int_{a\le L_3^{(3)}\le b} 3(m+2)\prod_{j=1}^{3} c_j^m(1+c_j)^{-(m+n+4)}\prod_{j'<j=2}^{3}(c_j-c_{j'})\prod_{j=1}^{3} dc_j -$$

$$\int_{a\le L_3^{(3)}\le b}(m+n+4)[\sum_{j=1}^{3} c_j/(1+c_j)]\prod_{j=1}^{3} c_j^m(1+c_j)^{-(m+n+4)}\prod_{j'<j=2}^{3}(c_j-c_{j'})\prod_{j=1}^{3} dc_j].$$

Now transform $\ell_j = c_j/(1+c_j)$, j = 1,2,3. Then (4.1) becomes

$$c(3,m,n)[\int_a^b\int_0^{\ell_3}\int_0^{\ell_2} 3(m+2)\prod_{j=1}^{3}\ell_j^m(1-\ell_j)^n\prod_{j'<j=2}^{3}(\ell_j-\ell_{j'})\prod_{j=1}^{3} d\ell_j -$$

$$\int_a^b\int_0^{\ell_3}\int_0^{\ell_2}(m+n+4)(\sum_{j=1}^{3}\ell_j)\prod_{j=1}^{3}\ell_j^m(1-\ell_j)^n\prod_{j'<j=2}^{3}(\ell_j-\ell_{j'})\prod_{j=1}^{3} d\ell_j].$$

Following the notation of Pillai [21], denote

$$U(x;m+1,n,m,n;t)=\int_0^x\int_0^{\ell_2}(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n e^{t(\ell_1+\ell_2)}(\ell_2-\ell_1)d\ell_1 d\ell_2.$$

Then from (3.4) we know that

$$(4.2)\quad U(x;m+1,n,m,n;0)=\int_0^x\int_0^{\ell_2}(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2$$

$$=[2B_x(2m+2,2n+2)-x^{m+1}(1-x)^{n+1}B_x(m+1,n+1)]/(m+n+2).$$

And also from (3.5), we have

$$(4.3)\quad \left.\frac{\partial U(x;m+1,n,m,n;t)}{\partial t}\right|_{t=0}=\int_0^x\int_0^{\ell_2}(\ell_1+\ell_2)(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)\,d\ell_1 d\ell_2$$

$$=[2B_x(2m+3,2n+2)-x^{m+2}(1-x)^{n+1}B_x(m+1,n+1)+(m+2)U(x;m+1,n,m,n;0)]/(m+n+3).$$

Now from Theorem 2 of Pillai [21]

$$U(x;m+2,n,m+1,n,m,n;t)$$

$$=\int_0^x\int_0^{\ell_3}\int_0^{\ell_2}\prod_{j=1}^{3}\ell_j^m(1-\ell_j)^n\exp(t\ell_j)\prod_{j'<j=2}^{3}(\ell_j-\ell_{j'})\prod_{j=1}^{3}d\ell_j$$

$$=[-x^{m+2}(1-x)^{n+1}e^{tx}U(x;m+1,n,m,n;t)+2B_x(2m+4,2n+2;2t)B_x(m+1,n+1;t)$$

$$-2B_x(2m+3,2n+2;2t)B_x(m+2,n+1;t)+tU(x;m+2,n+1,m+1,n,m,n;t)]/(m+n+3),$$

where $B_x(r,s;t)=\int_0^x u^{r-1}(1-u)^{s-1}e^{tu}du$. Therefore we have

$$(4.4)\quad U(x;m+2,n,m+1,n,m,n;0)$$

$$= \int_0^x \int_0^{\ell_3} \int_0^{\ell_2} \prod_{j=1}^{3} \ell_j^m (1-\ell_j)^n \prod_{j'<j=2}^{3} (\ell_j - \ell_{j'}) \prod_{j=1}^{3} d\ell_j$$

$$= [-x^{m+2}(1-x)^{n+1} U(x;m+1,n,m,n;0) + 2B_x(2m+4,2n+2)\cdot B_x(m+1,n+1)$$

$$-2B_x(2m+3,2n+2)\cdot B_x(m+2,n+1)]/(m+n+3).$$

And also $\left.\frac{\partial U(x;m+2,n,m+1,n,m,n;t)}{\partial t}\right|_{t=0}$

$$(4.5) \quad = \int_0^x \int_0^{\ell_3} \int_0^{\ell_2} [\sum_{j=1}^{3} \ell_j] \prod_{j=1}^{3} \ell_j^m (1-\ell_j)^n \prod_{j'<j=2}^{3} (\ell_j - \ell_{j'}) \prod_{j=1}^{3} d\ell_j$$

$$= [-x^{m+3}(1-x)^{n+1} U(x;m+1,n,m,n;0) - x^{m+2}(1-x)^{n+1} \frac{\partial}{\partial t} U(x,m+1,n,m,n;t)\Big|_{t=0}$$

$$+ 4B_x(2m+5,2n+2)B_x(m+1,n+1) - 2B_x(2m+4,2n+2)B_x(m+2,n+1)$$

$$- 2B_x(2m+3,2n+2)B_x(m+3,n+1) + U(x;m+2,n,m+1,n,m,n;0)$$

$$- U(x;m+3,n,m+1,n,m,n;0)]/(m+n+3).$$

Note that $\frac{\partial}{\partial t} U(x;m+2,n,m+1,n,m,n;t)\Big|_{t=0} = U(x;m+3,n,m+1,n,m,n;0)$. So (4.5) gives $\frac{\partial U(x;m+2,n,m+1,n,m,n;t)}{\partial t}$

$$(4.6) \quad = [-x^{m+3}(1-x)^{n+1} U(x;m+1,n,m,n;0) - x^{m+2}(1-x)^{n+1} \frac{\partial}{\partial t} U(x;m+1,n,m,n;t)\Big|_{t=0}$$

$$+ 4B_x(2m+5,2n+2)B_x(m+1,n+1) - 2B_x(2m+4,2n+2)B_x(m+2,n+1)$$

$$- 2B_x(2m+3,2n+2)B_x(m+3,n+1) + U(x;m+2,n,m+1,n,m,n;0)]/(m+n+4).$$

Therefore, we have the following theorem:

Theorem 8. The acceptance region based on Roy's largest root, $L_3^{(3)} = c_3/(1+c_3)$ with $\ell$up and $\alpha$ level is given by $a \leq L_3^{(3)} \leq b$ where a and b are so chosen as to satisfy

(i) $c(3,m,n)[U(b;m+2,n,m+1,n,m,n;0)-U(a;m+2,n,m+1,n,m,n;0)]=1-\alpha$ and

(ii) $c(3,m,n)\{3(m+2)[U(b;m+2,n,m+1,n,m,n;0)-U(a;m+2,n,m+1,n,m,n;0)]$

$$-(m+n+4)[\frac{\partial}{\partial t} U(b;m+2,n,m+1,n,m,n;t)\Big|_{t=0} - \frac{\partial}{\partial t} U(a,m+2,n,m+1,n,m,n;t)\Big|_{t=0}]\}$$

$= 0,$

where U and $\frac{\partial}{\partial t} U$ are given in (4.4) and (4.6) respectively.

2) Hotellings' trace, $U^{(3)} = c_1 + c_2 + c_3$. From Section 2, we have

(4.7) $P(a \leq U^{(3)} \leq b \mid \lambda_1 = \lambda_2 = \lambda_3 = \lambda)$

$$=(2\pi)^{-\frac{3}{2}(n_1+n_2)} \int_{a\leq c_1+c_2+c_3\leq b} \lambda^{-\frac{3}{2}n_1} \exp[-\frac{1}{2}\, tr(\lambda^{-1} \underset{\sim}{X}_1\underset{\sim}{X}_1' + \underset{\sim}{X}_2\underset{\sim}{X}_2')] d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Now transform $\lambda^{-\frac{1}{2}} \underset{\sim}{X}_1 = \underset{\sim}{Y}_1$ and $\underset{\sim}{X}_2 = \underset{\sim}{Y}_2$. $J(\underset{\sim}{Y}_1 : \underset{\sim}{X}_1) = \lambda^{-\frac{3}{2}n_1}$ and let $0 < d_1 \leq d_2 \leq d_3 < \infty$ be the characteristic roots of $|\underset{\sim}{Y}_1\underset{\sim}{Y}_1' - d\underset{\sim}{Y}_2\underset{\sim}{Y}_2'| = 0$. Then (4.7) becomes

(4.8) $$c(3,m,n) \int_{a\leq\lambda d_1+\lambda d_2+\lambda d_3\leq b} \prod_{j=1}^{3} d_j^m (1+d_j)^{-(m+n+4)} \prod_{j'<j=2}^{3} (d_j - d_{j'}) \prod_{j=1}^{3} dd_j.$$

After a proper transformation, (4.8) is $\int_{a\lambda^{-1}}^{b\lambda^{-1}} T_2(u)du$ where $T_2(u)$ is the probability density function of $U^{(3)}$. Then

$$\left.\frac{\partial P(a\le U^{(3)}\le b|\lambda_1 = \lambda_2 = \lambda_3 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1} = bT_2(b)-aT_2(a).$$

Note that $T_2(u)$, the density of $U^{(3)}$, and $T_1(u)$, the distribution function of $U^{(3)}$, are given in equation (4.9) and (4.11) respectively of Pillai and Sudjana [33]. Therefore we have the following theorem:

Theorem 9. The acceptance region based on $U^{(3)} = c_1+c_2+c_3$ with $\ell$up and $\alpha$ level is given by $a \le U^{(3)} \le b$ where a and b are so chosen as to satisfy (i) $T_1(b) - T_1(a) = 1-\alpha$ and (ii) $bT_2(b)-aT_2(a) = 0$.

5. $P(a\le[c_1/(1+c_1)]\le[c_2/(1+c_2)]\le b)$ in the non-null case

The non-null distribution of $c_1,\ldots,c_p$ was obtained by Khatri [12] in the form

(5.1) $$c(p,m,n)|\underset{\sim}{\Lambda}|^{-m-\frac{1}{2}p-\frac{1}{2}}|\underset{\sim}{C}|^{m}|\underset{\sim}{I}+\underset{\sim}{C}|^{-\frac{1}{2}\nu}\,{}_1F_0(\tfrac{1}{2}\nu;\underset{\sim}{I}-\underset{\sim}{\Lambda}^{-1},\underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{C})^{-1})$$

$$\prod_{i>j}(c_i-c_j),\ 0 < c_1 \le\ldots\le c_p < \infty,$$

where $\nu = n_1+n_2$ and the hypergeometric function of a matrix argument is defined by James [8]:

$${}_sF_t(a_1,\ldots,a_s;b_1,\ldots,b_t;\underset{\sim}{S},\underset{\sim}{T}) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_s)_\kappa}{(b_1)_\kappa\cdots(b_t)_\kappa}\frac{C_\kappa(\underset{\sim}{S})C_\kappa(\underset{\sim}{T})}{C_\kappa(\underset{\sim}{I})k!},.$$

where $a_1,\ldots,a_s$, $b_1,\ldots,b_t$ are real or complex constants and the coefficient $(a)_\kappa$ is defined by

$$(a)_\kappa = \prod_{i=1}^{p}(a - \frac{1}{2}(i-1))_{k_i},$$

where $(a)_k = a(a+1)\ldots(a+k-1)$ and $\kappa$ of k is a partition of k, $\kappa = (k_1,\ldots,k_p)$, $k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$ such that $k_1 + \ldots + k_p = k$ and the zonal polynomials $C_\kappa(\underset{\sim}{S})$ are expressible in terms of elementary symmetric functions of the characteristic roots of $\underset{\sim}{S}$ [James, 8].

Putting p = 2, the joint distribution of $c_1$, $c_2$ is

$$(5.2)\quad c(2,m,n)(\lambda_1\lambda_2)^{-m-\frac{3}{2}}(c_1c_2)^m[(1+c_1)(1+c_2)]^{-\frac{1}{2}\nu}\,{}_1F_0(\tfrac{1}{2}\nu;\underset{\sim}{I}-\underset{\sim}{\Lambda}^{-1},\underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{C})^{-1})(c_2-c_1).$$

Pillai and Jayachandran [28] have shown that by transforming $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$ and using zonal polynomials up to the sixth degree, the joint distribution of $\ell_1,\ell_2$ can be written as

$$K''\Big(\sum_{i+2j=k=0}^{6} c''_{ij}(\ell_1+\ell_2)^i(\ell_1\ell_2)^{j+m}+\ldots\Big)[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1),\ \ell_1<\ell_2$$

where $K'' = c(2,m,n)(\lambda_1\lambda_2)^{-m-\frac{3}{2}}$ and the $c''_{ij}$'s are functions of $\lambda_1$, $\lambda_2$, $n_1$, $n_2$ as given in terms of constants $A''_{ij}$ by Pillai and Jayachandran. The $A''_{ij}$'s and $c''_{ij}$'s are given in Appendix A. Now using the technique given by Pillai [19, 21, 22], we have

$$h_0 = 0,$$

$$h_j = \int_a^b\int_a^{\ell_2}(\ell_1\ell_2)^m[(1-\ell_1)(1-\ell_2)]^n(\ell_2^j-\ell_1^j)\,d\ell_1 d\ell_2$$

$$= [2B_{a,b}(2m+1+j,2n+2)-\{b^{m+j}(1-b)^{n+1}+a^{m+j}(1-a)^{n+1}\}B_{a,b}(m+1,n+1)+(m+j)h_{j-1}]/(m+n+1+j),$$

$$j = 1,\ldots,7.$$

Now defining $h_{ij} = \int_a^b \int_a^{\ell_2} (\ell_1+\ell_2)^i(\ell_1\ell_2)^{m+j}[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2$, then

$$h_{0j}=[2B_{a,b}(2m+2+2j,2n+2)-\{b^{m+1+j}(1-b)^{n+1}+a^{m+1+j}(1-a)^{n+1}\}B_{a,b}(m+1+j,n+1)]/(m+n+2+j),\quad j=1,2,3,$$

$$h_{1j}=[2B_{a,b}(2m+3+2j,2n+2)-\{b^{m+2+j}(1-b)^{n+1}+a^{m+2+j}(1-a)^{n+1}\}B_{a,b}(m+1+j,n+1)+(m+2+j)h_{0j}]/(m+n+3+j),\quad j=1,2,$$

$$h_{2j}=\{[2B_{a,b}(2m+4+2j,2n+2)-\{b^{m+3+j}(1-b)^{n+1}+a^{m+3+j}(1-a)^{n+1}\}B_{a,b}(m+1+j,n+1)+(m+3+j)h_{1j}]/(m+n+4+j)\}+h_{0j+1},\quad j=1,2,$$

$$h_{31}=\{[2B_{a,b}(2m+7,2n+2)-\{b^{m+5}(1-b)^{n+1}+a^{m+5}(1-a)^{n+1}\}B_{a,b}(m+2,n+1)+(m+5)(h_{21}-h_{02})]/(m+n+6)\}+2h_{12},$$

$$h_{41}=\{[2B_{a,b}(2m+8,2n+2)-\{b^{m+6}(1-b)^{n+1}+a^{m+6}(1-a)^{n+1}\}B_{a,b}(m+2,n+1)+(m+6)(h_{31}-h_{12})]/(m+n+7)\}+3h_{22}-h_{03},$$

where $B_{x,y}(\gamma,s) = \int_x^y t^{r-1}(1-t)^{s-1}dt$.

Then we have, in the non-null case,

$$P(a \le c_1/(1+c_1) \le c_2/(1+c_2) \le b) = P(a \le \ell_1 \le \ell_2 \le b)$$

$$=K''\sum_{i+2j=k=0}^{6} c''_{ij}\int_a^b \int_a^{\ell_2} (\ell_1+\ell_2)^i(\ell_1\ell_2)^{j+m}[(1-\ell_1)(1-\ell_2)]^n(\ell_2-\ell_1)d\ell_1 d\ell_2+\dots$$

$$=K''[c''_{00}h_1+c''_{10}h_2+c''_{20}(h_3+h_{01})+c''_{01}h_{01}+c''_{30}(h_4+2h_{11})+c''_{11}h_{11}$$
$$+c''_{40}(h_5+3h_{21}-h_{02})+c''_{21}h_{21}+c''_{02}h_{02}+c''_{50}(h_6+4h_{31}-3h_{12})+c''_{31}h_{31}$$
$$+c''_{12}h_{12}+c''_{60}(h_7+5h_{41}-6h_{22}+h_{03})+c''_{41}h_{41}+c''_{22}h_{22}+c''_{03}h_{03}]+\dots\ .$$

## 6. The likelihood ratio criterion, $Z^{(2)}$

In order to study the density of $Z^{(2)}$, starting from (5.2), Pillai and Jayachandran [29] have shown that by transforming $a_2 = \ell_1\ell_2$ and $w = (1-\ell_1)(1-\ell_2)$ and using zonal polynomials up to the sixth degree, the joint distribution of $a_2$ and w is given by $K'' \sum_{i+2j=k=0}^{6} c''_{ij} a_2^{j+m} w^n (1-w+a_2)^i + \ldots,\ 0 \le a_2^{\frac{1}{2}} + w^{\frac{1}{2}} \le 1$. Now $Z^{(2)}$ (to be denoted by Z in the rest of this section) $= (\ell_1\ell_2)^{\frac{1}{2}n_1}[(1-\ell_1)(1-\ell_2)]^{\frac{1}{2}n_2} = a_2^{\frac{1}{2}n_1} w^{\frac{1}{2}n_2}$. Let $y = w^{\frac{1}{2}}$. Then the joint distribution of Z and y is

$$(4/n_1)K'' \sum_{i+2j=k=0}^{6} c''_{ij} z^{\frac{2}{n_1}j-\frac{1}{n_1}} y^{\frac{-2n_2}{n_1}j+\frac{n_2}{n_1}-2} (1-y^2+z^{\frac{2}{n_1}} y^{\frac{-2n_2}{n_1}})^i + \ldots,$$

where $y^{\frac{n_2}{n_1}+1} - y^{\frac{n_2}{n_1}} + z^{\frac{1}{n_1}} \le 0$.

Now putting $n_2 = 2n_1$, the joint distribution of Z and y can be written as

$$(4/n_1)K'' \sum_{i+2j=k=0}^{6} c''_{ij} z^{\frac{2}{n_1}j-\frac{1}{n_1}} y^{-4j}(1-y^2+z^{\frac{2}{n_1}} y^{-4})^i + \ldots,$$

where $y^3 - y^2 + z^{\frac{1}{n_1}} \le 0$.

Note that the area bounded by $y^3 - y^2 + z^{\frac{1}{n_1}} \le 0$ is the area bounded by $0 \le z \le (4/27)^{n_1}$ and

$\frac{2}{3}\cos(\frac{\theta}{3} + \frac{4\pi}{3}) + \frac{1}{3} \le y \le \frac{2}{3}\cos(\frac{\theta}{3}) + \frac{1}{3}$ where $\cos\theta = 1 - \frac{27}{2} z^{\frac{1}{n_1}}$.

Now integrate out y to obtain the density of the m$\ell$r Z:

$$f(z) = (4/n_1)K'' \sum_{i+2j=k=0}^{6} c''_{ij} z^{\frac{2}{n_1}j - \frac{1}{n_1}} \int_a^b y^{-4j}(1-y^2+z^{\frac{2}{n_1}}y^{-4})^i dy + \ldots,$$

where $a = \frac{2}{3}\cos(\frac{\theta}{3} + \frac{4\pi}{3}) + \frac{1}{3}$, $b = \frac{2}{3}\cos(\frac{\theta}{3}) + \frac{1}{3}$ with $\cos\theta = 1 - \frac{27}{2} z^{\frac{1}{n_1}}$ and $0 \leq z \leq (4/27)^{n_1}$. Now let

$$g_{ij}(z) = \int_a^b y^{-4j}(1-y^2 + z^{\frac{2}{n_1}}y^{-4})^i dy.$$

Then the density of Z can be written as

$$f(z)=(4/n_1)K'' \sum_{i+2j=k=0}^{6} c''_{ij} z^{\frac{2}{n_1}j - \frac{1}{n_1}} g_{ij}(z)+\ldots, 0 \leq z \leq (4/27)^{n_1}.$$

The expressions for $g_{ij}(z)$ may be found in Appendix B. The method for determining the $g_{ij}(z)$'s will be illustrated by considering $g_{10}(z)$ which can be written as

$$\begin{aligned} g_{10}(z) &= \int_a^b (1-y^2+z^{\frac{2}{n_1}}y^{-4})dy \\ &=(b-a)- \frac{1}{3}(b^3-a^3)- \frac{1}{3} z^{\frac{2}{n_1}}(b^{-3}-a^{-3}) \\ &=(b-a)- \frac{1}{3}(b^3-a^3)- \frac{1}{3} z^{\frac{2}{n_1}}[z^{-\frac{2}{n_1}}(b-a)-z^{\frac{-2}{n_1}}(b^2-a^2)] \\ &= \frac{2}{3}(b-a), \end{aligned}$$

where we have made use of the relations

$$b^k-a^k = (b^{k-1}-a^{k-1})-z^{\frac{1}{n_1}}(b^{k-3}-a^{k-3}) \text{ for any integer } k.$$

Note that the $g_{ij}(z)$'s are expressed in terms of z, $b-a= \frac{2}{\sqrt{3}} \cos(\frac{\theta}{3} + \frac{\pi}{6})$ and $b^2-a^2= \frac{4}{3\sqrt{3}} \cos(\frac{\theta}{3} + \frac{\pi}{6})[\cos(\frac{\pi}{3} - \frac{\theta}{3})+1]$ where $\cos \theta = 1- \frac{27}{2} z^{\frac{1}{n_1}}$.

## 7. Numerical study of power

The results in the previous sections were used to obtain five percent points for the tests of $H_0$: $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$ based on criteria 1) to 4) in the unbiased as well as equal tail areas cases and criterion 5) in the unbiased case for p = 2, values of m = 0, 1, 2, 5 and n = 5, 10, 15, 20, 25, 30, 40, 60, 80, 100, and are given in Table 1. For m$\ell$r $Z^{(2)}$, lower five percent points were obtained for $n_2 = 2n_1$ and for values of $n_1$ = 3, 5, 7, 13 and the five percentage points were also computed for tests 1) to 5) in the unbiased case and 1) to 4) in the equal tails case for the same values of $n_1$ and all these are presented in Table 2. In addition, five percent points were also computed for $LS_1^{(2)}$, $LS_2^{(2)}$ and $LS_3^{(2)}$ for m = 0,1,2,5 and n = 5, 15, 30, 60 and are given in Table 3. Finally, for p = 3, five percentage points were computed for test 1), namely, Roy's largest root, in the unbiased as well as equal tail areas cases, which are also presented in Table 1.

The next step was to compute the powers of the various tests using the percentage points evaluated and the non-null distributions. For tests 1) to 4), non-null distributions were available in Pillai and Jayachandran [29] and for tests 5) and 6), they have been obtained in Sections 5 and 6. Before computing the power for a specific value of $(\lambda_1,\lambda_2)$ using series involving zonal polynomials of degree 0 to 6, the total probability in that case over the whole range of the respective statistic for all the terms included in the formula was calculated and the number of decimal places included in the tables was determined depending on the number of places of accuracy obtained in the total probability at least as many decimal places as in the tables. Powers for tests 1) to 5) in the unbiased as well as equal tail areas cases for $p = 2$, for values of $m = 0, 1, 2, 5$, $n = 5, 15, 30, 60$ and various $(\lambda_1,\lambda_2)$ are presented in Table 4, (to be continued in the Appendix C). Further, powers for tests 1) and 6) under the condition $n_2 = 2n_1$ are given in Table 5 for $p = 2$, for values of $n_1 = 3, 5, 7, 13$ and various $(\lambda_1,\lambda_2)$ in the unbiased case and also in the equal tail areas case for tests 1) to 4). Again, in order to compare the largest-smallest root tests $(LS^{(P)})$ with the approximations, a tabulation of powers is presented in Table 6 for $LS^{(2)}$, $LS_1^{(2)}$, $LS_2^{(2)}$, $LS_3^{(2)}$ and Roy's largest root $L_2^{(2)}$ for m, n as in Table 6 and various $(\lambda_1,\lambda_2)$.

A few findings seem to emerge from tabulations of powers in Tables 4, 5, 6.

1. $\lambda_1 \geq 1$, $\lambda_2 \geq 1$. It may be seen from Table 4 that equal tail areas tests based on 1) to 4) generally seem to perform better than corresponding unbiased ones except when very close to $H_0$ in which case bias is observed in some instances, mostly when m is close to n.

2. $\lambda_1 < 1$, $\lambda_2 > 1$ or $\lambda_1 > 1$, $\lambda_2 < 1$. For tests 1) to 4), unbiased test is better than equal tails except when $\lambda_1 + \lambda_2 > 2$. When $\lambda_1 + \lambda_2 \leq 2$, bias is observed though small.

3. $\lambda_1 < 1$, $\lambda_2 < 1$. For tests 1) to 4), unbiased test seems to be better than equal tails. There exists some bias when close to $H_0$.

4. $L_2^{(2)}$ seems to be least biased, then $U^{(2)}$, then $W^{(2)}$ and lastly $V^{(2)}$.

5. $\lambda_1 \geq 1$, $\lambda_2 \geq 1$. In regard to comparative performance of the criteria, findings in the equal tail areas case are as in the one-sided case for 1) to 4) described by Pillai and Jayachandran [29], in the unbiased case when $\lambda_1$ and $\lambda_2$ are far apart but both greater than unity, in terms of power, $U^{(2)} > Z^{(2)} > W^{(2)} > V^{(2)} > L_2^{(2)} > LS^{(2)}$, but with only one large positive deviation, $L_2^{(2)} > U^{(2)} > W^{(2)} > Z^{(2)} > LS^{(2)} > V^{(2)}$. But if $\lambda_1$ and $\lambda_2$ are close, then $V^{(2)} > W^{(2)} > U^{(2)} > L_2^{(2)} > LS^{(2)} > Z^{(2)}$.

6. $\lambda_1 < 1$, $\lambda_2 > 1$ or $\lambda_1 > 1$, $\lambda_2 < 1$. In the unbiased case, $U^{(2)} > W^{(2)} > L_2^{(2)} > V^{(2)} > Z^{(2)} > LS^{(2)}$ when $\lambda_1 + \lambda_2 < 2$, $Z^{(2)} > LS^{(2)} > L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$ when $\lambda_1 + \lambda_2 = 2$, $L_2^{(2)} > U^{(2)} > W^{(2)} > LS^{(2)} > Z^{(2)} > V^{(2)}$ when $\lambda_1 + \lambda_2 > 2$. In equal tail areas case, $L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$.

7. $\lambda_1 < 1$, $\lambda_2 < 1$. In the unbiased case, $V^{(2)} > W^{(2)} > U^{(2)} > L_2^{(2)} > Z^{(2)} > LS^{(2)}$, and in the equal tail areas case, $L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$.

8. In comparing $LS^{(2)}$ with the approximate methods $LS_1^{(2)}$, $LS_2^{(2)}$ and $LS_3^{(2)}$ from Table 6, it is observed that $LS_1^{(2)}$ to $LS_3^{(3)}$ are

all seriously biased and $LS_3^{(2)}$ especially so.

9. The bias in the tests 1) to 4), $LS_1^{(2)}$ and $LS_2^{(2)}$ disappears gradually with increasing m. Tests are practically unbiased when m = 5 i.e., $n_1 = 13$, $LS_3^{(2)}$ does not seem to become unbiased with large m.

10. If a single test has to be recommended on an overall basis over the whole parameter space, Roy's largest root seems to be the proper candidate. In the two-sided case as well as when both $\lambda_1$ and $\lambda_2$ are less than unity, among tests 1) to 4), largest root performs best in the equal tail areas case. Since the largest root is the least biased, even equal tail areas could be adequate. However, for the two-sided case, the unbiased largest root test compare favorably with $Z^{(2)}$ and $LS^{(2)}$ when $\lambda_1 + \lambda_2 = 2$ and is even the best when $\lambda_1 + \lambda_2 > 2$.

Table 1

Percentage points of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $(W^{(2)})^{1/2}$, $LS^{(2)}$ and $L_3^{(3)}$

| m \ n | | 5 | 10 | 15 | 20 | 25 | 30 | 40 | 60 | 80 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 5% points of $L_2^{(2)}$ with $\ell$up | | | | | |
| 0 | a | .078233 | .046349 | .032920 | .025523 | .020839 | .017608 | .013440 | .0091213 | .0069031 | .0055527 |
| | b | .631304 | .436723 | .332050 | .267488 | .223832 | .192383 | .150139 | .104288 | .079879 | .064727 |
| 1 | a | .158169 | .097653 | .070633 | .055325 | .045470 | .038596 | .029635 | .020237 | .015365 | .012384 |
| | b | .697949 | .505015 | .392957 | .321009 | .271141 | .234607 | .184730 | .129553 | .099737 | .081071 |
| 2 | a | .229499 | .146966 | .108125 | .085530 | .070748 | .060323 | .046593 | .032019 | .024390 | .019697 |
| | b | .743316 | .556725 | .441531 | .365025 | .310846 | .270561 | .214756 | .151938 | .117522 | .095809 |
| 5 | a | .391002 | .272387 | .209192 | .169814 | .142937 | .123500 | .096624 | .068000 | .051235 | .041585 |
| | b | .821942 | .659518 | .545565 | .463759 | .402798 | .355941 | .287492 | .209072 | .161090 | .132237 |
| | | | | | | Lower 2.5% points of $L_2^{(2)}$ | | | | | |
| 0 | | .073548 | .042195 | .029589 | .022783 | .018523 | .015606 | .011867 | .0080232 | .0060602 | .0048690 |
| 1 | | .156683 | .094226 | .067399 | .052468 | .042954 | .036362 | .027822 | .018931 | .014346 | .011550 |
| 2 | | .231254 | .144878 | .105550 | .083029 | .068433 | .058203 | .044808 | .030686 | .023333 | .018823 |
| 5 | | .398551 | .273826 | .208810 | .168804 | .141682 | .122077 | .095624 | .066719 | .051235 | .041585 |
| | | | | | | Upper 2.5% points of $L_2^{(2)}$ | | | | | |
| 0 | | .617908 | .417886 | .313858 | .250942 | .208930 | .178926 | .138963 | .096023 | .073347 | .059332 |
| 1 | | .695660 | .496309 | .382840 | .310983 | .261643 | .225739 | .177051 | .123631 | .094955 | .077072 |
| 2 | | .745265 | .552837 | .435546 | .358447 | .304254 | .264180 | .208985 | .147294 | .113692 | .092565 |
| 5 | | .827152 | .661201 | .544999 | .462038 | .400460 | .353132 | .285373 | .205996 | .161090 | .132237 |
| | | | | | | 5% points of $U^{(2)}$ with $\ell$up | | | | | |
| 0 | a | .106792 | .061166 | .042822 | .032934 | .026752 | .022523 | .017110 | .011550 | .0087301 | .0070119 |
| | b | 1.89953 | .877542 | .567392 | .418687 | .331591 | .274431 | .204017 | .134728 | .100717 | .080357 |
| 1 | a | .258332 | .149172 | .104797 | .080757 | .065683 | .055349 | .042100 | .028468 | .021505 | .017278 |
| | b | 2.67383 | 1.21937 | .784457 | .577371 | .456546 | .377454 | .280257 | .184932 | .137980 | .110038 |
| 2 | a | .429501 | .249669 | .175920 | .135792 | .110564 | .093230 | .070988 | .048050 | .036316 | .029188 |
| | b | 3.43163 | 1.55037 | .993530 | .729694 | .576218 | .475913 | .352973 | .232625 | .173459 | .138279 |
| 5 | a | .988636 | .582690 | .413327 | .320295 | .261460 | .220928 | .167803 | .114412 | .084408 | .067858 |
| | b | 5.66433 | 2.51423 | 1.59841 | 1.16857 | .920040 | .758492 | .558353 | .368596 | .268521 | .213779 |

Table 1 (continued)

| m \ n | | 5 | 10 | 15 | 20 | 25 | 30 | 40 | 60 | 80 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | Lower 2.5% points of $U^{(2)}$ | | | | | | |
| 0 | | .098891 | .054938 | .038041 | .029095 | .023555 | .019788 | .014993 | .010098 | .0076128 | .0061093 |
| 1 | | .252579 | .141904 | .098716 | .075691 | .061378 | .051618 | .039164 | .026417 | .019931 | .016002 |
| 2 | | .428758 | .242889 | .169574 | .130283 | .105782 | .089040 | .067636 | .045678 | .034483 | .027696 |
| 5 | | 1.00831 | .580524 | .408370 | .315129 | .256603 | .216431 | .164842 | .111643 | .084408 | .067858 |
| | | | | | Upper 2.5% points of $U^{(2)}$ | | | | | | |
| 0 | | 1.78212 | .808820 | .520238 | .382990 | .302930 | .250509 | .186070 | .122836 | .091673 | .073120 |
| 1 | | 2.61830 | 1.16901 | .747279 | .548380 | .432890 | .357515 | .265110 | .174722 | .130290 | .103870 |
| 2 | | 3.42571 | 1.51260 | .962659 | .704750 | .555510 | .458321 | .339420 | .223403 | .166470 | .132660 |
| 5 | | 5.78781 | 2.50510 | 1.58046 | 1.15140 | .904690 | .744770 | .549980 | .360907 | .268520 | .213780 |
| | | | | | 5% points of $V^{(2)}$ with $\ell$up | | | | | | |
| 0 | a | .100390 | .058783 | .041597 | .032193 | .026258 | .022172 | .016909 | .011483 | .0086742 | .0069757 |
| | b | .797466 | .532218 | .399122 | .319178 | .265879 | .227819 | .177096 | .122689 | .093654 | .075794 |
| 1 | a | .224051 | .136509 | .098263 | .076778 | .063008 | .053429 | .040973 | .027945 | .021204 | .017083 |
| | b | .982657 | .677643 | .516977 | .417800 | .350511 | .301871 | .236273 | .164678 | .126375 | .102527 |
| 2 | a | .343947 | .217315 | .159019 | .125424 | .103560 | .088192 | .068012 | .046663 | .035516 | .028667 |
| | b | 1.11565 | .793692 | .615388 | .502339 | .424316 | .367244 | .289365 | .203160 | .156519 | .127291 |
| 5 | a | .639300 | .439525 | .335369 | .271287 | .227762 | .196333 | .150650 | .107400 | .080524 | .065324 |
| | b | 1.36073 | 1.03938 | .838627 | .702397 | .603950 | .529710 | .418538 | .304509 | .232806 | .190511 |
| | | | | | Lower 2.5% points of $V^{(2)}$ | | | | | | |
| 0 | | .092062 | .052764 | .036986 | .028473 | .023146 | .019498 | .014826 | .010022 | .0075696 | .0060814 |
| 1 | | .216162 | .129637 | .092620 | .072054 | .058965 | .049901 | .038167 | .025960 | .019669 | .015833 |
| 2 | | .338074 | .210867 | .153321 | .120472 | .099221 | .084346 | .064892 | .044410 | .033756 | .027225 |
| 5 | | .639285 | .435841 | .331081 | .267032 | .223784 | .192607 | .150650 | .104948 | .080524 | .065324 |

Table 1 (continued)

| m \ n | | 5 | 10 | 15 | 20 | 25 | 30 | 40 | 60 | 80 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | Upper 2.5% points of $V^{(2)}$ | | | | | |
| 0 | | .769068 | .504284 | .374451 | .297612 | .246883 | .210907 | .163290 | .112476 | .085776 | .069319 |
| 1 | | .966662 | .659036 | .499376 | .401796 | .336047 | .288760 | .225311 | .156498 | .119875 | .097140 |
| 2 | | 1.10667 | .780503 | .601943 | .489614 | .412520 | .356360 | .280053 | .196040 | .150788 | .122506 |
| 5 | | 1.36071 | 1.03441 | .831923 | .695107 | .596708 | .522614 | .418538 | .299220 | .232806 | .190511 |
| | | | | | | 5% points of $(W^{(2)})^{1/2}$ with ℓup | | | | | |
| 0 | a | .556476 | .714448 | .789734 | .833659 | .862420 | .882707 | .909426 | .937780 | .952615 | .961739 |
| | b | .949462. | .970439 | .979108 | .983845 | .986831 | .988885 | .991528 | .994259 | .995658 | .996509 |
| 1 | a | .469095 | .641567 | .730065 | .783638 | .819506 | .845188 | .879497 | .916517 | .936140 | .948295 |
| | b | .886413 | .931071 | .950501 | .961381 | .968339 | .973172 | .979446 | .985996 | .989380 | .991447 |
| 2 | a | .406686 | .583893 | .680522 | .740932 | .782193 | .812142 | .852687 | .897124 | .920972 | .935846 |
| | b | .824861 | .889985 | .919742 | .936816 | .947895 | .955667 | .965852 | .976601 | .982203 | .985641 |
| 5 | a | .292315 | .462786 | .568819 | .640356 | .691685 | .730252 | .783620 | .846003 | .882630 | .904098 |
| | b | .672940 | .776766 | .830277 | .863039 | .885176 | .901140 | .922357 | .946053 | .959621 | .967261 |
| | | | | | | Lower 2.5% points of $(W^{(2)})^{1/2}$ | | | | | |
| 0 | | .571871 | .730027 | .803227 | .845268 | .872528 | .891629 | .916626 | .942951 | .956643 | .965035 |
| 1 | | .476229 | .651321 | .739589 | .792083 | .827095 | .852032 | .885175 | .920712 | .939455 | .951032 |
| 2 | | .409925 | .590469 | .687466 | .747539 | .788314 | .817777 | .857484 | .900765 | .923889 | .938274 |
| 5 | | .291242 | .464886 | .572104 | .644001 | .695577 | .733864 | .787620 | .848795 | .882630 | .904098 |
| | | | | | | Upper 2.5% points of $(W^{(2)})^{1/2}$ | | | | | |
| 0 | | .953421 | .973441 | .981420 | .985712 | .988393 | .990227 | .992573 | .994983 | .996212 | .996957 |
| 1 | | .889830 | .934452 | .953322 | .963752 | .970370 | .974944 | .980855 | .986992 | .990149 | .992073 |
| 2 | | .827014 | .893092 | .922573 | .939296 | .950073 | .957600 | .967420 | .977733 | .983086 | .986364 |
| 5 | | .671792 | .778335 | .832356 | .865137 | .887171 | .903007 | .924257 | .947326 | .959620 | .967261 |

Table 1 (continued)

| m \ n | | 5 | 10 | 15 | 20 | 25 | 30 | 40 | 60 | 80 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | 5% points of $LS^{(2)}$ with ℓup | | | | | |
| 0 | a | .0031051 | .0018394 | .0013046 | .0010104 | .00082430 | .00069606 | .00053086 | .00035996 | .00027228 | .00021895 |
| | b | .673421 | .477232 | .367102 | .297749 | .250271 | .215783 | .169105 | .117981 | .090577 | .073499 |
| 1 | a | .024594 | .015004 | .010781 | .0084102 | .0068935 | .0058395 | .0044727 | .0030460 | .0023093 | .0018596 |
| | b | .719427 | .528510 | .414542 | .340302 | .288383 | .250098 | .197564 | .139025 | .107225 | .087256 |
| 2 | a | .058204 | .036531 | .026595 | .020905 | .017219 | .014636 | .011261 | .0077048 | .0058562 | .0047227 |
| | b | .756681 | .573029 | .457318 | .379590 | .324137 | .282670 | .224995 | .159650 | .123692 | .100940 |
| 5 | a | .173052 | .116768 | .088137 | .070801 | .059169 | .050803 | .039320 | .027524 | .010231 | .0080733 |
| | b | .826948 | .667352 | .554116 | .472327 | .411050 | .363530 | .293502 | .214186 | .149354 | .122410 |
| | | | | | | 5% points of $L_3^{(3)}$ with ℓup | | | | | |
| 0 | a | .185837 | .115332 | .083607 | .065570 | .053934 | .045805 | .035195 | .024053 | .018269 | .014728 |
| | b | .711361 | .517835 | .404052 | .330596 | .279524 | .242034 | .190756 | .133911 | .103144 | .083867 |
| 1 | a | .273346 | .176655 | .130502 | .103471 | .085716 | .073163 | .056587 | .038938 | .029795 | .023242 |
| | b | .761110 | .575177 | .458147 | .379726 | .323904 | .282258 | .224386 | .158997 | .123519 | .097939 |
| 2 | a | .344427 | .230426 | .173140 | .138670 | .115620 | .098954 | .076035 | .052528 | .040124 | .032459 |
| | b | .795605 | .618927 | .501474 | .420252 | .361181 | .316042 | .250795 | .178756 | .138810 | .113441 |
| 5 | a | .493845 | .356089 | .281400 | .230543 | .195261 | .169347 | .133828 | .094281 | .072776 | .059260 |
| | b | .856577 | .705650 | .596680 | .512244 | .447975 | .397695 | .324414 | .236690 | .186188 | .153412 |
| | | | | | | Lower 2.5% points of $L_3^{(3)}$ | | | | | |
| 0 | | .186066 | .112743 | .080893 | .063078 | .051693 | .043792 | .033539 | .022843 | .017319 | .013947 |
| 1 | | .277532 | .176022 | .128915 | .101705 | .083982 | .071520 | .055153 | .037836 | .028795 | .023242 |
| 2 | | .351212 | .231413 | .172602 | .137635 | .114452 | .097954 | .076035 | .052528 | .040124 | .032459 |
| 5 | | .503945 | .361083 | .281400 | .230543 | .195261 | .169347 | .133828 | .094281 | .072776 | .059260 |
| | | | | | | Upper 2.5% points of $L_3^{(3)}$ | | | | | |
| 0 | | .711674 | .511979 | .396465 | .322770 | .271953 | .234871 | .184458 | .128982 | .099137 | .080502 |
| 1 | | .765208 | .574130 | .454863 | .375584 | .319495 | .277842 | .220239 | .155561 | .120210 | .097940 |
| 2 | | .801042 | .620274 | .500551 | .418225 | .358696 | .313809 | .250795 | .178756 | .138810 | .113441 |
| 5 | | .862473 | .710482 | .596680 | .512244 | .447976 | .397695 | .324414 | .236690 | .186188 | .153412 |

Table 2

Percentage points of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $(W^{(2)})^{\frac{1}{2}}$, $LS^{(2)}$ and $Z^{(2)}$

| test \ $n_1$ | | 3 | 5 | 7 | 13 |
|---|---|---|---|---|---|
| 5% points of tests 1) to 5) with $\ell$up | | | | | |
| $L_2^{(2)}$ | a | .150415 | .194303 | .217283 | .249727 |
| | b | .879749 | .782155 | .720098 | .621018 |
| $U^{(2)}$ | a | .221589 | .330719 | .400672 | .518881 |
| | b | 7.76039 | 4.07220 | 3.06799 | 2.14719 |
| $V^{(2)}$ | a | .202030 | .277900 | .324936 | .402030 |
| | b | 1.23671 | 1.13620 | 1.07231 | .969926 |
| $(W^{(2)})^{\frac{1}{2}}$ | a | .287479 | .381477 | .430628 | .499561 |
| | b | .899426 | .858903 | .834674 | .796084 |
| $LS^{(2)}$ | a | .0058353 | .030358 | .054955 | .106398 |
| | b | .903133 | .800316 | .734105 | .629220 |
| Lower 2.5% points of tests 1) to 4) | | | | | |
| $L_2^{(2)}$ | | .153770 | .195723 | .218213 | .250412 |
| $U^{(2)}$ | | .225627 | .330052 | .398204 | .515278 |
| $V^{(2)}$ | | .192957 | .270486 | .318795 | .398016 |
| $(W^{(2)})^{\frac{1}{2}}$ | | .290421 | .385738 | .434498 | .502147 |
| Upper 2.5% points of tests 1) to 4) | | | | | |
| $L_2^{(2)}$ | | .882622 | .783731 | .721217 | .621892 |
| $U^{(2)}$ | | 7.90736 | 4.06403 | 3.04951 | 2.13282 |
| $V^{(2)}$ | | 1.21806 | 1.12340 | 1.06252 | 0.96421 |
| $(W^{(2)})^{\frac{1}{2}}$ | | .901012 | .861421 | .837114 | .797900 |
| Lower 5% points of $Z^{(2)}$ | | | | | |
| $Z^{(2)}$ | | 1.32301 (-5) | 6.38366 (-7) | 1.82925 (-8) | 2.53203 (-12) |

The numbers in parentheses indicate the power of 10 by which the tabulated values are to be multiplied.

Table 3

Percentage points of $LS_1^{(2)}$, $LS_2^{(2)}$ and $LS_3^{(2)}$

| m \ n | | 5 | 15 | 30 | 60 |
|---|---|---|---|---|---|
| | | 5% points of $LS_1^{(2)}$ | | | |
| 0 | a | .0019456 | .00076691 | .00040179 | .00020581 |
| | b | .616853 | .313126 | .178468 | .095766 |
| 1 | a | .020893 | .0086368 | .0045961 | .0023746 |
| | b | .694902 | .382234 | .225337 | .123397 |
| 2 | a | .054045 | .023401 | .012658 | .0066008 |
| | b | .744652 | .435001 | .263802 | 147066 |
| 5 | a | .172848 | .084453 | .047943 | .025734 |
| | b | .826744 | .544551 | .352784 | .205771 |
| | | 5% points of $LS_2^{(2)}$ | | | |
| 0 | a | .0019751 | .00077843 | .00040782 | .00020890 |
| | b | .617908 | .313858 | .178926 | .096023 |
| 1 | a | .021036 | .0086953 | .0046272 | .0023907 |
| | b | .695660 | .382804 | .225739 | .123631 |
| 2 | a | .054288 | .023506 | .012715 | .0066305 |
| | b | .745265 | .435546 | .264180 | .147294 |
| 5 | a | .173256 | .084655 | .048060 | .025797 |
| | b | .827152 | .544999 | .353132 | .205996 |
| | | 5% points of $LS_3^{(2)}$ | | | |
| 0 | 1-b | .0039379 | .0015531 | .00081385 | .00041693 |
| | b | .9960621 | .9984469 | .99918615 | .99958307 |
| 1 | 1-b | .030424 | .012612 | .0067177 | .0034725 |
| | b | .969576 | .987388 | .9932823 | .9965275 |
| 2 | 1-b | .070509 | .030681 | .016625 | .0086772 |
| | b | .929491 | .969319 | .983375 | .9913228 |
| 5 | 1-b | .173052 | .098895 | .056328 | .030295 |
| | b | .826948 | .901105 | .943672 | .969705 |

Table 4

Powers of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and $LS^{(2)}$ in the unbiased and equal tail areas cases for testing $\lambda_1=1$, $\lambda_2=1$ against different simple two-sided alternative hypotheses, $\alpha=.05$

$m = 0$, $n = 5$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050011 | .050014 | .050017 | .050016 |
| 1 | 1.1 | .050586 | .050594 | .050511 | .050576 | .050347 | .051691 | .051959 | .052229 | .052156 |
| 1.05 | 1.05 | .050434 | .050475 | .050509 | .050499 | .050203 | .051534 | .051843 | .052247 | .052089 |
| 1 | 1.5 | .0624 | .0624 | .0602 | .0619 | .0576 | .0678 | .0690 | .0682 | .0695 |
| 1.25 | 1.25 | .0594 | .0604 | .0612 | .0610 | .0547 | .0647 | .0671 | .0696 | .0687 |
| 1 | 2 | .0919 | .0917 | .0827 | .0899 | .0773 | .1022 | .1042 | .0980 | .1042 |
| 1.333 | 1.5 | .0739 | .0765 | .0780 | .0780 | .0630 | .0826 | .0874 | .0917 | .0906 |
| 1 | 4 | .267 | .265 | .224 | | .220 | .288 | .290 | .256 | |
| 1 | 5 | .353 | .350 | .301 | | .300 | .375 | .376 | .336 | |
| 2 | 4 | .342 | .361 | .357 | | .281 | .367 | .391 | .393 | |
| 3 | 3 | .339 | .364 | .371 | | .273 | .364 | .396 | .409 | |
| 1 | 8 | .546 | .542 | .491 | | .494 | .566 | .566 | .523 | |
| 4.5 | 4.5 | .579 | .606 | .610 | | .516 | .601 | .633 | .642 | |
| 1 | 11 | .665 | .662 | .616 | | .621 | .681 | .681 | .664 | |
| 6 | 6 | .731 | .752 | .754 | | .683 | .747 | .772 | .777 | |
| 1.00001 | 0.9 | .050649 | .050663 | .050590 | .050648 | .050379 | .049520 | .049260 | .048793 | .049015 |
| 1.00001 | 0.8 | .052787 | .052856 | .052600 | .052808 | .051643 | .050483 | .049989 | .048896 | .049462 |
| 1.01 | 0.99 | .050006 | .050005 | .050001 | .050004 | .050006 | .050007 | .050005 | .050000 | .050003 |
| 1.1 | 0.9 | .050658 | .050532 | .050080 | .050369 | .050621 | .050671 | .050515 | .049995 | .050327 |
| 1.1 | 0.8 | .052114 | .051902 | .050976 | .051582 | .051782 | .051001 | .050452 | .048984 | .049847 |
| 1.05 | 0.95 | .050164 | .050133 | .050019 | .050092 | .050154 | .050168 | .050128 | .049998 | .050081 |
| 1.2 | 0.99 | .0521 | .0521 | .0517 | .0520 | .0513 | .0542 | .0547 | .0549 | .0550 |
| 1.2 | 0.8 | .0526 | .0520 | .0503 | .0514 | .0524 | .0526 | .0520 | .0499 | .0512 |
| 2 | 0.9 | .091 | .092 | .077 | .081 | .077 | .099 | .103 | .089 | .094 |
| 2 | 0.7 | .091 | .087 | .072 | .081 | .084 | .098 | .095 | .080 | .090 |
| 3 | 0.9 | .15 | .15 | .12 | .15 | .13 | .19 | .19 | .12 | .17 |
| 5 | 0.9 | .34 | .33 | .31 | .32 | .28 | .36 | .36 | .30 | .35 |
| 0.99999 | 0.9 | .050649 | .050663 | .050590 | .050648 | .050379 | .049520 | .049260 | .048793 | .049015 |
| 0.99999 | 0.7 | .0589 | .0591 | .0592 | .0591 | .0567 | .0551 | .0545 | .0531 | .0537 |
| 0.999 | 0.9 | .050656 | .050670 | .050601 | .050657 | .050381 | .049515 | .049253 | .048786 | .049007 |
| 0.9 | 0.9 | .051973 | .052142 | .052289 | .052241 | .050861 | .049667 | .049307 | .048697 | .048955 |
| 0.9 | 0.8 | .054955 | .055317 | .055566 | .055504 | .052280 | .051414 | .050969 | .050050 | .050467 |
| 0.9 | 0.76 | .0569 | .0574 | .0577 | .0576 | .0535 | .0528 | .0524 | .0513 | .0518 |
| 0.85 | 0.9 | .053216 | .053476 | .053683 | .053621 | .051417 | .050303 | .049898 | .049150 | .049477 |
| 0.85 | 0.8 | .0567 | .0572 | .0577 | .0575 | .0530 | .0525 | .0521 | .0512 | .0516 |
| 0.81 | 0.9 | .054557 | .054897 | .055138 | .055076 | .052071 | .051144 | .050707 | .049823 | .050222 |
| 0.8 | 0.8 | .0593 | .0599 | .0605 | .0603 | .0543 | .0544 | .0539 | .0530 | .0534 |

Table 4 (continued)

m = 0, n = 15

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050023 | .050026 | .050027 | .050027 |
| 1 | 1.1 | .050811 | .050800 | .050745 | .050778 | .050469 | .053126 | .053455 | .053473 | .053487 |
| 1.05 | 1.05 | .050586 | .050643 | .050648 | .050647 | .050280 | .052884 | .053300 | .053387 | .053361 |
| 1 | 1.5 | .0677 | .0672 | .0659 | .0667 | .0611 | .0793 | .0803 | .0792 | .0800 |
| 1.25 | 1.25 | .0630 | .0644 | .0646 | .0646 | .0569 | .0744 | .0776 | .0782 | .0780 |
| 1 | 2 | .1103 | .1083 | .1034 | .1071 | .0907 | .1322 | .1328 | .1287 | .1320 |
| 1.333 | 1.5 | .0838 | .0872 | .0874 | .0876 | .0692 | .1023 | .1088 | .1096 | .1096 |
| 1 | 4 | .334 | .326 | .287 | | .280 | .373 | .370 | .357 | |
| 1 | 5 | .428 | .419 | .400 | | .371 | .467 | .463 | .448 | |
| 2 | 4 | .418 | .439 | .436 | | .348 | .463 | .489 | .488 | |
| 3 | 3 | .418 | .447 | .448 | | .343 | .465 | .500 | .502 | |
| 1 | 8 | .620 | .612 | .593 | | .569 | .653 | .649 | .634 | |
| 4.5 | 4.5 | .658 | .687 | .687 | | .592 | .696 | .727 | .728 | |
| 1 | 11 | .728 | .721 | .705 | | .686 | .754 | .750 | .738 | |
| 6 | 6 | .792 | .813 | .813 | | .744 | .818 | .840 | .841 | |
| 1.00001 | 0.9 | .050873 | .050866 | .050820 | .050850 | .050489 | .048573 | .048200 | .048064 | .048123 |
| 1.00001 | 0.8 | .053663 | .053663 | .053473 | .053603 | .052043 | .049053 | .048295 | .047921 | .048102 |
| 1.01 | 0.99 | .050009 | .050007 | .050004 | .050006 | .050007 | .050010 | .050006 | .050004 | .050005 |
| 1.1 | 0.9 | .050941 | .050686 | .050452 | .050585 | .050790 | .051014 | .050679 | .050409 | .050561 |
| 1.1 | 0.8 | .052872 | .052402 | .051919 | .052202 | .052186 | .050708 | .049708 | .049051 | .049409 |
| 1.05 | 0.95 | .050235 | .050171 | .050112 | .050146 | .050197 | .050254 | .050169 | .050102 | .050140 |
| 1.2 | 0.99 | .0530 | .0529 | .0526 | .0528 | .0518 | .0574 | .0579 | .0578 | .0579 |
| 1.2 | 0.8 | .0537 | .0527 | .0517 | .0523 | .0531 | .0540 | .0526 | .0516 | .0522 |
| 2 | 0.9 | .106 | .101 | .095 | .098 | .091 | .126 | .123 | .118 | .122 |
| 2 | 0.7 | .105 | .097 | .088 | .093 | .094 | .121 | .113 | .105 | .109 |
| 3 | 0.9 | .21 | .20 | .19 | .20 | .17 | .25 | .24 | .23 | .25 |
| 5 | 0.9 | .42 | .40 | .38 | .40 | .36 | .46 | .45 | .43 | .50 |
| 0.99999 | 0.9 | .050873 | .050866 | .050820 | .050851 | .050489 | .048573 | .048200 | .048063 | .048123 |
| 0.99999 | 0.7 | .0600 | .0602 | .0601 | .0601 | .0567 | .0527 | .0517 | .0511 | .0514 |
| 0.999 | 0.9 | .050881 | .050876 | .050832 | .050861 | .050491 | .048557 | .048184 | .048048 | .048107 |
| 0.9 | 0.9 | .052597 | .052806 | .052844 | .052836 | .051137 | .047926 | .047463 | .047339 | .047380 |
| 0.9 | 0.8 | .056406 | .056828 | .056866 | .056869 | .052878 | .049332 | .048744 | .048529 | .048612 |
| 0.9 | 0.76 | .0587 | .0593 | .0593 | .0593 | .0541 | .0506 | .0500 | .0497 | .0498 |
| 0.85 | 0.9 | .054208 | .054522 | .054568 | .054562 | .051844 | .048345 | .047823 | .047664 | .047721 |
| 0.85 | 0.8 | .0586 | .0592 | .0593 | .0593 | .0537 | .0503 | .0497 | .0495 | .0496 |
| 0.81 | 0.9 | .055912 | .056312 | .056352 | .056353 | .052636 | .049083 | .048509 | .048306 | .048384 |
| 0.8 | 0.8 | .0616 | .0624 | .0625 | .0625 | .0551 | .0520 | .0514 | .0513 | .0513 |

Table 4 (continued)

m = 0, n = 30

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050028 | .050031 | .050031 | .050031 |
| 1 | 1.1 | .050905 | .050886 | .050844 | .050862 | .050522 | .053698 | .054012 | .053999 | .054013 |
| 1.05 | 1.05 | .050649 | .050715 | .050707 | .050707 | .050313 | .053418 | .053840 | .053866 | .053859 |
| 1 | 1.5 | .0700 | .0692 | .0683 | .0688 | .0626 | .0840 | .0846 | .0839 | .0844 |
| 1.25 | 1.25 | .0646 | .0660 | .0660 | .0660 | .0578 | .0783 | .0817 | .0818 | .0818 |
| 1 | 2 | .1179 | .1148 | .1120 | .1143 | .0967 | .1440 | .1435 | .1410 | .1432 |
| 1.333 | 1.5 | .0878 | .0915 | .0914 | .0916 | .0720 | .1102 | .1169 | .1172 | .1172 |
| 1 | 4 | .358 | .348 | .309 | | .304 | .402 | .396 | .388 | |
| 1 | 5 | .455 | .444 | .433 | | .399 | .498 | .492 | .483 | |
| 2 | 4 | .446 | .466 | .464 | | .376 | .496 | .521 | .520 | |
| 3 | 3 | .447 | .476 | .476 | | .372 | .500 | .533 | .533 | |
| 1 | 8 | .646 | .635 | .625 | | .596 | .680 | .674 | .666 | |
| 4.5 | 4.5 | .686 | .713 | .713 | | .622 | .727 | .754 | .754 | |
| 1 | 11 | .749 | .740 | .732 | | .709 | .776 | .770 | .764 | |
| 6 | 6 | .812 | .833 | .833 | | .767 | .840 | .860 | .860 | |
| 1.00001 | 0.9 | .050963 | .050935 | .050915 | .050932 | .050533 | .048214 | .047823 | .047763 | .047790 |
| 1.00001 | 0.8 | .054006 | .053943 | .053858 | .053917 | .052197 | .048535 | .047709 | .047528 | .047615 |
| 1.01 | 0.99 | .050010 | .050007 | .050006 | .050007 | .050008 | .050011 | .050007 | .050005 | .050006 |
| 1.1 | 0.9 | .051058 | .050741 | .050607 | .050679 | .050859 | .051164 | .050744 | .050589 | .050672 |
| 1.1 | 0.8 | .053175 | .052570 | .052302 | .052457 | .052343 | .050634 | .049450 | .049094 | .049284 |
| 1.05 | 0.95 | .050264 | .050185 | .050151 | .050169 | .050214 | .050291 | .050186 | .050147 | .050167 |
| 1.2 | 0.99 | .0533 | .0532 | .0530 | .0531 | .0520 | .0587 | .0591 | .0590 | .0591 |
| 1.2 | 0.8 | .0541 | .0529 | .0524 | .0527 | .0534 | .0546 | .0529 | .0523 | .0526 |
| 2 | 0.9 | .113 | .107 | .103 | .114 | .094 | .143 | .133 | .130 | .137 |
| 2 | 0.7 | .111 | .101 | .096 | .099 | .099 | .130 | .120 | .115 | .118 |
| 3 | 0.9 | .23 | .22 | .21 | .22 | .19 | .27 | .26 | .25 | .27 |
| 5 | 0.9 | .44 | .43 | .42 | .43 | .39 | .49 | .48 | .47 | .49 |
| 0.99999 | 0.9 | .050963 | .050935 | .050915 | .050932 | .050533 | .048214 | .047822 | .047763 | .047789 |
| 0.99999 | 0.7 | .0604 | .0605 | .0604 | .0605 | .0566 | .0519 | .0508 | .0505 | .0506 |
| 0.999 | 0.9 | .050972 | .050946 | .050927 | .050944 | .050535 | .048194 | .047803 | .047744 | .047770 |
| 0.9 | 0.9 | .052844 | .053043 | .053072 | .053070 | .051248 | .047274 | .046822 | .046784 | .046796 |
| 0.9 | 0.8 | .056972 | .057366 | .057393 | .057398 | .053110 | .048581 | .048000 | .047921 | .047949 |
| 0.9 | 0.76 | .0594 | .0599 | .0599 | .0599 | .0544 | .0498 | .0492 | .0491 | .0491 |
| 0.85 | 0.9 | .054599 | .054896 | .054929 | .054929 | .052014 | .047624 | .047114 | .047062 | .047080 |
| 0.85 | 0.8 | .0593 | .0599 | .0600 | .0600 | .0540 | .0495 | .0490 | .0489 | .0489 |
| 0.81 | 0.9 | .056441 | .056816 | .056845 | .056849 | .052857 | .048336 | .047771 | .047697 | .047724 |
| 0.8 | 0.8 | .0625 | .0633 | .0634 | .0633 | .0554 | .0511 | .0507 | .0506 | .0506 |

Table 4 (continued)

m = 0, n = 60

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050030 | .050034 | .050034 | .050034 |
| 1 | 1.1 | .050961 | .050959 | .050857 | .050914 | .050554 | .054037 | .054328 | .054314 | .054323 |
| 1.05 | 1.05 | .050687 | .050780 | .050787 | .050744 | .050333 | .053733 | .054147 | .054154 | .054152 |
| 1 | 1.5 | .0714 | .0705 | .0696 | .0701 | .0636 | .0868 | .0871 | .0867 | .0869 |
| 1.25 | 1.25 | .0655 | .0672 | .0672 | .0670 | .0584 | .0807 | .0840 | .0841 | .0840 |
| 1 | 2 | .1224 | .1189 | .1166 | .1187 | .1003 | .1509 | .1495 | .1482 | .1499 |
| 1.333 | 1.5 | .0903 | .0942 | .0940 | .0940 | .0737 | .1150 | .1216 | .1216 | .1217 |
| 1 | 4 | .372 | .360 | .355 | | .319 | .418 | .411 | .407 | |
| 1 | 5 | .470 | .458 | .451 | | .415 | .515 | .507 | .503 | |
| 2 | 4 | .462 | .482 | .479 | | .392 | .515 | .538 | .537 | |
| 3 | 3 | .464 | .493 | .493 | | .389 | .520 | .550 | .551 | |
| 1 | 8 | .659 | .648 | .643 | | .612 | .694 | .687 | .684 | |
| 4.5 | 4.5 | .702 | .728 | .728 | | .640 | .743 | .768 | .769 | |
| 1 | 11 | .760 | .751 | .747 | | .723 | .787 | .781 | .778 | |
| 6 | 6 | .824 | .843 | .843 | | .780 | .851 | .869 | .870 | |
| | | | | | | | | | | |
| 1.00001 | 0.9 | .051016 | .050952 | .050927 | .050981 | .050561 | .048007 | .047611 | .047585 | .047597 |
| 1.00001 | 0.8 | .054209 | .054058 | .054053 | .054101 | .052288 | .048242 | .047385 | .047298 | .047340 |
| 1.01 | 0.99 | .050011 | .050008 | .050006 | .050007 | .050009 | .050012 | .050008 | .050007 | .050007 |
| 1.1 | 0.9 | .051129 | .050771 | .050699 | .050737 | .050901 | .051257 | .050781 | .050698 | .050741 |
| 1.1 | 0.8 | .053355 | .052641 | .052572 | .052610 | .052455 | .050602 | .049308 | .049122 | .049222 |
| 1.05 | 0.95 | .050282 | .050192 | .050174 | .050184 | .050224 | .050315 | .050195 | .050174 | .050185 |
| 1.2 | 0.99 | .0536 | .0534 | .0532 | .0533 | .0522 | .0595 | .0598 | .0598 | .0598 |
| 1.2 | 0.8 | .0545 | .0531 | .0527 | .0529 | .0536 | .0550 | .0530 | .0527 | .0529 |
| 2 | 0.9 | .116 | .111 | .109 | .119 | .099 | .144 | .139 | .137 | .105 |
| 2 | 0.7 | .116 | .103 | .100 | .102 | .102 | .136 | .124 | .121 | .120 |
| 3 | 0.9 | .24 | .25 | .22 | .23 | .20 | .28 | .27 | .27 | .28 |
| 5 | 0.9 | .36 | .44 | .44 | .44 | .41 | .50 | .49 | .49 | .52 |
| | | | | | | | | | | |
| 0.99999 | 0.9 | .051017 | .050952 | .050927 | .050981 | .050561 | .048006 | .047611 | .047585 | .047597 |
| 0.99999 | 0.7 | .0607 | .0607 | .0608 | .0607 | .0567 | .0515 | .0503 | .0501 | .0502 |
| 0.999 | 0.9 | .051026 | .050964 | .050939 | .050993 | .050562 | .047984 | .047589 | .047563 | .047576 |
| 0.9 | 0.9 | .052991 | .053134 | .053211 | .053208 | .051315 | .046899 | .046468 | .046457 | .046460 |
| 0.9 | 0.8 | .057305 | .057606 | .057738 | .057708 | .053247 | .048157 | .047597 | .047567 | .047576 |
| 0.9 | 0.76 | .0599 | .0602 | .0604 | .0603 | .0546 | .0494 | .0488 | .0487 | .0487 |
| 0.85 | 0.9 | .054829 | .055055 | .055255 | .055145 | .052115 | .047212 | .046726 | .046709 | .046715 |
| 0.85 | 0.8 | .0598 | .0603 | .0605 | .0604 | .0542 | .0490 | .0485 | .0485 | .0485 |
| 0.81 | 0.9 | .056753 | .057039 | .057265 | .057139 | .052988 | .047913 | .047369 | .047342 | .047351 |
| 0.8 | 0.8 | .0631 | .0637 | .0639 | .0638 | .0556 | .0507 | .0502 | .0503 | .0502 |

Table 5

Powers of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $W^{(2)}$, $LS^{(2)}$ (in the unbiased and equal tail areas cases) and $Z^{(2)}$ for testing $\lambda_1 = 1$, $\lambda_2 = 1$ against different simple two-sided alternative hypotheses, $\alpha = .05$

| $n_1 = 3$ | | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $Z^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .0500 | .050000 | .050000 | .050000 | .050000 | .050000 | .049997 | .049997 | .050007 | .050002 |
| 1 | 1.1 | .0502 | .050340 | .050350 | .050317 | .050354 | .050216 | .050053 | .050139 | .051065 | .050586 |
| 1.05 | 1.05 | .0501 | .050262 | .050281 | .050368 | .050327 | .050122 | .049975 | .050069 | .051133 | .050561 |
| 1 | 1.5 | .0552 | .0568 | .0569 | .0558 | .0569 | .0543 | .0555 | .0560 | .0590 | .0580 |
| 1.25 | 1.25 | .0520 | .0554 | .0559 | .0578 | .0569 | .0527 | .0541 | .0549 | .0614 | .0580 |
| 1 | 2 | .0691 | .0724 | .0729 | .0675 | .0725 | .0648 | .0700 | .0711 | .0731 | .0744 |
| 1.333 | 1.5 | .0562 | .0635 | .0646 | .0692 | .0673 | .0571 | .0614 | .0631 | .0749 | .0691 |
| 1.5 | 1.5 | .0584 | .0683 | .0699 | .0766 | | .0597 | .0659 | .0681 | .0833 | |
| 2 | 2 | .0724 | .1096 | .1148 | .1338 | | .0872 | .1053 | .1115 | .1455 | |
| 1.00001 | 0.9 | .0502 | .050391 | .050405 | .050384 | .050414 | .050251 | .050696 | .050631 | .049567 | .050166 |
| 1.00001 | 0.8 | .0517 | .051739 | .051806 | .051758 | .051850 | .051141 | .052375 | .052277 | .050038 | .051333 |
| 1.01 | 0.99 | .0500 | .050003 | .050003 | .050001 | .050001 | .050004 | .050003 | .050003 | .049997 | .050001 |
| 1.1 | 0.9 | .0505 | .050353 | .050323 | .050018 | .050160 | .050426 | .050357 | .050329 | .049782 | .050148 |
| 1.1 | 0.8 | .0520 | .051244 | .051211 | .050303 | .050908 | .051273 | .051563 | .051452 | .049325 | .050634 |
| 1.05 | 0.95 | .0501 | .050088 | .050080 | .050032 | .050039 | .050105 | .050089 | .050082 | .049945 | .050036 |
| 1.2 | 0.99 | .0508 | .0512 | .0512 | .0510 | .0512 | .0508 | .0506 | .0508 | .0523 | .0516 |
| 1.2 | 0.8 | .0527 | .0513 | .0512 | .0510 | .0505 | .0516 | .0513 | .0512 | .0490 | .0505 |
| 2 | 0.9 | .071 | .076 | .075 | .066 | .069 | .073 | .074 | .074 | .070 | .069 |
| 2 | 0.7 | | .076 | .076 | .062 | .071 | .075 | .075 | .074 | .063 | .072 |
| 3 | 0.9 | .09 | .10 | .10 | .07 | .09 | .09 | .10 | .10 | .08 | .10 |
| 5 | 0.9 | | .23 | .23 | .18 | .22 | .20 | .23 | .23 | .19 | .22 |
| 0.99999 | 0.9 | .0502 | .050391 | .050405 | .050384 | .050414 | .050251 | .050696 | .050631 | .049567 | .050166 |
| 0.99999 | 0.7 | | .0574 | .0576 | .0579 | .0577 | .0562 | .0584 | .0583 | .0551 | .0569 |
| 0.999 | 0.9 | .0502 | .050395 | .050410 | .050393 | .050420 | .050251 | .050704 | .050638 | .049568 | .050170 |
| 0.9 | 0.9 | .0504 | .051234 | .051318 | .051696 | .051514 | .050544 | .051861 | .051780 | .050063 | .051011 |
| 0.9 | 0.8 | .0521 | .053190 | .053385 | .054157 | .053793 | .051548 | .054171 | .054106 | .051617 | .053012 |
| 0.9 | 0.76 | .0556 | .0547 | .0549 | .0559 | .0554 | .0526 | .0558 | .0558 | .0530 | .0545 |
| 0.85 | 0.9 | .0508 | .052027 | .052161 | .052732 | .052459 | .050914 | .052826 | .052758 | .050659 | .051821 |
| 0.85 | 0.8 | .0530 | .0544 | .0546 | .0558 | .0552 | .0520 | .0555 | .0555 | .0528 | .0543 |
| 0.81 | 0.9 | .0517 | .052914 | .053096 | .053826 | .053482 | .051386 | .053857 | .053789 | .051382 | .052730 |
| 0.8 | 0.8 | .0556 | .0563 | .0567 | .0582 | .0574 | .0531 | .0577 | .0577 | .0547 | .0563 |

Table 5 (continued)

$n_1 = 5$

| $\lambda_1$ | $\lambda_2$ | $Z^{(2)}$ | With local unbiasedness property: $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas: $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .0500 | .050000 | .050000 | .050000 | .050000 | .050000 | .049998 | .050000 | .050009 | .050005 |
| 1 | 1.1 | .0505 | .050676 | .050713 | .050635 | .050703 | .050542 | .050477 | .050763 | .051536 | .051267 |
| 1.05 | 1.05 | .0502 | .050527 | .050604 | .050713 | .050675 | .050321 | .050329 | .050653 | .051633 | .051244 |
| 1 | 1.5 | .0659 | .0641 | .0646 | .0619 | .0641 | .0611 | .0632 | .0649 | .0659 | .0667 |
| 1.25 | 1.25 | .0562 | .0611 | .0628 | .0653 | .0645 | .0571 | .0601 | .0631 | .0697 | .0672 |
| 1 | 2 | .1001 | .0993 | .1003 | .0885 | .0987 | .0899 | .0976 | .1008 | .0955 | .1035 |
| 1.333 | 1.5 | .0679 | .0779 | .0824 | .0881 | .0867 | .0690 | .0764 | .0827 | .0951 | .0911 |
| 1.5 | 1.5 | .0727 | .0880 | .0945 | .1031 | | .0763 | .0863 | .0949 | .1114 | |
| 2 | 2 | .0851 | .1799 | .1993 | .2215 | | .1530 | .1769 | .2001 | .2355 | |
| 1.00001 | 0.9 | .0506 | .050759 | .050811 | .050759 | .050811 | .050627 | .050967 | .050758 | .049784 | .050212 |
| 1.00001 | 0.8 | .0534 | .053544 | .053797 | .053693 | .053829 | .053071 | .053978 | .053687 | .051645 | .052578 |
| 1.01 | 0.99 | .0500 | .050006 | .050004 | .050001 | .050001 | .050009 | .050006 | .050004 | .049997 | .050001 |
| 1.1 | 0.9 | .0513 | .050639 | .050502 | .050068 | .050193 | .050998 | .050636 | .050501 | .049715 | .050169 |
| 1.1 | 0.8 | .0533 | .052306 | .052143 | .050821 | .051580 | .053105 | .052512 | .052088 | .049674 | .050923 |
| 1.05 | 0.95 | .0503 | .050159 | .050125 | .050012 | .050048 | .050248 | .050159 | .050125 | .049928 | .050042 |
| 1.2 | 0.99 | .0522 | .0524 | .0525 | .0520 | .0524 | .0520 | .0520 | .0526 | .0537 | .0534 |
| 1.2 | 0.8 | .0543 | .0520 | .0515 | .0502 | .0503 | .0535 | .0520 | .0515 | .0483 | .0502 |
| 2 | 0.9 | .090 | .093 | .092 | .074 | .086 | .086 | .091 | .092 | .080 | .090 |
| 2 | 0.7 | | .111 | .107 | .086 | .098 | .114 | .110 | .108 | .088 | .101 |
| 3 | 0.9 | | .20 | .20 | .16 | .19 | .18 | .19 | .20 | .17 | .20 |
| 5 | 0.9 | | .45 | .45 | .40 | .46 | .43 | .45 | .45 | .41 | .46 |
| 0.99999 | 0.9 | .0506 | .050759 | .050811 | .050759 | .050811 | .050627 | .050968 | .050758 | .049785 | .050212 |
| 0.99999 | 0.7 | | .0852 | .0859 | .0890 | .0867 | .0861 | .0861 | .0857 | .0851 | .0842 |
| 0.999 | 0.9 | .0506 | .050768 | .050822 | .050776 | .050825 | .050629 | .050979 | .050769 | .049792 | .050220 |
| 0.9 | 0.9 | .0512 | .052491 | .052823 | .053283 | .053115 | .051442 | .052924 | .052715 | .051331 | .051900 |
| 0.9 | 0.8 | .0567 | .057398 | .058121 | .059044 | .058689 | .055077 | .058080 | .057951 | .055977 | .056779 |
| 0.9 | 0.76 | | .0648 | .0657 | .0669 | .0663 | .0622 | .0657 | .0654 | .0653 | .0640 |
| 0.85 | 0.9 | .0533 | .054174 | .054692 | .055377 | .055170 | .052507 | .054725 | .054555 | .052897 | .053586 |
| 0.85 | 0.8 | .0607 | .0619 | .0628 | .0640 | .0635 | .0585 | .0627 | .0626 | .0604 | .0612 |
| 0.81 | 0.9 | .0558 | .056489 | .057173 | .058042 | .057718 | .054292 | .057143 | .057010 | .055101 | .055887 |
| 0.8 | 0.8 | .0727 | .0737 | .0744 | .0755 | .0750 | .0695 | .0747 | .0742 | .0710 | .0721 |

Table 5 (continued)

$n_1 = 7$

| $\lambda_1$ | $\lambda_2$ | $Z^{(2)}$ | With local unbiasedness property $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .0500 | .050000 | .050000 | .050000 | .050000 | .050000 | .049998 | .050002 | .050010 | .050007 |
| 1 | 1.1 | .0508 | .051018 | .051084 | .050970 | .051058 | .050893 | .050837 | .051322 | .051952 | .051792 |
| 1.05 | 1.05 | .0504 | .050796 | .050948 | .051069 | .051033 | .050538 | .050616 | .051186 | .052071 | .051773 |
| 1 | 1.5 | .0722 | .0720 | .0727 | .0685 | .0717 | .0687 | .0711 | .0739 | .0730 | .0751 |
| 1.25 | 1.25 | .0613 | .0668 | .0704 | .0732 | .0724 | .0621 | .0660 | .0715 | .0780 | .0760 |
| 1 | 2 | .1269 | .1298 | .1307 | .1133 | .1284 | .1196 | .1282 | .1328 | .1210 | .1347 |
| 1.333 | 1.5 | .0792 | .0928 | .1018 | .1081 | .1073 | .0824 | .0915 | .1037 | .1158 | .1131 |
| 1.5 | 1.5 | .0835 | .1087 | .1216 | .1312 | | .0951 | .1071 | .1238 | .1404 | |
| 2 | 2 | .2001 | .2568 | .2918 | .3138 | | .2291 | .2543 | .2953 | .3278 | |
| 1.00001 | 0.9 | .0511 | .051121 | .051222 | .051152 | .051214 | .051034 | .051310 | .050973 | .050095 | .050437 |
| 1.00001 | 0.8 | .0563 | .056304 | .056813 | .056801 | .056863 | .056199 | .056699 | .056286 | .054555 | .055220 |
| 1.01 | 0.99 | .0500 | .050009 | .050006 | .050002 | .050002 | .050015 | .050009 | .050006 | .049996 | .050001 |
| 1.1 | 0.9 | .0521 | .050914 | .050619 | .050159 | .050209 | .051607 | .050907 | .050616 | .049678 | .050179 |
| 1.1 | 0.8 | .0558 | .053536 | .053202 | .051708 | .052493 | .055325 | .053714 | .052941 | .050471 | .051640 |
| 1.05 | 0.95 | .0504 | .050229 | .050154 | .050039 | .050052 | .050399 | .050227 | .050154 | .049918 | .050044 |
| 1.2 | 0.99 | .0532 | .0537 | .0538 | .0531 | .0536 | .0533 | .0533 | .0542 | .0549 | .0549 |
| 1.2 | 0.8 | .0530 | .0512 | .0501 | .0500 | .0501 | .0539 | .0511 | .0501 | .0463 | .0490 |
| 2 | 0.9 | | .121 | .118 | .093 | .110 | .115 | .120 | .120 | .100 | .115 |
| 2 | 0.7 | | .157 | .149 | .123 | .137 | .163 | .156 | .150 | .126 | .140 |
| 3 | 0.9 | | .30 | .29 | .24 | .29 | .28 | .29 | .29 | .25 | .30 |
| 5 | 0.9 | | .63 | .63 | .58 | .64 | .62 | .63 | .63 | .59 | .65 |
| 0.99999 | 0.9 | .0511 | .051122 | .051223 | .051153 | .051214 | .051034 | .051310 | .050973 | .050096 | .050438 |
| 0.99999 | 0.7 | | .2227 | .2242 | .2400 | .2286 | .2308 | .2241 | .2224 | .2330 | .2231 |
| 0.999 | 0.9 | .0511 | .051135 | .051241 | .051177 | .051236 | .051037 | .051325 | .050989 | .050110 | .050452 |
| 0.9 | 0.9 | .0528 | .053832 | .054486 | .054980 | .054823 | .052492 | .054225 | .053974 | .052856 | .053245 |
| 0.9 | 0.8 | .0676 | .068522 | .069654 | .070665 | .070187 | .066041 | .069177 | .068799 | .067137 | .067558 |
| 0.9 | 0.76 | | .1115 | .1118 | .1149 | .1122 | .1104 | .1124 | .1106 | .1090 | .1084 |
| 0.85 | 0.9 | .0548 | .057079 | .058067 | .055784 | .058550 | .055007 | .057583 | .057410 | .056065 | .056529 |
| 0.85 | 0.8 | | .0927 | .0931 | .0937 | .0932 | .0896 | .0936 | .0920 | .0891 | .0897 |
| 0.81 | 0.9 | .0631 | .064557 | .065708 | .066629 | .066246 | .062078 | .065174 | .064903 | .063304 | .063771 |
| 0.8 | 0.8 | | .1836 | .1792 | .1762 | .1764 | .1830 | .1850 | .1774 | .1687 | .1707 |

Table 5 (continued)

$n_1 = 13$

| | | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $Z^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .0500 | .050000 | .050000 | .050000 | .050000 | .050000 | .049997 | .050006 | .050011 | .050010 |
| 1 | 1.1 | .0519 | .052054 | .052203 | .052011 | .052134 | .051988 | .051807 | .052771 | .053102 | .053088 |
| 1.05 | 1.05 | .0509 | .051596 | .052022 | .052157 | .052122 | .051214 | .051355 | .052591 | .053268 | .053085 |
| 1 | 1.5 | .0971 | .0981 | .0984 | .0905 | .0960 | .0942 | .0968 | .1012 | .0954 | .1005 |
| 1.25 | 1.25 | .0757 | .0842 | .0943 | .0976 | .0969 | .0777 | .0830 | .0971 | .1030 | .1016 |
| 1 | 2 | .2315 | .2372 | .2340 | .2044 | .2359 | .2261 | .2350 | .2386 | .2125 | .2438 |
| 1.333 | 1.5 | .1245 | .1402 | .1652 | .1717 | .1734 | .1267 | .1384 | .1696 | .1803 | .1811 |
| 1.5 | 1.5 | .1517 | .1754 | .2107 | .2210 | | .1583 | .1733 | .2159 | .2309 | |
| 2 | 2 | | .4916 | .5559 | .5717 | | .4657 | .4891 | .5616 | .5822 | |
| 1.00001 | 0.9 | .0523 | .052223 | .052505 | .052422 | .052484 | .052358 | .052473 | .051912 | .051253 | .051479 |
| 1.00001 | 0.8 | | .128 | .130 | .133 | .131 | .131 | .129 | .128 | .130 | .128 |
| 1.01 | 0.99 | .0500 | .050017 | .050008 | .050002 | .050002 | .050034 | .050016 | .050008 | .049996 | .050002 |
| 1.1 | 0.9 | .0541 | .051694 | .050795 | .050206 | .050215 | .053485 | .051672 | .050792 | .049614 | .050180 |
| 1.1 | 0.8 | | .0651 | .0647 | .0641 | .0643 | .0705 | .0653 | .0640 | .0626 | .0631 |
| 1.05 | 0.95 | .0511 | .050428 | .050202 | .050042 | .050057 | .050865 | .050422 | .050201 | .049905 | .050048 |
| 1.2 | 0.99 | .0572 | .0576 | .0577 | .0566 | .0573 | .0574 | .0571 | .0588 | .0586 | .0591 |
| 1.2 | 0.8 | | .0758 | .0773 | .0816 | .0804 | .0807 | .0758 | .0770 | .0811 | .0801 |
| 2 | 0.9 | | .222 | .207 | .168 | .197 | .216 | .220 | .211 | .175 | .204 |
| 2 | 0.7 | | .341 | .322 | .299 | .310 | .354 | .340 | .324 | .301 | .312 |
| 3 | 0.9 | | .58 | .56 | .52 | .58 | .57 | .58 | .57 | .52 | .59 |
| 5 | 0.9 | | .91 | .91 | .90 | .93 | .91 | .91 | .91 | .90 | .93 |
| 0.99999 | 0.9 | .0523 | .052224 | .052505 | .052423 | .052485 | .052358 | .052473 | .051912 | .051253 | .051479 |
| 0.99999 | 0.7 | | | | | | | | | | |
| 0.999 | 0.9 | .0522 | .052254 | .052549 | .052474 | .052533 | .052369 | .052507 | .051950 | .051293 | .051517 |
| 0.9 | 0.9 | .0626 | .064878 | .056278 | .066678 | .066562 | .062975 | .065432 | .064998 | .064203 | .064411 |
| 0.9 | 0.8 | | | | | | | | | | |
| 0.9 | 0.76 | | | | | | | | | | |
| 0.85 | 0.9 | | .148 | .147 | .147 | .146 | .147 | .149 | .144 | .143 | .142 |

Table 6

Powers of $LS_1^{(2)}$, $LS_2^{(2)}$, $LS_3^{(2)}$, $L_2^{(2)}$ (unbiased) and $LS^{(2)}$ for testing $\lambda_1 = 1$, $\lambda_2 = 1$

against different simple two-sided alternative hypotheses, $\alpha = .05$

| $\lambda_1$ | $\lambda_2$ | $LS_1^{(2)}$ | $LS_2^{(2)}$ | $LS_3^{(2)}$ | $L_2^{(2)}$ | $LS^{(2)}$ | $LS_1^{(2)}$ | $LS_2^{(2)}$ | $LS_3^{(2)}$ | $L_2^{(2)}$ | $LS^{(2)}$ | $LS_1^{(2)}$ | $LS_2^{(2)}$ | $LS_3^{(2)}$ | $L_2^{(2)}$ | $LS^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m = 0, n = 5 | | | | | m = 0, n = 15 | | | | | m = 0, n = 30 | | | | |
| 1 | 1.001 | .050029 | .050028 | .049975 | .050000 | .050000 | .050041 | .050041 | .049975 | .050000 | .050000 | .050046 | .050045 | .049975 | .050000 | .050000 |
| 1.05 | 1.05 | .053180 | .053108 | .047681 | .050434 | .050203 | .054582 | .054498 | .047678 | .050586 | .050286 | .055132 | .055043 | .047677 | .050649 | .050313 |
| 1 | 1.5 | .0736 | .0733 | .0418 | .0624 | .0576 | .0852 | .0848 | .0418 | .0677 | .0611 | .0899 | .0895 | .0418 | .0700 | .0626 |
| 1.25 | 1.25 | .0704 | .0701 | .0402 | .0594 | .0547 | .0802 | .0798 | .0402 | .0630 | .0569 | .0842 | .0837 | .0402 | .0646 | .0578 |
| 1.333 | 1.5 | .0900 | .0893 | .0358 | .0739 | .0630 | .1097 | .1090 | .0358 | .0838 | .0692 | .1177 | .1170 | .0357 | .0878 | .0720 |
| 1 | 4 | .298 | .296 | .090 | .267 | .220 | .381 | .380 | .090 | .334 | .280 | .410 | .409 | .090 | .358 | .304 |
| 3 | 3 | .371 | .369 | .115 | .339 | .273 | .472 | .470 | .115 | .418 | .343 | .506 | .505 | .115 | .447 | .372 |
| 1 | 8 | .573 | .571 | .335 | .546 | .494 | .658 | .657 | .335 | .620 | .569 | .685 | .684 | .335 | .646 | .596 |
| 1.00001 | 0.9 | .047498 | .047569 | .052697 | .050649 | .050379 | .046469 | .046550 | .052700 | .050873 | .050489 | .046085 | .046170 | .052702 | .050963 | .050533 |
| 1.1 | 0.8 | .048935 | .049006 | .053909 | .052114 | .051782 | .048521 | .048598 | .053915 | .052872 | .052186 | .048409 | .048488 | .053918 | .053175 | .052343 |
| 1.05 | 0.95 | .05165 | .050165 | .050121 | .050164 | .050154 | .050247 | .050247 | .050121 | .050235 | .050197 | .050283 | .050283 | .050122 | .050264 | .050214 |
| 2 | 0.7 | .1064 | .1059 | .0600 | .0914 | .0848 | .1291 | .1285 | .0600 | .1057 | .0947 | .1384 | .1378 | .0600 | .1118 | .0992 |
| 3 | 0.9 | .195 | .193 | .035 | .151 | .133 | .259 | .258 | .035 | .216 | .176 | .282 | .281 | .035 | .235 | .193 |
| 6 | 0.9 | .373 | .372 | .116 | .340 | .286 | .468 | .467 | .116 | .421 | .364 | .499 | .498 | .116 | .449 | .393 |
| 0.99999 | 0.8 | .045947 | .046088 | .056110 | .052787 | .051643 | .044305 | .044464 | .056119 | .053663 | .052043 | .043721 | .043886 | .056122 | .054006 | .052197 |
| 0.99999 | 0.7 | .0474 | .0476 | .0646 | .0589 | .0567 | .0446 | .0448 | .0646 | .0600 | .0567 | .0436 | .0439 | .0646 | .0604 | .0566 |
| 0.999 | 0.9 | .047471 | .047542 | .052721 | .050656 | .050381 | .046430 | .046512 | .052725 | .050881 | .050491 | .046042 | .046128 | .052726 | .050972 | .050535 |
| 0.9 | 0.8 | .043844 | .044053 | .058877 | .054955 | .052280 | .041412 | .041647 | .058889 | .056406 | .052878 | .040551 | .040796 | .058893 | .056972 | .053110 |
| 0.85 | 0.9 | .044358 | .044532 | .056978 | .053216 | .051417 | .042140 | .042338 | .056988 | .054208 | .051844 | .041337 | .041544 | .056991 | .054599 | .052014 |
| 0.8 | 0.8 | .0450 | .0433 | .0650 | .0593 | .0543 | .0401 | .0404 | .0630 | .0616 | .0551 | .0391 | .0394 | .0630 | .0625 | .0554 |
| | | m = 1, n = 5 | | | | | m = 1, n = 15 | | | | | m = 1, n = 30 | | | | |
| 1 | 1.001 | .050021 | .050021 | .049956 | .050000 | .050000 | .050038 | .050037 | .049955 | .050000 | .050000 | .050045 | .050044 | .049955 | .050000 | .050000 |
| 1.05 | 1.05 | .052570 | .052492 | .045878 | .050621 | .050374 | .054459 | .054347 | .045842 | .050901 | .050536 | .055247 | 055148 | .045830 | .051027 | .05061[illegible] |
| 1 | 1.5 | .0747 | .0743 | .0368 | .0672 | .0633 | .0921 | .0916 | .0367 | .0770 | .0704 | .1001 | .0995 | .0367 | .0816 | .0739 |
| 1.25 | 1.25 | .0697 | .0693 | .0336 | .0632 | .0585 | .0837 | .0833 | .0335 | .0698 | .0628 | .0901 | .0896 | .0334 | .0729 | .0649 |
| 1.333 | 1.5 | .0918 | .0912 | .0274 | .0835 | .0729 | .1212 | .1204 | .0273 | .1011 | .0853 | .1346 | .1337 | .0273 | .1092 | .0914 |
| 1 | 4 | .596 | .595 | .207 | .584 | .554 | .506 | .504 | .207 | .475 | .436 | .548 | .547 | .207 | .513 | .472 |
| 3 | 3 | .494 | .493 | .278 | .486 | .446 | .610 | .608 | .278 | .581 | .532 | .654 | .653 | .278 | .620 | .570 |
| 1 | 8 | .747 | .746 | .621 | .740 | .722 | .817 | .817 | .621 | .802 | .779 | .842 | .842 | .621 | .825 | .803 |
| 1.00001 | 0.9 | .048512 | .048590 | .055019 | .050895 | .050713 | .047228 | .047318 | .055073 | .051296 | .050955 | .046715 | .046808 | .055090 | .051471 | .051057 |
| 1.1 | 0.8 | .051058 | .051142 | .057874 | .052727 | .053447 | .050706 | .050797 | .057983 | .053974 | .054365 | .050651 | .050744 | .058018 | .054523 | .054736 |
| 1.05 | 0.95 | .050270 | .050270 | .050266 | .050198 | .050281 | .050404 | .050404 | .050270 | .050318 | .050376 | .050471 | .050471 | .050272 | .050373 | .050418 |
| 2 | 0.7 | .1328 | .1324 | .0846 | .1190 | .1197 | .1659 | .1653 | .0848 | .1423 | .1381 | .1814 | .1808 | .0849 | .1534 | .1470 |
| 3 | 0.9 | .243 | .242 | .081 | .228 | .206 | .340 | .338 | .081 | .307 | .273 | .379 | .377 | .0813 | .339 | .304 |
| 5 | 0.9 | .505 | .504 | .311 | .492 | .465 | .616 | .615 | .311 | .588 | .552 | .656 | .655 | .311 | .625 | .589 |
| 0.99999 | 0.8 | .048872 | .049034 | .062079 | .054072 | .053393 | .046846 | .047027 | .062222 | .055576 | .054249 | .046083 | .046272 | .062269 | .056216 | .054589 |
| 0.99999 | 0.7 | .0751 | .0754 | .1084 | .0942 | .0850 | .0643 | .0647 | .1087 | .0803 | .0805 | .0601 | .0605 | .1088 | .0785 | .0782 |
| 0.999 | 0.9 | .048492 | .048572 | .055064 | .050905 | .050715 | .047194 | .047285 | .055118 | .051309 | .050958 | .046673 | .046769 | .055135 | .051487 | .051061 |
| 0.9 | 0.8 | .048769 | .049010 | .068489 | .058288 | .055503 | .045575 | .045845 | .068688 | .060781 | .056629 | .044371 | .044653 | .068753 | .061829 | .057070 |
| 0.05 | 0.9 | .047353 | .047549 | .063572 | .054819 | .052833 | .044635 | .044858 | .063519 | .056662 | .053743 | .043591 | .043824 | .063567 | .057449 | .054121 |
| 0.8 | 0.8 | .0597 | .0601 | .0901 | .0747 | .0697 | .0535 | .0539 | .0904 | .0771 | .0699 | .0511 | .0515 | .0905 | .0780 | .0698 |

Table 6 (continued)

| $\lambda_1$ | $\lambda_2$ | $LS_1^{(2)}$ | $LS_2^{(2)}$ | $LS_3^{(2)}$ | $L_2^{(2)}$ | $LS^{(2)}$ | $LS_1^{(2)}$ | $LS_2^{(2)}$ | $LS_3^{(2)}$ | $L_2^{(2)}$ | $LS^{(2)}$ | $LS_1^{(2)}$ | $LS_2^{(2)}$ | $LS_3^{(2)}$ | $L_2^{(2)}$ | $LS^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | m = 2, n = 5 | | | | | m = 2, n = 15 | | | | | m = 2, n = 30 | | | | |
| 1 | 1.001 | .050015 | .050014 | .049940 | .050000 | .050000 | .050034 | .050033 | .049939 | .050000 | .050000 | .050043 | .050042 | .049939 | .050000 | .050000 |
| 1.05 | 1.05 | .052031 | .051948 | .044525 | .050760 | .050516 | .054246 | .054145 | .044405 | .051168 | .050768 | .055281 | .055173 | .044356 | .051367 | .050895 |
| 1 | 1.5 | .0754 | .0750 | .0338 | .0708 | .0678 | .0979 | .0974 | .0336 | .0850 | .0788 | .1093 | .1087 | .0335 | .0923 | .0845 |
| 1.25 | 1.25 | .0691 | .0686 | .0292 | .0660 | .0615 | .0865 | .0860 | .0228 | .0755 | .0681 | .0952 | .0947 | .0287 | .0803 | .0716 |
| 1.333 | 1.5 | .0934 | .0928 | .0229 | .0907 | .0808 | .1310 | .1302 | .0224 | .1158 | .0997 | .1498 | .1489 | .0223 | .1285 | .1096 |
| 1 | 4 | .502 | .500 | .351 | .499 | .481 | .612 | .611 | .350 | .594 | .567 | .660 | .659 | .350 | .636 | .607 |
| 3 | 3 | .620 | .618 | .460 | .622 | .599 | .723 | .722 | .459 | .710 | .679 | .767 | .766 | .459 | .748 | .716 |
| 1 | 8 | .865 | .865 | .806 | .864 | .858 | .908 | .907 | .806 | .902 | .893 | .924 | .924 | .806 | .917 | .908 |
| 1.00001 | 0.9 | .049437 | .049523 | .056903 | .051071 | .050998 | .048006 | .048105 | .057087 | .051644 | .051387 | .047396 | .047501 | .057147 | .051915 | .051560 |
| 1.1 | 0.8 | .053363 | .053465 | .061767 | .053395 | .055185 | .053096 | .053203 | .062158 | .054999 | .056646 | .053106 | .053214 | .062292 | .055763 | .057264 |
| 1.05 | 0.95 | .050367 | .050368 | .050406 | .050215 | .050385 | .050550 | .050550 | .050423 | .050378 | .050539 | .050650 | .050650 | .050430 | .050461 | .050610 |
| 2 | 0.7 | .1688 | .1684 | .1238 | .1553 | .1614 | .2079 | .2073 | .1243 | .1844 | .1862 | .2284 | .2278 | .1246 | .1998 | .1992 |
| 3 | 0.9 | .298 | .297 | .145 | .292 | .277 | .413 | .412 | .144 | .389 | .362 | .464 | .463 | .144 | .433 | .404 |
| 5 | 0.9 | .632 | .631 | .498 | .628 | .614 | .730 | .729 | .498 | .714 | .693 | .770 | .769 | .498 | .751 | .729 |
| 0.99999 | 0.8 | .052736 | .052922 | .068600 | .056131 | .056080 | .050265 | .050473 | .069111 | .058015 | .057274 | .049257 | .049473 | .069283 | .058864 | .057746 |
| 0.99999 | 0.7 | .219 | .220 | .299 | .225 | .233 | .172 | .173 | .301 | .192 | .203 | .150 | .151 | .301 | .175 | .187 |
| 0.999 | 0.9 | .049425 | .049512 | .056964 | .051084 | .051001 | .047977 | .048078 | .057149 | .051663 | .051392 | .047360 | .047466 | .057210 | .051937 | .051566 |
| 0.9 | 0.8 | .060728 | .061017 | .085981 | .068291 | .065952 | .055596 | .055922 | .086697 | .070673 | .066687 | .053390 | .053729 | .086935 | .071670 | .066852 |
| 0.85 | 0.9 | .050915 | .051135 | .069676 | .056841 | .054881 | .047801 | .048051 | .070193 | .059454 | .056228 | .046528 | .046790 | .070365 | .060647 | .056808 |
| 0.8 | 0.8 | .1722 | .1729 | .2354 | .1845 | .1839 | .1458 | .1465 | .2366 | .1732 | .1720 | .1332 | .1339 | .2370 | .1671 | .1654 |
| | | m = 5, n = 5 | | | | | m = 5, n = 15 | | | | | m = 5, n = 30 | | | | |
| 1 | 1.001 | .050000 | .049999 | .050000 | .050000 | .050000 | .050023 | .050022 | .049904 | .050000 | .050000 | .050036 | .050035 | .049902 | .050000 | .050000 |
| 1.05 | 1.05 | .050869 | .050773 | .050821 | .051026 | .050821 | .053705 | .053585 | .041388 | .051776 | .051347 | .055291 | .055158 | .041203 | .052154 | .051662 |
| 1 | 1.5 | .0783 | .0778 | .0781 | .0789 | .0781 | .1126 | .1120 | .0301 | .1048 | .0999 | .1343 | .1336 | .0299 | .1209 | .1138 |
| 1.25 | 1.25 | .0680 | .0676 | .0678 | .0713 | .0678 | .0932 | .0926 | .0207 | .0885 | .0811 | .1086 | .1079 | .0203 | .0987 | .0893 |
| 1.333 | 1.5 | .1001 | .0994 | .0998 | .1072 | .0998 | .1564 | .1554 | .0188 | .1516 | .1361 | .1910 | .1899 | .0184 | .1731 | .1589 |
| 1 | 4 | .777 | .776 | .777 | .780 | .777 | .839 | .838 | .720 | .836 | .828 | .872 | .871 | .720 | .866 | .857 |
| 3 | 3 | .877 | .876 | .876 | .883 | .876 | .919 | .919 | .825 | .920 | .912 | .940 | .940 | .825 | .938 | .931 |
| 1 | 8 | .986 | .986 | .986 | .986 | .986 | .991 | .991 | .982 | .991 | .990 | .993 | .993 | .982 | 993 | .992 |
| 1.00001 | 0.9 | .051650 | .051755 | .051702 | .051453 | .051702 | .050159 | .050282 | .061873 | .052484 | .052555 | .049422 | .049555 | .062219 | .053123 | .052984 |
| 1.1 | 0.8 | .0589 | .0690 | .0689 | .0645 | .0669 | .0677 | .0679 | .0842 | .0653 | .0709 | .0670 | .0672 | .0851 | .0656 | .0716 |
| 1.05 | 0.95 | .050606 | .050609 | .050607 | .050234 | .050607 | .050931 | .050932 | .050857 | .050496 | .050946 | .051139 | .051139 | .050899 | .050658 | .051130 |
| 2 | 0.7 | .3320 | .3317 | .3318 | .3189 | .3318 | .3708 | .3704 | .3060 | .3498 | .3620 | .3963 | .3958 | .3072 | .3699 | .3810 |
| 3 | 0.9 | .510 | .509 | .510 | .513 | .510 | .617 | .616 | .414 | .609 | .597 | .677 | .676 | .414 | .663 | .649 |
| 5 | 0.9 | .891 | .891 | .891 | .892 | .891 | .924 | .924 | .861 | .922 | .919 | .942 | .942 | .861 | .939 | .935 |
| 0.99999 | 0.8 | .1350 | .1354 | .1352 | .1329 | .1352 | .1217 | .1221 | .1695 | .1266 | .1298 | .1133 | .1137 | .1706 | .1223 | .1256 |
| 0.99999 | 0.7 | | | | | | | | | | | | | | | |
| 0.999 | 0.9 | .051658 | .051764 | .051711 | .051475 | .051711 | .050147 | .050271 | .061977 | .052518 | .052566 | .049401 | .049534 | .062325 | .053164 | .052997 |
| 0.9 | 0.8 | | | | | | | | | | | | | | | |
| 0.85 | 0.9 | .1501 | .1505 | .1503 | .1507 | .1503 | .1373 | .1378 | .1890 | .1477 | .1466 | .1297 | .1302 | .1900 | .1456 | .1438 |
| 0.8 | 0.8 | | | | | | | | | | | | | | | |

# CHAPTER II

# SOME RESULTS ON TRANSFORMATIONS IN THE TWO-SAMPLE COMPLEX GAUSSIAN CASE AND EXACT POWER COMPARISONS OF TWO-SIDED TESTS OF EQUALITY OF TWO COVARIANCE MATRICES

## 1. Introduction and Summary

In the complex multivariate normal theory [Goodman, 6 ], the study of distribution problems concerning MANOVA, canonical correlation and equality of two covariance matrices was made by several authors, notably by Khatri [10] and James [8 ]. The non-central distributions of the characteristic roots concerning the various test procedures were explicitly given by James [8 ] in terms of zonal polynomials of complex hermitian matrices. Here, we consider the problem of testing the equality of covariance matrices of two complex normal populations.

Let $\underset{\sim}{X}_1(p \times n_1)$ and $\underset{\sim}{X}_2(p \times n_2)$ $p \leq n_1, n_2$, be independent complex matrix variates, columns of $\underset{\sim}{X}_1$ being independently distributed as $CN(\underset{\sim}{0},\underset{\sim}{\Sigma}_1)$ and those of $\underset{\sim}{X}_2$ independently distributed as $CN(\underset{\sim}{0},\underset{\sim}{\Sigma}_2)$. Let $0 \leq c_1 \leq \ldots \leq c_p < \infty$ be the characteristic roots of $|X_1\bar{X}_1' - cX_2\bar{X}_2'| = 0$ and $\lambda_1,\ldots,\lambda_p$, the characteristic roots of $|\underset{\sim}{\Sigma}_1 - \lambda\underset{\sim}{\Sigma}_2| = 0$. To test $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ or equivalently $\lambda_1 = \ldots = \lambda_p = 1$ against $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$ (two-sided), the following five criteria are considered:

1) Roy's largest root, $c_p$, [Roy, 36] or $L_p^{(p)} = c_p/(1+c_p)$,

2) Hotelling's trace, $U^{(p)} = \sum_{i=1}^{p} c_i$, [Pillai, 20],

3) Pillai's trace, $V^{(p)} = \sum_{i=1}^{p} [c_i/(1+c_i)]$, [Pillai, 20],

4) Wilks' criterion, $W^{(p)} = \prod_{i=1}^{p} (1+c_i)^{-1}$, [Wilks, 43],

5) Roy's largest-smallest roots, $c_1(L_1^{(p)} = c_1/(1+c_1))$ and $c_p(L_p^{(p)} = c_p/(1+c_p))$, to be denoted by $LS^{(p)}$ defined in terms of $L_p^{(p)}$ and $L_1^{(p)}$, [Roy, 37].

Power studies of tests of $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against the alternative of a one-sided nature:

$$\lambda_i \geq 1, \quad \sum_{i=1}^{p} \lambda_i > p, \quad i = 1,\ldots,p,$$

were carried out by Pillai and Hsu [27] based on the first four criteria. Exact power tabulations were made in the two-roots case for various $(\lambda_1,\lambda_2)$ and different degrees of freedom $n_1$ and $n_2$ (actually in terms of $m = n_1-2$ and $n = n_2-2$).

In this chapter, a power comparison study has been attempted for tests of the hypothesis $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$ (two-sided) based on the above five criteria. A theorem which is similar to the one in the real case is proved first obtaining the condition of local unbiasedness for a class of tests of which 1) to 5) are special cases. Using the theorem, relations between the two critical values for each of the five tests are obtained as special cases for tests 1) to 5) for the two-roots case. Further, critical values for level $\alpha = .05$ (five percent points) for the five tests are computed for $p = 2$ and values of $n_1 = 2, 3, 4, 7$ and $n_2 = 7$, 17, 32, 62 and are given in Table 7. Also, powers of the criteria 1) to 5) have been tabulated for various values of $(\lambda_1, \lambda_2)$, $n_1 = 2, 3, 4, 7$ and $n_2 = 7, 17, 32, 62$ and these are presented in Table 8. In addition, power tabulations have also been carried out from the equal tail areas point of view, of tests 1) to 4) which are observed to be biased. These tabulations are also available in Table 8 for the same values of $(\lambda_1, \lambda_2)$, $n_1$ and $n_2$ as before facilitating comparisons with powers in the unbiased case. The critical values in this case are also given in Table 7.

A few findings seem to emerge from the numerical results of powers tabulated and there is a general agreement with those discussed before in the real case. In general it is observed that the largest root has some power advantage over the other criteria studied. These findings are presented in Section 6.

Some results on transformations and Jacobians in the two-sample complex Gaussian case are proved in the next section which are needed in the sequel.

## 2. Some results on transformations and Jacobians

Lemma 1. If $\underset{\sim}{M}(p \times p)$ is hermitian and at least positive semi-definite of rank $r(\leq p)$, then there exists a unitary matrix $\underset{\sim}{A}$ with real first row such that $\underset{\sim}{M} = \underset{\sim}{A}\, \underset{\sim}{D}_{c_k}\, \bar{\underset{\sim}{A}}'$ where $\underset{\sim}{D}_{c_k}$ is diagonal with elements the characteristic roots $c_k$'s of $\underset{\sim}{M}$.

Proof. From linear algebra, we know that there exists a unitary matrix $\underset{\sim}{B}$ such that $\underset{\sim}{M} = \underset{\sim}{B}\,\underset{\sim}{D}_{c_k}\,\bar{\underset{\sim}{B}}'$ where the columns of $\underset{\sim}{B}$ are the unit characteristic vectors of $\underset{\sim}{M}$. If each column of $\underset{\sim}{B}$ is multiplied by some constant of absolute value unity, the resulting matrix preserves the properties of $\underset{\sim}{B}$, i.e. the resulting matrix is unitary and diagonalizes $\underset{\sim}{M}$. Therefore, if we multiple the jth column of $\underset{\sim}{B} = (b_{ij})$ by $\bar{b}_{1j}/|b_{1j}|$ for $j = 1,\ldots,p$, and denote the resulting matrix by $\underset{\sim}{A} = (a_{ij})$, $(a_{ij} = b_{ij}\bar{b}_{1j}/|b_{1j}|)$, then $\underset{\sim}{A}$ is unitary and diagonalizes $\underset{\sim}{M}$, and moreover, the first row of $\underset{\sim}{A}$ is real. Hence the lemma.

Lemma 2. If $\underset{\sim}{M}_1(pxp)$ is hermitian and at least positive semi-definite of rank $r(\leq p)$ and $\underset{\sim}{M}_2(pxp)$ is hermitian positive definite, then there exists a non-singular matrix $\underset{\sim}{A}$ with real first **row** such that $\underset{\sim}{M}_1 = \underset{\sim}{A}\,\underset{\sim}{D}_{c_k}\,\bar{\underset{\sim}{A}}'$ and $\underset{\sim}{M}_2 = \underset{\sim}{A}\bar{\underset{\sim}{A}}'$, where $c_k'$ are the roots of the equation $|\underset{\sim}{M}_1 - c\underset{\sim}{M}_2| = 0$.

Proof. Since $\underset{\sim}{M}_2$ is hermitian positive definite, there exists a lower triangular matrix $\underset{\sim}{T}$ with real diagonal elements such that $\underset{\sim}{M}_2 = \underset{\sim}{T}\bar{\underset{\sim}{T}}'$. Now $\underset{\sim}{T}^{-1}\underset{\sim}{M}_1\bar{\underset{\sim}{T}}'^{-1}$ is hermitian and at least positive semi-definite. By Lemma 1, there exists a unitary matrix $\underset{\sim}{B}$ with real first row such that $\underset{\sim}{T}^{-1}\underset{\sim}{M}_1\bar{\underset{\sim}{T}}'^{-1} = \underset{\sim}{B}\,\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{B}}'$ where $c_k$'s are the characteristic roots of $\underset{\sim}{M}_1\underset{\sim}{M}_2^{-1}$. Let $\underset{\sim}{A} = \underset{\sim}{T}\underset{\sim}{B}$. $\underset{\sim}{A}$ is non-singular with real first row. Then $\underset{\sim}{M}_1 = \underset{\sim}{T}\underset{\sim}{B}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{B}}'\bar{\underset{\sim}{T}}' = \underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}'$, $\underset{\sim}{M}_2 = \underset{\sim}{T}\bar{\underset{\sim}{T}}' = \underset{\sim}{T}\underset{\sim}{B}\bar{\underset{\sim}{B}}'\bar{\underset{\sim}{T}}' = \underset{\sim}{A}\bar{\underset{\sim}{A}}'$.

Lemma 3. The matrix $\underset{\sim}{A}$ of Lemma 2 will be unique, except for a post-factor $\underset{\sim}{D}_k = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$, if $\underset{\sim}{M}_1$ is positive definite and all the characteristic roots $c_k$'s are distinct.

Proof. Suppose there are two non-singular $\underset{\sim}{A}$'s with real first row, say $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$, satisfying the conditions of Lemma 2. Then we have

$$\underset{\sim}{M}_1 = \underset{\sim}{A}_1 \underset{\sim}{D}_{c_k} \bar{\underset{\sim}{A}}_1' = \underset{\sim}{A}_2 \underset{\sim}{D}_{c_k} \bar{\underset{\sim}{A}}_2',$$

$$\underset{\sim}{M}_2 = \underset{\sim}{A}_1 \bar{\underset{\sim}{A}}_1' = \underset{\sim}{A}_2 \bar{\underset{\sim}{A}}_2'.$$

Thus $\quad \underset{\sim}{A}_2^{-1} \underset{\sim}{A}_1 \underset{\sim}{D}_{c_k} = \underset{\sim}{D}_{c_k} \bar{\underset{\sim}{A}}_2' \bar{\underset{\sim}{A}}_1'^{-1} = \underset{\sim}{D}_{c_k} \underset{\sim}{A}_2^{-1} \underset{\sim}{A}_1,$

i.e. $\quad \underset{\sim}{B}\underset{\sim}{D}_{c_k} = \underset{\sim}{D}_{c_k}\underset{\sim}{B}$ where $\underset{\sim}{B} = \underset{\sim}{A}_2^{-1}\underset{\sim}{A}_1 = (\beta_{ij})$.

And this implies $\beta_{ij}c_j = c_i\beta_{ij}$.

So we have $\quad \beta_{ij} = 0$ if $i \neq j$ and $c_i \neq c_j$.

Thus $\quad \underset{\sim}{B} = \underset{\sim}{D}_{\alpha_k}$ (say) where $\alpha_k = a_k + ib_k$, $k = 1,\ldots,p$.

And $\quad \underset{\sim}{D}_{|\alpha_k|^2} = \underset{\sim}{D}_{\alpha_k}\underset{\sim}{D}_{\bar{\alpha}_k} = \underset{\sim}{B}\bar{\underset{\sim}{B}}' = \underset{\sim}{A}_2^{-1}\underset{\sim}{A}_1\bar{\underset{\sim}{A}}_1'\bar{\underset{\sim}{A}}_2'^{-1} = \underset{\sim}{A}_2^{-1}\underset{\sim}{A}_2\bar{\underset{\sim}{A}}_2'\bar{\underset{\sim}{A}}_2'^{-1} = \underset{\sim}{I}.$

So $\quad |\alpha_k| = 1$ for $k = 1,\ldots,p$.

Therefore $\underset{\sim}{A}_1 = \underset{\sim}{A}_2\underset{\sim}{D}_{\alpha_k}$ where $|\alpha_k| = 1$, $k = 1,\ldots,p$.

Since the first row elements of both $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ are real, so $\alpha_k$ must be real, i.e. $b_k = 0$ and $a_k = \pm 1$, $k = 1,\ldots,p$. Thefore $\underset{\sim}{A}_1 = \underset{\sim}{A}_2\underset{\sim}{D}_k$ where $\underset{\sim}{D}_k = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$. We note that $\underset{\sim}{A}$ can thus be made unique by choosing the real first row positive. The transformation is now one to one.

Lemma 4. If $\underset{\sim}{X}_1(p \times n_1)$, $\underset{\sim}{X}_2(p \times n_2)$, $(p \leq n_1, n_2)$ are each of rank p, then there exists a transformation $\underset{\sim}{X}_1(p \times n_1) = \underset{\sim}{A}(p \times p)\underset{\sim}{D}_{\sqrt{c_k}}(p \times p)\underset{\sim}{L}_1(p \times n_1)$

and $\underset{\sim}{X}_2(p\text{x}n_2) = \underset{\sim}{A}(p\text{x}p)\underset{\sim}{L}_2(p\text{x}n_2)$ where $\underset{\sim}{A}$ is non-singular with real first row, $c_k$'s are the roots (all positive) of the equation $|\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1' - c\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2'| = 0$ and $\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1' = \underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2' = \underset{\sim}{I}$. If all $c_k$'s are distinct, then this transformation is unique except for a post-factor $\underset{\sim}{D}_k$ to go with $\underset{\sim}{A}$.

Proof. By Lemma 2, there exists a non-singular $\underset{\sim}{A}$ with real first row such that $\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1' = \underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}'$ and $\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2' = \underset{\sim}{A}\bar{\underset{\sim}{A}}'$. We now define $\underset{\sim}{L}_1(p\text{x}n_1)$ and $\underset{\sim}{L}_2(p\text{x}n_2)$ by $\underset{\sim}{X}_1 = \underset{\sim}{A}\underset{\sim}{D}_{\sqrt{c_k}}\underset{\sim}{L}_1$ and $\underset{\sim}{X}_2 = \underset{\sim}{A}\underset{\sim}{L}_2$ and note that, given $\underset{\sim}{X}_1, \underset{\sim}{X}_2$ and $c_k$'s and $\underset{\sim}{A}$, $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ are uniquely solvable. Also $\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1' = \underset{\sim}{D}_{\frac{1}{\sqrt{c_k}}}\underset{\sim}{A}^{-1}\underset{\sim}{X}_1 \times \bar{\underset{\sim}{X}}_1'\bar{\underset{\sim}{A}}'^{-1}\underset{\sim}{D}_{\frac{1}{\sqrt{c_k}}} = \underset{\sim}{I}$ and $\underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2' = \underset{\sim}{A}^{-1}\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2'\bar{\underset{\sim}{A}}'^{-1} = \underset{\sim}{I}$. This proves the existence of the transformation. Notice that if all $c_k$'s are distinct, then by Lemma 3, $\underset{\sim}{A}$ is unique except for a post-factor $\underset{\sim}{D}_k$ and that $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ will go with $\underset{\sim}{A}$ being defined by $\underset{\sim}{L}_1 = \underset{\sim}{D}_{\frac{1}{\sqrt{c_k}}}\underset{\sim}{A}^{-1}\underset{\sim}{X}_1$ and $\underset{\sim}{L}_2 = \underset{\sim}{A}^{-1}\underset{\sim}{X}_2$.

Lemma 5. If $\underset{\sim}{X}_1(p\text{x}n_1)$, $\underset{\sim}{X}_2(p\text{x}n_2)(p \leq n_1, n_2)$ have the joint density:

$$\pi^{-p(n_1+n_2)}|\underset{\sim}{\Sigma}_1|^{-n_1}|\underset{\sim}{\Sigma}_2|^{-n_2}\exp[-\text{tr}(\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1' + \underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')],$$

where $\underset{\sim}{\Sigma}_1$, $\underset{\sim}{\Sigma}_2$ are hermitian positive definite, then the distribution of the characteristic roots of $(\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1')(\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')^{-1}$ involves as parameters only the characteristic roots of $\underset{\sim}{\Sigma}_1\underset{\sim}{\Sigma}_2^{-1}$ (to be called $\lambda_k$'s).

Proof. Since $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{\Sigma}_2$ are hermitian positive definite, by Lemma 2, $\underset{\sim}{\Sigma}_1 = \underset{\sim}{A}\underset{\sim}{D}_{\lambda_k}\bar{\underset{\sim}{A}}'$ and $\underset{\sim}{\Sigma}_2 = \underset{\sim}{A}\bar{\underset{\sim}{A}}'$, where $\underset{\sim}{A}$ is non-singular and all $\lambda_k$'s are positive. Now $tr(\Sigma_1^{-1}X_1\bar{X}_1'+\Sigma_2^{-1}X_2\bar{X}_2') = tr(D_{\frac{1}{\lambda_k}}A^{-1}X_1\bar{X}_1'\bar{A}'^{-1}+A^{-1}X_2\bar{X}_2'\bar{A}'^{-1})$. Transform $X_1 = AY_1$ and $X_2 = AY_2$, then the characteristic roots of $(X_1\bar{X}_1')(X_2\bar{X}_2')^{-1}$ are the same as the characterisitc roots of $(Y_1\bar{Y}_1')(Y_2\bar{Y}_2')^{-1}$. Now $J(X_1,X_2: Y_1,Y_2) = |(\det A)|^{2(n_1+n_2)}$. In view of the fact that $|\Sigma_1| = |A\bar{A}'|\prod_{j=1}^{p}\lambda_j = |\det (A)|^2 \prod_{j=1}^{p}\lambda_j$ and $|\Sigma_2| = |A\bar{A}'| = |(\det A)|^2$, $Y_1$ and $Y_2$ have the joint density:

$$\pi^{-p(n_1+n_2)}\prod_{i=1}^{p}\lambda_i^{-n_1}\exp[-tr(D_{\lambda_k^{-1}}Y_1\bar{Y}_1'+Y_2\bar{Y}_2')].$$

This proves the lemma. If we are interested in the distribution of the roots $c_k$'s, we can, without loss of generality, start directly from the above form, which will thus be called the canonical form.

Lemma 6. The Jacobian of the transformation $X_1(p\times n_1) = A(p\times p)D_{\sqrt{c_k}}(p\times p)L_1(p\times n_1)$, $X_2(p\times n_2) = A(p\times p)L_2(p\times n_2)$ where $A$ is non-singular with positive and real first row and $L_1\bar{L}_1' = L_2\bar{L}_2' = I$ is given by

$$J(X_1,X_2: A, c_k\text{'s}, L_{1I},L_{2I})$$

$$= \left|\frac{\partial(X_1,X_2,L_1\bar{L}_1',L_2\bar{L}_2')}{\partial(A,c_k\text{'s},L_1,L_2)}\right|_{A,c_k\text{'s},L_{1I},L_{2I}} \Bigg/ \left|\frac{\partial(L_1\bar{L}_1')}{\partial(L_{1D})}\right|_{L_{1I}}\left|\frac{\partial(L_2\bar{L}_2')}{(L_{2D})}\right|_{L_{2I}}$$

$$= 2^P|\det(A)|^{2(n_1+n_2-P+\frac{1}{2})}\prod_{j=1}^{P}c_j^{n_1-P}\prod_{j>k}(c_j-c_k)^2 \Bigg/ \left|\frac{\partial(L_1\bar{L}_1')}{\partial(L_{1D})}\right|_{L_{1I}}\left|\frac{\partial(L_2\bar{L}_2')}{\partial(L_{2D})}\right|_{L_{2I}}.$$

Proof. To evaluate $\left|\frac{\partial(X_1,X_2,L_1\bar{L}_1',L_2\bar{L}_2')}{\partial(A,c_k\text{'s},L_1,L_2)}\right|_{A,c_k\text{'s},L_{1I},L_{2I}}$ we proceed as follows. Let $X_1 = (x_{1jk}+iy_{1jk})$, $X_2 = (x_{2jk}+iy_{2jk})$, $A = (a_{jk}+ib_{jk})$

(note that $b_{1k} = 0$, $k = 1,\ldots,p$), $\underset{\sim}{L}_1 = (\ell_{1jk}+im_{1jk})$, $\underset{\sim}{L}_2=(\ell_{2jk}+im_{2jk})$, $\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1' = (k_{1jk}+ig_{1jk})$ (note that $g_{1jj} = 0$, $j = 1,\ldots,p$), $\underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2' = (k_{2jk}+ig_{2jk})$(note that $g_{2jj} = 0$, $j = 1,\ldots,p$) and $t_j = c_j^{\frac{1}{2}}$, $j=1,\ldots,p$. Let $x_{1j}' = (x_{1j1}, y_{1j1},\ldots,x_{1jn_1},y_{1jn_1})$, $x_{2j}' = (x_{2j1},y_{2j1},\ldots,x_{2jn_2}, y_{2jn_2})$, $\ell_{1j}' = (\ell_{1j1}, m_{1j1},\ldots,\ell_{1jn_1},m_{1jn_1})$, $m_{1j}' = (-m_{1j1},\ell_{1j1},\ldots, -m_{1jn_1},\ell_{1jn_1})$, $\ell_{2j}' = (\ell_{2j1},m_{2j1},\ldots,\ell_{2jn_2},m_{2jn_2})$, $m_{2j}' = (-m_{2j1}, \ell_{2j1},\ldots,-m_{2jn_2},\ell_{2jn_2})$, $j = 1,\ldots,p$. The scheme of partial differentiation is given below.

| | $a'$ | $t'$ | $\ell_1'$ | $\ell_2'$ |
|---|---|---|---|---|
| $x_1$ | $\underset{\sim}{M}_{11}$ | $\underset{\sim}{M}_{12}$ | $\underset{\sim}{M}_{13}$ | $\underset{\sim}{0}$ |
| $x_2$ | $\underset{\sim}{M}_{21}$ | $\underset{\sim}{0}$ | $\underset{\sim}{0}$ | $\underset{\sim}{M}_{24}$ |
| $k_1$ | $\underset{\sim}{0}$ | $\underset{\sim}{0}$ | $\underset{\sim}{M}_{33}$ | $\underset{\sim}{0}$ |
| $k_2$ | $\underset{\sim}{0}$ | $\underset{\sim}{0}$ | $\underset{\sim}{0}$ | $\underset{\sim}{M}_{44}$ |

where $a' = (a_{11},\ldots,a_{1p}, a_{21}, b_{21},\ldots,a_{2p},b_{2p},\ldots,a_{p1},b_{p1},\ldots,a_{pp},b_{pp})$, $t' = (t_1,\ldots,t_p)$, $\ell_1' = (\ell_{11}',\ldots,\ell_{1p}')$, $\ell_2' = (\ell_{21}',\ldots,\ell_{2p}')$,

$$x_1=\begin{bmatrix}x_{11}\\ \vdots\\ x_{1p}\end{bmatrix}, \quad x_2=\begin{bmatrix}x_{21}\\ \vdots\\ x_{2p}\end{bmatrix}, \quad k_1=\begin{bmatrix}k_{111}\\ \vdots\\ k_{1pp}\\ k_{112}\\ g_{112}\\ \vdots\\ k_{11p}\\ g_{11p}\\ \vdots\\ k_{1p-1p}\\ g_{1p-1p}\end{bmatrix}, \quad k_2=\begin{bmatrix}k_{211}\\ \vdots\\ k_{2pp}\\ k_{212}\\ g_{212}\\ \vdots\\ k_{21p}\\ g_{21p}\\ \vdots\\ k_{2p-1p}\\ g_{2p-1p}\end{bmatrix},$$

$$\underset{\sim}{M}_{11} = \begin{bmatrix} t_1\ell_{11}\cdots t_p\ell_{1p} & 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 \\ 0 \cdots\cdots 0 & t_1\ell_{11}\ t_1m_{11}\cdots\cdots t_p\ell_{1p}\ t_pm_{1p} 0\cdots\cdots\cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 & t_1\ell_{11}\ t_1m_{11}\cdots t_p\ell_{1p}\ t_pm_{1p} \end{bmatrix},$$

$$\underset{\sim}{M}_{12} = \begin{bmatrix} a_{11}\ell_{11} & \cdots\cdots\cdots\cdots\cdots & a_{1p}\ell_{1p} \\ a_{21}\ell_{11}+b_{21}m_{11} & \cdots\cdots\cdots\cdots\cdots & a_{2p}\ell_{1p}+b_{2p}m_{1p} \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ a_{p1}\ell_{11}+b_{p1}m_{11} & \cdots\cdots\cdots\cdots\cdots & a_{pp}\ell_{1p}+b_{pp}m_{1p} \end{bmatrix},$$

$$\underset{\sim}{M}_{13} = \begin{bmatrix} \underset{\sim}{D}_{a_{11}t_1}(n_1) & \cdots\cdots\cdots\cdots\cdots & \underset{\sim}{D}_{a_{1p}t_p}(n_1) \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ \underset{\sim}{D}_{a_{p1}t_1}(n_1) & \cdots\cdots\cdots\cdots\cdots & \underset{\sim}{D}_{a_{pp}t_p}(n_1) \end{bmatrix},$$

$$\underset{\sim}{M}_{21} = \begin{bmatrix} \ell_{21}\cdots\ell_{2p} & 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 \\ 0\cdots\cdots 0 & \ell_{21}\ m_{21}\cdots\ell_{2p}\ m_{2p}\ 0\cdots\cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ 0\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots 0 & \ell_{21}\ m_{21}\cdots\ell_{2p}\ m_{2p} \end{bmatrix},$$

$$\underset{\sim}{M}_{24} = \begin{bmatrix} \underset{\sim}{D}_{a_{11}}(n_2) & \cdots\cdots\cdots\cdots\cdots & \underset{\sim}{D}_{a_{1p}}(n_2) \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ \underset{\sim}{D}_{a_{p1}}(n_2) & \cdots\cdots\cdots\cdots\cdots & \underset{\sim}{D}_{a_{pp}}(n_2) \end{bmatrix},$$

$$\underset{\sim}{M}_{33} = \left[\begin{array}{l} 2\ell'_{11} \; 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \; 2\ell'_{1p} \\ \ell'_{12} \; \ell'_{11} \; 0 \cdots\cdots\cdots\cdots\cdots 0 \\ m'_{12} \; m'_{11} \; 0 \cdots\cdots\cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ell'_{1p} \; 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \; \ell'_{11} \\ m'_{1p} \; 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \; m'_{11} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \cdots\cdots\cdots\cdots\cdots 0 \; \ell'_{1p} \; \ell'_{1p-1} \\ 0 \cdots\cdots\cdots\cdots\cdots 0 \; m'_{1p} \; m'_{1p-1} \end{array}\right]$$

$$\underset{\sim}{M}_{44} = \left[\begin{array}{l} 2\ell'_{21} \; 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \; 2\ell'_{2p} \\ \ell'_{22} \; \ell'_{21} \; 0 \cdots\cdots\cdots\cdots\cdots 0 \\ m'_{22} \; m'_{21} \; 0 \cdots\cdots\cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ell'_{2p} \; 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \; \ell'_{21} \\ m'_{2p} \; 0 \cdots\cdots\cdots\cdots\cdots\cdots 0 \; m'_{21} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \cdots\cdots\cdots\cdots\cdots 0 \; \ell'_{2p} \; \ell'_{2p-1} \\ 0 \cdots\cdots\cdots\cdots\cdots 0 \; m'_{2p} \; m'_{2p-1} \end{array}\right]$$

$$\text{where } \underset{\sim}{D}_{a_{jk}t_k}(n_1) = \begin{bmatrix} a_{jk}t_k & -b_{jk}t_k & 0 & \cdots & & 0 \\ b_{jk}t_k & a_{jk}t_k & & & & \\ & & a_{jk}t_k & -b_{jk}t_k & & \\ 0 & & & & 0 \cdots & 0 \\ & & b_{jk}t_k & a_{jk}t_k & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & a_{jk}t_k & -b_{jk}t_k \\ 0 & \cdots & & 0 & & \\ & & & & b_{jk}t_k & a_{jk}t_k \end{bmatrix}_{2n_1 \times 2n_1},$$

$$\text{and } \underset{\sim}{D}_{a_{jk}}(n_2) = \begin{bmatrix} a_{jk} & -b_{jk} & 0 & \cdots & & 0 \\ b_{jk} & a_{jk} & & & & \\ & & a_{jk} & -b_{jk} & & \\ 0 & & & & 0 \cdots & 0 \\ & & b_{jk} & a_{jk} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & a_{jk} & -b_{jk} \\ 0 & \cdots & & 0 & & \\ & & & & b_{jk} & a_{jk} \end{bmatrix}_{2n_2 \times 2n_2}.$$

We are interested in the absolute value of the determinant of the

$$\text{matrix } \begin{bmatrix} \underset{\sim}{M}_{11} & \underset{\sim}{M}_{12} & \underset{\sim}{M}_{13} & \underset{\sim}{0} \\ \underset{\sim}{M}_{21} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{M}_{24} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{M}_{33} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{M}_{44} \end{bmatrix} \quad \left(\text{which is } \left|\frac{\partial(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \underset{\sim}{L}_1\underset{\sim}{\Gamma}_1', \underset{\sim}{L}_2\underset{\sim}{\Gamma}_2')}{\partial(\underset{\sim}{A}, t_k\text{'s}, \underset{\sim}{L}_1, \underset{\sim}{L}_2)}\right|_{\underset{\sim}{A}, t_k\text{'s}, \underset{\sim}{L}_{1I}, \underset{\sim}{L}_{2I}}\right)$$

and which is $(2p^2+2pn_1+2pn_2)\times(2p^2+2pn_1+2pn_2)$. After some obvious manipulations we can take out a factor $2^{2p} \prod_{j=1}^{p} t_j^{2n_1-2p+1}$, so that the above determinant becomes

$$2^{2p} \prod_{j=1}^{p} t_j^{2n_1-2p+1} \begin{vmatrix} \underset{\sim}{M}_{11} & \underset{\sim}{M}_{12} & \underset{\sim}{M}^*_{13} & \underset{\sim}{0} \\ \underset{\sim}{M}_{21} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{M}_{24} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{M}^*_{33} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{M}^*_{44} \end{vmatrix} ,$$

$$\text{where} \quad \underset{\sim}{M}^*_{13} = \begin{bmatrix} \underset{\sim}{D}_{a_{11}}(n_1) \cdots\cdots\cdots \underset{\sim}{D}_{a_{1p}}(n_1) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \underset{\sim}{D}_{a_{p1}}(n_1) \cdots\cdots\cdots \underset{\sim}{D}_{a_{pp}}(n_1) \end{bmatrix} ,$$

$$\underset{\sim}{M}^*_{33} = \begin{bmatrix} \ell'_{11} & 0 \cdots\cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \cdots\cdots\cdots\cdots 0 \ \ell'_{1p} \\ \ell'_{12}t_2 \ \ \ell'_{11}t_1 \ \ 0 \cdots\cdots\cdots 0 \\ m'_{12}t_2 \ \ m'_{11}t_1 \ \ 0 \cdots\cdots\cdots 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \ell'_{1p}t_p \ \ 0 \cdots\cdots\cdots 0 \ \ \ell'_{11}t_1 \\ m'_{1p}t_p \ \ 0 \cdots\cdots\cdots 0 \ \ m'_{11}t_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \cdots\cdots\cdots 0 \ \ \ell'_{1p}t_p \ \ \ell'_{1p-1}t_{p-1} \\ 0 \cdots\cdots\cdots 0 \ \ m'_{1p}t_p \ \ m'_{1p-1}t_{p-1} \end{bmatrix} ,$$

$$\underset{\sim}{M}^*_{44} = \begin{bmatrix} \ell'_{21} & 0 \ldots\ldots\ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 \ldots\ldots\ldots\ldots\ldots\ldots 0 & \ell'_{2p} \\ \ell'_{22} \;\; \ell'_{21} & 0 \ldots\ldots\ldots\ldots\ldots 0 \\ m'_{22} \;\; m'_{21} & 0 \ldots\ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ell'_{2p} & 0 \ldots\ldots\ldots\ldots 0 \;\; \ell'_{21} \\ m'_{2p} & 0 \ldots\ldots\ldots\ldots 0 \;\; m'_{21} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 \ldots\ldots\ldots\ldots\ldots & 0 \;\; \ell'_{2p} \;\; \ell'_{2p-1} \\ 0 \ldots\ldots\ldots\ldots\ldots & 0 \;\; m'_{2p} \;\; m'_{2p-1} \end{bmatrix} .$$

Now we have

$$\begin{vmatrix} \underset{\sim}{M}_{11} & \underset{\sim}{M}_{12} & \underset{\sim}{M}^*_{13} & \underset{\sim}{0} \\ \underset{\sim}{M}_{21} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{M}_{24} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{M}^*_{33} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{M}^*_{44} \end{vmatrix} = \begin{vmatrix} \underset{\sim}{M}^*_{13} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{M}_{24} \end{vmatrix} \left| \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} - \begin{bmatrix} \underset{\sim}{M}^*_{33} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{M}^*_{44} \end{bmatrix} \begin{bmatrix} \underset{\sim}{M}^{*-1}_{13} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{M}^{-1}_{24} \end{bmatrix} \begin{bmatrix} \underset{\sim}{M}_{11} & \underset{\sim}{M}_{12} \\ \underset{\sim}{M}_{21} & \underset{\sim}{0} \end{bmatrix} \right|$$

$$= |\underset{\sim}{M}^*_{13}||\underset{\sim}{M}_{24}| \begin{vmatrix} \underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{11} & \underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{12} \\ \underset{\sim}{M}^*_{44}\underset{\sim}{M}^{-1}_{24}\underset{\sim}{M}_{21} & \underset{\sim}{0} \end{vmatrix}$$

$$= |(\det \underset{\sim}{A})|^{2n_1+2n_2} \begin{vmatrix} \underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{11} & \underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{12} \\ \underset{\sim}{M}^*_{44}\underset{\sim}{M}^{-1}_{24}\underset{\sim}{M}_{21} & \underset{\sim}{0} \end{vmatrix} .$$

The structure of $\begin{vmatrix} \underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{11} & \underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{12} \\ \underset{\sim}{M}^*_{44}\underset{\sim}{M}^{-1}_{24}\underset{\sim}{M}_{21} & \underset{\sim}{0} \end{vmatrix}$ can be visualized by considering $p = 3$ which will make immediately obvious the corresponding structure for the general case. Below is given the case of $p = 3$.

$$
\underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{11} = \begin{bmatrix}
t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3 & 0 \\
0 & t_2^2 & 0 & t_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & t_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2^2 \\
0 & 0 & t_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & t_1^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_3^2 & 0 & 0 & 0 & t_1^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3^2 & 0 & 0 & 0 & t_2^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3^2 & 0 & 0 & 0 & t_2^2 & 0 & 0
\end{bmatrix} \underset{\sim}{E},
$$

where $|\underset{\sim}{E}| = |(\det \underset{\sim}{A}^{-1})|^5 = |(\det \underset{\sim}{A})|^{-5}$,

$$
\underset{\sim}{M}^*_{44}\underset{\sim}{M}^{-1}_{24}\underset{\sim}{M}_{21} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \underset{\sim}{E},
$$

$$
\underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{12} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\multicolumn{3}{c}{\underset{\sim}{0}(6\text{x}3)}
\end{bmatrix}.
$$

Therefore, in the case of p = 3,

$$
\begin{vmatrix}
\underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{11} & \underset{\sim}{M}^*_{33}\underset{\sim}{M}^{*-1}_{13}\underset{\sim}{M}_{12} \\
\underset{\sim}{M}^*_{44}\underset{\sim}{M}^{-1}_{24}\underset{\sim}{M}_{21} & 0
\end{vmatrix} = |(\det \underset{\sim}{A})|^{-5} \prod_{j>k} (t_j^2 - t_k^2)^2 .
$$

In the general case, we have

$$
\left| \frac{\partial(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \underset{\sim}{L}_1\underset{\sim}{L}_1', \underset{\sim}{L}_2\underset{\sim}{L}_2')}{\partial(\underset{\sim}{A}, t_k\text{'s}, \underset{\sim}{L}_1, \underset{\sim}{L}_2)} \right|_{\underset{\sim}{A}, t_k\text{'s}, \underset{\sim}{L}_{1I}, \underset{\sim}{L}_{2I}}
$$

$$= 2^{2p} \prod_{j=1}^{p} t_j^{2n_1-2p+1} |(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} \prod_{j>k} (t_j^2-t_k^2)^2.$$

Remembering $t_j^2 = c_j$, so we have

$$\left|\frac{\partial(\underset{\sim}{X}_1,\underset{\sim}{X}_2,\underset{\sim}{L}_1\underset{\sim}{L}_1',\underset{\sim}{L}_2\underset{\sim}{L}_2')}{\partial(\underset{\sim}{A},c_k\text{'s},\underset{\sim}{L}_1,\underset{\sim}{L}_2)}\right|_{\underset{\sim}{A},c_k\text{'s},\underset{\sim}{L}_{1I},\underset{\sim}{L}_{2I}}$$

$$= 2^p \prod_{j=1}^{p} c_j^{n_1-p} |(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} \prod_{j>k} (c_j-c_k)^2.$$

Thus the Jacobian is

$$J(\underset{\sim}{X}_1,\underset{\sim}{X}_2: \underset{\sim}{A},\ c_k\text{'s},\ \underset{\sim}{L}_{1I},\ \underset{\sim}{L}_{2I})$$

$$= 2^p \prod_{j=1}^{p} c_j^{n_1-p} |(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} \prod_{j>k} (c_j-c_k)^2 \Big/ \left|\frac{\partial(\underset{\sim}{L}_1\underset{\sim}{L}_1')}{\partial(\underset{\sim}{L}_{1D})}\right|_{\underset{\sim}{L}_{1I}} \left|\frac{\partial(\underset{\sim}{L}_2\underset{\sim}{L}_2')}{\partial(\underset{\sim}{L}_{2D})}\right|_{\underset{\sim}{L}_{2I}}.$$

## 3. The condition of local unbiasedness

The acceptance regions based on criteria 1) to 5) with local unbiasedness property and $\alpha$ level of significance can be written in one form:

$$R:\quad a(p,n_1,n_2) \le \omega(c_1,\dots,c_p) \le b(p,n_1,n_2),$$

where a and b are so chosen to satisfy

(3.1) (i) $P(a \le \omega(c_1,\dots,c_p) \le b \mid \lambda_1 = \dots = \lambda_p = 1) = 1-\alpha,$

(3.2) (ii) $\left.\dfrac{\partial P(a \le \omega(c_1,\dots,c_p) \le b \mid \lambda_1,\dots,\lambda_p)}{\partial \lambda_i}\right|_{\lambda_1=\dots=\lambda_p=1} = 0,\ i=1,\dots,p,$

where $\omega(c_1,\ldots,c_p) = c_p/(1+c_p)$ for test 1), $\sum_{i=1}^{p} c_i$ for 2), $\sum_{i=1}^{p} [c_i/(1+c_i)]$ for 3), $\prod_{i=1}^{p} (1+c_i)^{-1}$ for 4) and $c_1/(1+c_1)$, $c_p/(1+c_p)$ for 5).

In this section, we will show that the p equations given in (ii) are really equivalent to one equation and are in turn equivalent to

$$(3.3) \quad (ii') \quad \left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial\lambda}\right|_{\lambda=1} = 0.$$

We call (ii) or equivalently (ii') "the condition of local unbiasedness".

Theorem 1. The p equations given in (ii) are equivalent to one equation and are in turn equivalent to (ii').

Proof. The joint density of $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ is given by

$$\pi^{-p(n_1+n_2)}|\underset{\sim}{\Sigma}_1|^{-n_1}|\underset{\sim}{\Sigma}_2|^{-n_2}\exp[-tr(\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1'+\underset{\sim}{\Sigma}_2^{-1}\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')].$$

By Lemma 5, we may, without loss of generality, start directly from the following canonical form:

$$\pi^{-p(n_1+n_2)}\prod_{i=1}^{p}\lambda_i^{-n_1}\exp[-tr(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1' + \underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')],$$

where $\underset{\sim}{D}_{\lambda_k^{-1}} = \text{diag}\,(1/\lambda_k)$. Then

$$P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)$$
$$= \pi^{-p(n_1+n_2)}\int_{a\le\omega(c_1,\ldots,c_p)\le b}\prod_{i=1}^{p}\lambda_i^{-n_1}\exp[-tr(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1'+\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')]d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Hence

$$(3.4) \qquad \frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial\lambda_i^{-1}}$$

$$= \pi^{-p(n_1+n_2)} \int_{a \le \omega(c_1,\ldots,c_p) \le b} \prod_{j=1}^{p} \lambda_j^{-n_1} [n_1\lambda_i - (\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1')_{ii}] \exp[-\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}} \underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1' + \underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')] d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

By Lemma 4, we may transform $\underset{\sim}{X}_1 = \underset{\sim}{A}\underset{\sim}{D}_{\sqrt{c_k}}\underset{\sim}{L}_1$ and $\underset{\sim}{X}_2 = \underset{\sim}{A}\underset{\sim}{L}_2$ where $\underset{\sim}{A} = (a_{jk}+ib_{jk})$ is non-singular with real first row and $\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1' = \underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2' = \underset{\sim}{I}$. And by Lemma 6, we have

$$J(\underset{\sim}{X}_1,\underset{\sim}{X}_2: \underset{\sim}{A}, c_k\text{'s}, \underset{\sim}{L}_{1I}, \underset{\sim}{L}_{2I}) = 2^p |(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j - c_{j'})^2$$
$$\div \left| \frac{\partial(\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1')}{\partial(\underset{\sim}{L}_{1D})} \right|_{\underset{\sim}{L}_{1I}} \left| \frac{\partial(\underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2')}{(\underset{\sim}{L}_{2D})} \right|_{\underset{\sim}{L}_{2I}}.$$

Therefore, (3.4) becomes

$$(3.5)\quad 2^p \pi^{-p(n_1+n_2)} \int_{R^*} \prod_{j=1}^{p} \lambda_j^{-n_1} [n_1\lambda_i - (\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}')_{ii}] \exp[-\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}' + \underset{\sim}{A}\bar{\underset{\sim}{A}}')]$$
$$\times |(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} d\underset{\sim}{A} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j$$
$$\times \int_{\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1' = \underset{\sim}{I}} \frac{d\underset{\sim}{L}_{1I}}{\left| \frac{\partial(\underset{\sim}{L}_1\bar{\underset{\sim}{L}}_1')}{\partial(\underset{\sim}{L}_{1D})} \right|_{\underset{\sim}{L}_{1I}}} \int_{\underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2' = \underset{\sim}{I}} \frac{d\underset{\sim}{L}_{2I}}{\left| \frac{\partial(\underset{\sim}{L}_2\bar{\underset{\sim}{L}}_2')}{(\underset{\sim}{L}_{2D})} \right|_{\underset{\sim}{L}_{2I}}}$$

$$= \frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{R^*} \prod_{j=1}^{p} \lambda_j^{-n_1} [n_1\lambda_i - (\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}')_{ii}] \exp[-\mathrm{tr}(\underset{\sim}{D}_{\lambda_k^{-1}}\underset{\sim}{A}\underset{\sim}{D}_{c_k}\bar{\underset{\sim}{A}}' + \underset{\sim}{A}\bar{\underset{\sim}{A}}')]$$
$$\times |(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} d\underset{\sim}{A} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j,$$

where $R^*$ is $a \le \omega(c_1,\ldots,c_p) \le b$ and $-\infty <$ all $a_{jk}, b_{jk} < \infty$ and $\tilde{\Gamma}_p(n) = \pi^{\frac{1}{2}p(p-1)} \prod_{j=1}^{p} \Gamma(n-j+1)$. Thus we have

$$(3.6)\quad \left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial \lambda_i^{-1}}\right|_{\lambda_1=\ldots=\lambda_p=1}$$

$$= \frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{R^*} [n_1-(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\underset{\sim}{\bar{A}}')_{ii}]\exp[-tr(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\underset{\sim}{\bar{A}}'+\underset{\sim}{A}\underset{\sim}{\bar{A}}')]$$

$$|(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} d\underset{\sim}{A} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j.$$

The only term in the above integrand which depends on i is $(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\underset{\sim}{\bar{A}}')_{ii} = (a_{i1}^2c_1+b_{i1}^2c_1+\ldots+a_{ip}^2c_p+b_{ip}^2c_p)$ and the integral is taken over R*. Thus the integral is invariant under a change of the subscript i for i = 2,...,p. For i = 1, the integral is half of the integral for i = 2 (note that $b_{1k} = 0$ for k = 1,...,p). Hence the p equations $\left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1,\ldots,\lambda_p)}{\partial \lambda_i^{-1}}\right|_{\lambda_1=\ldots=\lambda_p=1} = 0$ are really equivalent to one equation. Now adding up integrals like (3.6) over i = 1,...,p, we have

$$(3.7)\quad \frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{R^*} [n_1p-tr(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\underset{\sim}{\bar{A}}')]\exp[-tr(\underset{\sim}{A}\underset{\sim}{D}_{c_k}\underset{\sim}{\bar{A}}'+\underset{\sim}{A}\underset{\sim}{\bar{A}}')]$$

$$|(\det \underset{\sim}{A})|^{2n_1+2n_2-2p+1} d\underset{\sim}{A} \prod_{j=1}^{p} c_j^{n_1-p} \prod_{j>j'} (c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j.$$

It is easy to see that (3.7) is actually equal to

$$\left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1=\ldots=\lambda_p=\lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}.$$

Hence, (ii) is equivalent to (ii').

Theorem 2. Condition (ii') can be written as

$$C(p)\Big[\int_{a\le\omega(c_1,\dots,c_p)\le b} n_1 p \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)} \prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j -$$

$$\int_{a\le\omega(c_1,\dots,c_p)\le b} (n_1+n_2)\Big[\sum_{j=1}^{p} c_j/(1+c_j)\Big] \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)}$$

$$\prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j\Big] = 0,$$

where $C(p) = \dfrac{\pi^{p(p-1)}\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)}$.

Proof. Starting from (3.7) and transforming $\underset{\sim}{A} = \underset{\sim}{B}\underset{\sim}{D}_{(1+c_k)^{-\frac{1}{2}}}$, where $\underset{\sim}{B}$ is non-singular with real and positive first row if the matrix $\underset{\sim}{A}$ in Theorem 1 is taken with real and positive first row. $J(\underset{\sim}{A}\colon \underset{\sim}{B}) = |\underset{\sim}{D}_{(1+c_k)^{-\frac{1}{2}}}|^{2p-1} = \prod_{j=1}^{p}(1+c_j)^{-p+\frac{1}{2}}$. Then (3.7) becomes

$$(3.8)\quad \frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{R^{**}} [n_1 p - \mathrm{tr}(\underset{\sim}{B}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\bar{\underset{\sim}{B}}')]\exp[-\mathrm{tr}(\underset{\sim}{B}\bar{\underset{\sim}{B}}')]\, |(\det \underset{\sim}{B})|^{2n_1+2n_2-2p+1}\, d\underset{\sim}{B}$$

$$\times \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)} \prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j,$$

where $R^{**}$ is $a \le \omega(c_1,\dots,c_p) \le b$ and $\underset{\sim}{B}$ is non-singular with real and positive first row.

Further, transform $\underset{\sim}{B} = \underset{\sim}{T}\underset{\sim}{L}$ where $\underset{\sim}{T}$ is lower triangular with real and positive diagonal and $\underset{\sim}{L}$ is semi-unitary with real and positive first row. We shall now find the Jacobian $J(\underset{\sim}{B}\colon \underset{\sim}{T},\underset{\sim}{L})$. Differentiating both sides of $\underset{\sim}{B} = \underset{\sim}{T}\underset{\sim}{L}$, we have $(d\underset{\sim}{B}) = (d\underset{\sim}{T})\underset{\sim}{L} + \underset{\sim}{T}(d\underset{\sim}{L})$. Pre-multiplying by $\underset{\sim}{T}^{-1}$, we get $\underset{\sim}{T}^{-1}(d\underset{\sim}{B}) = \underset{\sim}{T}^{-1}(d\underset{\sim}{T})\underset{\sim}{L} + (d\underset{\sim}{L})$. Putting $\underset{\sim}{U} = \underset{\sim}{T}^{-1}(d\underset{\sim}{B})$ and $\underset{\sim}{V} = \underset{\sim}{T}^{-1}(d\underset{\sim}{T})$, we have $\underset{\sim}{U} = \underset{\sim}{V}\underset{\sim}{L} + (d\underset{\sim}{L})$.

Hence

$$
\begin{aligned}
J(\underset{\sim}{B}: \underset{\sim}{T},\underset{\sim}{L}) &= J(d\underset{\sim}{B}: d\underset{\sim}{T},d\underset{\sim}{L}) \\
&= J(d\underset{\sim}{B}: \underset{\sim}{U})J(\underset{\sim}{U}: \underset{\sim}{V},(d\underset{\sim}{L}))J(\underset{\sim}{V},(d\underset{\sim}{L}): (d\underset{\sim}{T}), (d\underset{\sim}{L})) \\
&= |\underset{\sim}{T}|^{2p-1}2^p \left(d\underset{\sim}{L}_I \Big/ \left|\frac{\partial(\underset{\sim}{L}\bar{\underset{\sim}{L}}')}{\partial(\underset{\sim}{L}_D)}\right|_{\underset{\sim}{L}_I}\right) \prod_{i=1}^{p} t_{ii}^{-2i+1} \\
&= \left(\prod_{i=1}^{p} t_{ii}^{2p-2i}\right)2^p \left(d\underset{\sim}{L}_I \Big/ \left|\frac{\partial(\underset{\sim}{L}\bar{\underset{\sim}{L}}')}{\partial(\underset{\sim}{L}_D)}\right|_{\underset{\sim}{L}_I}\right).
\end{aligned}
$$

Then

$$
\frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{\underset{\sim}{B}} [n_1 p - \mathrm{tr}(\underset{\sim}{B}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\bar{\underset{\sim}{B}}')]\exp[-\mathrm{tr}(\underset{\sim}{B}\bar{\underset{\sim}{B}}')] |(\det \underset{\sim}{B})|^{\frac{2n_1+2n_2-2p+1}{2}} d\underset{\sim}{B}
$$

$$
\begin{aligned}
(3.9) = &\frac{2^{2p}}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{\underset{\sim}{L}\bar{\underset{\sim}{L}}'=\underset{\sim}{I}} \int_{\underset{\sim}{T}} [n_1 p - \mathrm{tr}(\underset{\sim}{T}\underset{\sim}{L}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\bar{\underset{\sim}{L}}'\bar{\underset{\sim}{T}}')]\exp[-\mathrm{tr}(\underset{\sim}{T}\bar{\underset{\sim}{T}}')] \\
&\times |\underset{\sim}{T}\bar{\underset{\sim}{T}}'|^{n_1+n_2-p+\frac{1}{2}} \prod_{i=1}^{p} t_{ii}^{2p-2i} d\underset{\sim}{T} d\underset{\sim}{L}_I \Big/ \left|\frac{\partial(\underset{\sim}{L}\bar{\underset{\sim}{L}}')}{\partial(\underset{\sim}{L}_D)}\right|_{\underset{\sim}{L}_I}
\end{aligned}
$$

$$
\begin{aligned}
= &\frac{2^{2p}}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{\underset{\sim}{L}\bar{\underset{\sim}{L}}'=\underset{\sim}{I}} \int_{\underset{\sim}{T}} [n_1 p - \mathrm{tr}(\underset{\sim}{T}\underset{\sim}{L}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\bar{\underset{\sim}{L}}'\bar{\underset{\sim}{T}}')]\exp[-\mathrm{tr}(\underset{\sim}{T}\bar{\underset{\sim}{T}}')] \\
&\times |\underset{\sim}{T}\bar{\underset{\sim}{T}}'|^{n_1+n_2-p} \prod_{i=1}^{p} t_{ii}^{2p-2i+1} d\underset{\sim}{T} d\underset{\sim}{L}_I \Big/ \left|\frac{\partial(\underset{\sim}{L}\bar{\underset{\sim}{L}}')}{\partial(\underset{\sim}{L}_D)}\right|_{\underset{\sim}{L}_I}.
\end{aligned}
$$

Transform $\underset{\sim}{S} = \bar{\underset{\sim}{T}}'\underset{\sim}{T}$, $J(\underset{\sim}{T}:\underset{\sim}{S}) = 2^{-p} \prod_{i=1}^{p} t_{ii}^{-2p+2i-1}$. Then (3.9) is

$$
\begin{aligned}
(3.10)\ &\frac{2^p}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{\underset{\sim}{L}\bar{\underset{\sim}{L}}'=\underset{\sim}{I}} \int_{\underset{\sim}{S}>0} [n_1 p - \mathrm{tr}(\underset{\sim}{S}\underset{\sim}{L}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\bar{\underset{\sim}{L}}')]\exp[-\mathrm{tr}\underset{\sim}{S}] \\
&|\underset{\sim}{S}|^{n_1+n_2-p} d\underset{\sim}{S}\, d\underset{\sim}{L}_I \Big/ \left|\frac{\partial(\underset{\sim}{L}\bar{\underset{\sim}{L}}')}{\partial(\underset{\sim}{L}_D)}\right|_{\underset{\sim}{L}_I}.
\end{aligned}
$$

Apply the following equation of Khatri [14]:

$$\int_{\underset{\sim}{S}>\underset{\sim}{0}} \exp[-\mathrm{tr}\underset{\sim}{S}]|\underset{\sim}{S}|^{r-p}\tilde{C}_\kappa(\underset{\sim}{A}\underset{\sim}{S})d\underset{\sim}{S} = \tilde{\Gamma}_p(r,\kappa)\tilde{C}_\kappa(A),$$

where $\tilde{\Gamma}_p(r,\kappa) = \tilde{\Gamma}_p(r)[r]_\kappa$, then (3.10) becomes

$$\frac{2^p\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)} \int_{\underset{\sim}{L}\underset{\sim}{\bar{L}}'=\underset{\sim}{I}} [n_1p-(n_1+n_2)\mathrm{tr}(\underset{\sim}{L}\underset{\sim}{D}_{c_k(1+c_k)^{-1}}\underset{\sim}{\bar{L}}')]d\underset{\sim}{L}_I \Big/ \left|\frac{\partial(\underset{\sim}{L}\underset{\sim}{\bar{L}}')}{\partial(\underset{\sim}{L}_D)}\right|_{\underset{\sim}{L}_I}$$

$$= \frac{\pi^{p(p-1)}\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)} [n_1p-(n_1+n_2)\sum_{j=1}^{p} c_j/(1+c_j)].$$

Therefore, we get

$$\left.\frac{\partial P(a \le \omega(c_1,\ldots,c_p) \le b|\lambda_1 = \ldots = \lambda_p = \lambda)}{\partial\lambda^{-1}}\right|_{\lambda = 1}$$

$$= C(p)\Big[\int_{a\le\omega(c_1,\ldots,c_p)\le b} n_1p \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)} \prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j -$$

$$\int_{a\le\omega(c_1,\ldots,c_p)\le b} (n_1+n_2)\Big[\sum_{j=1}^{p} c_j/(1+c_j)\Big] \prod_{j=1}^{p} c_j^{n_1-p}(1+c_j)^{-(n_1+n_2)} \prod_{j>j'}(c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j\Big].$$

Now equating the above to zero, we have Theorem 2.

Thus, the acceptance region based on criterion $\omega(c_1,\ldots,c_p)$ with local unbiasedness property ($\ell$up) and $\alpha$ level of significance can be written as

$$R: \quad a(p,n_1,n_2) \le \omega(c_1,\ldots,c_p) \le b(p,n_1,n_2),$$

where a and b are so chosen as to satisfy:

$$\text{(i)}\quad C(p) \int_{a \le \omega(c_1,\ldots,c_p) \le b} \prod_{j=1}^{p} c_j^{n_1-p} (1+c_j)^{-(n_1+n_2)} \prod_{j>j'} (c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j = 1-\alpha$$

and

$$\text{(ii)}\quad C(p)\Big[ \int_{a \le \omega(c_1,\ldots,c_p) \le b} n_1 p \prod_{j=1}^{p} c_j^{n_1-p} (1+c_j)^{-(n_1+n_2)} \prod_{j>j'} (c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j - \int_{a \le \omega(c_1,\ldots,c_p) \le b} (n_1+n_2)\Big[\sum_{j=1}^{p} c_j/(1+c_j)\Big] \prod_{j=1}^{p} c_j^{n_1-p} (1+c_j)^{-(n_1+n_2)} \prod_{j>j'} (c_j-c_{j'})^2 \prod_{j=1}^{p} dc_j\Big] = 0,$$

where $C(p) = \dfrac{\pi^{p(p-1)}\tilde{\Gamma}_p(n_1+n_2)}{\tilde{\Gamma}_p(n_1)\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(p)}$.

## 4. The acceptance regions based on the five criteria with ℓup for p = 2

In this section, we will consider the acceptance regions of tests 1) to 5) in that order.

1) Roy's largest root, $L_2^{(2)} = c_2/(1+c_2)$. By using Theorem 2 in the previous section, we know that

$$(4.1)\qquad \left.\frac{\partial P(a \le L_2^{(2)} \le b \mid \lambda_1 = \lambda_2 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$= C(2)\Big[ \int_{a \le L_2^{(2)} \le b} g_2(c_1,c_2;n_1,n_2)dc_1dc_2 - \int_{a \le L_2^{(2)} \le b} h_2(c_1,c_2;n_1,n_2)dc_1dc_2\Big],$$

where $g_2(c_1,c_2;n_1,n_2) = 2n_1(c_1c_2)^{n_1-2}[(1+c_1)(1+c_2)]^{-(n_1+n_2)}(c_1-c_2)^2$ and

$$h_2(c_1,c_2;n_1,n_2) = (n_1+n_2)[c_1/(1+c_1)+c_2/(1+c_2)](c_1c_2)^{n_1-2}$$

$$\times [(1+c_1)(1+c_2)]^{-(n_1+n_2)}(c_1-c_2)^2.$$

Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then (4.1) becomes

$$(4.2) \qquad C(2)[\int_a^b \int_0^{\ell_2} g_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2 - \int_a^b \int_0^{\ell_2} h_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2],$$

where $g_3(\ell_1,\ell_2;n_1,n_2)=2n_1(\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2$ and

$$h_3(\ell_1,\ell_2;n_1,n_2)=(n_1+n_2)(\ell_1+\ell_2)(\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2.$$

Further, note that by making the same transformation,

$$(4.3) \qquad P(a \le L_2^{(2)} \le b|\lambda_1=\lambda_2=1)$$

$$= C(2)\int_a^b \int_0^{\ell_2} (\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2.$$

Khatri [9] has shown that

$$\int_0^x \int_0^{\ell_2} (\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$$

$$= B_x(n_1-1,n_2-1)B_x(n_1+1,n_2-1)-B_x(n_1,n_2-1)B_x(n_1,n_2-1),$$

where $B_x(r,s) = \int_0^x t^{r-1}(1-t)^{s-1}dt.$

Now $\int_0^x \int_0^{\ell_2} (\ell_1+\ell_2)(\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$

$$=\int_0^x \int_0^{\ell_2} [1-(1-\ell_1)(1-\ell_2)+\ell_1\ell_2](\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$$

$= B_x(n_1-1,n_2-1)B_x(n_1+1,n_2-1)-B_x(n_1,n_2-1)B_x(n_1,n_2-1)$

$- B_x(n_1-1,n_2)B_x(n_1+1,n_2)+B_x(n_1,n_2)B_x(n_1,n_2)$

$+ B_x(n_1,n_2-1)B_x(n_1+2,n_2-1)-B_x(n_1+1,n_2-1)B_x(n_1+1,n_2-1).$

Thus, we have proved the following:

Theorem 3. Let $T_1(x) = B_x(n_1-1,n_2-1)B_x(n_1+1,n_2-1)-B_x(n_1,n_2-1)B_x(n_1,n_2-1)$ and $T_2(x)=T_1(x)-B_x(n_1-1,n_2)B_x(n_1+1,n_2)+B_x(n_1,n_2)B_x(n_1,n_2)+B_x(n_1,n_2-1)$ $B_x(n_1+2,n_2-1)-B_x(n_1+1,n_2-1)B_x(n_1+1,n_2-1)$. Then the acceptance region based on Roy's largest root, $L_2^{(2)} = c_2/(1+c_2)$ with $\ell$up and $\alpha$ level is given by $a \le L_2^{(2)} \le b$ where a and b are so chosen as to satisfy:

(i) $C(2)[T_1(b)-T_1(a)] = 1-\alpha$ and

(ii) $C(2)\{2n_1[T_1(b)-T_1(a)]-(n_1+n_2)[T_2(b)-T_2(a)]\} = 0.$

2) Hotelling's trace, $U^{(2)} = c_1+c_2$. From the previous section,

(4.4) $P(a \le U^{(2)} \le b|\lambda_1 = \lambda_2 = \lambda)$

$$= \pi^{-2(n_1+n_2)} \int_{a\le c_1+c_2\le b} \lambda^{-2n_1} \exp[-tr(\lambda^{-1}\underset{\sim}{X}_1\bar{\underset{\sim}{X}}_1'+\underset{\sim}{X}_2\bar{\underset{\sim}{X}}_2')]d\underset{\sim}{X}_1 d\underset{\sim}{X}_2.$$

Now transform $\lambda^{-\frac{1}{2}} \underset{\sim}{X}_1 = \underset{\sim}{Y}_1$ and $\underset{\sim}{X}_2 = \underset{\sim}{Y}_2$. $J(\underset{\sim}{Y}_1:\underset{\sim}{X}_1) = \lambda^{-2n_1}$ and let $0 < d_1 \le d_2 < \infty$ be the characteristic roots of $|\underset{\sim}{Y}_1\bar{\underset{\sim}{Y}}_1'-d\underset{\sim}{Y}_2\bar{\underset{\sim}{Y}}_2'| = 0$. Then (4.4) becomes

(4.5) $$C(2) \int_{a\le\lambda d_1+\lambda d_2\le b} (d_1d_2)^{n_1-2}[(1+d_1)(1+d_2)]^{-(n_1+n_2)}(d_1-d_2)^2 dd_1 dd_2.$$

Let $u = d_1 + d_2$ and $g = d_1d_2$. We get

$$P(a \le U^{(2)} \le b | \lambda_1 = \lambda_2 = \lambda)$$

$$= C(2)\int_{a\lambda^{-1}}^{b\lambda^{-1}} \int_0^{\frac{1}{4}u^2} g^{n_1-2}(1+u+g)^{-(n_1+n_2)}(u^2-4g)^{\frac{1}{2}} dg\,du.$$

Thus

$$\left.\frac{\partial P(a \le U^{(2)} \le b | \lambda_1 = \lambda_2 = \lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$= C(2)\Big[b\int_0^{\frac{1}{4}b^2} g^{n_1-2}(1+b+g)^{-(n_1+n_2)}(b^2-4g)^{\frac{1}{2}} dg - a\int_0^{\frac{1}{4}a^2} g^{n_1-2}(1+a+g)^{-(n_1+n_2)}(a^2-4g)^{\frac{1}{2}} dg\Big]$$

$$= bT_2(b) - aT_2(a),$$

where $T_2(u)$, the density function of $U^{(2)}$, has been found by Pillai and Jouris [30] as follows:

$$(4.6)\quad T_2(u) = C(2)\sum_{r=0}^{\infty}(-1)^r \frac{\binom{-(n_1+n_2)}{r}(n_1-2)!\,u^{2r+2n_1-1}}{4^{r+n_1-1}(1+u/2)^{2r+2n_1+2n_2}} \times \frac{1}{(r+3/2)(r+5/2)\ldots[r+(2(n_1-3)+5)/2]}.$$

Furthermore,

$$P(a \le U^{(2)} \le b | \lambda_1 = \lambda_2 = 1)$$

$$= C(2)\int_a^b \int_0^{\frac{1}{4}u^2} g^{n_1-2}(1+u+g)^{-(n_1+n_2)}(u^2-4g)^{\frac{1}{2}} dg\,du$$

$$= T_1(b) - T_1(a),$$

where $T_1(u)$ is defined as follows:

$$(4.7)\quad T_1(u)=C(2)\sum_{r=0}^{\infty}(-1)^r\binom{-(n_1+n_2)}{r}(n_1-2)!\;4B_{\frac{u}{2+u}}(2r+2n_1,2n_2)$$

$$\times\frac{1}{(r+3/2)(r+5/2)\ldots[r+(2n_1-1)/2]}.$$

Therefore, we have proved the following:

Theorem 4. The acceptance region based on $U^{(2)} = c_1+c_2$ with $\ell$up and $\alpha$ level is given by $a \le U^{(2)} \le b$ where a and b are chosen as to satisfy:

(i) $T_1(b)-T_1(a) = 1-\alpha$ and (ii) $bT_2(b)-aT_2(a) = 0$, where $T_1(u)$ and $T_2(u)$ are defined in (4.7) and (4.6) respectively.

3) Pillai's trace, $V^{(2)} = [c_1/(1+c_1)]+[c_2/(1+c_2)]$. From Theorem 2, we obtain $\left.\frac{\partial P(a\le V^{(2)}\le b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$ by replacing $L_2^{(2)}$ in the limits of the integrals in (4.1) by $V^{(2)}$. Now transform $\ell_1=c_1/(1+c_1)$ and $\ell_2=c_2/(1+c_2)$. Then

$$(4.8)\qquad \left.\frac{\partial P(a\le V^{(2)}\le b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$$

$$=C(2)[\int_{a\le\ell_1+\ell_2\le b} g_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2-\int_{a\le\ell_1+\ell_2\le b} h_3(\ell_1,\ell_2;$$

$$n_1,n_2)d\ell_1 d\ell_2].$$

Let $v = \ell_1+\ell_2$ and $g = \ell_1\ell_2$. Then (4.8) becomes

$$C(2)[2n_1\int_a^b\int_0^{\frac{1}{4}v^2} g^{n_1-2}(1-v+g)^{n_2-2}(v^2-4g)^{\frac{1}{2}}dgdv$$

$$-(n_1+n_2)\int_a^b \int_0^{\frac{1}{4}v^2} v g^{n_1-2}(1-v+g)^{n_2-2}(v^2-4g)^{\frac{1}{2}}dg\,dv].$$

And $P(a \le V^{(2)} \le b|\lambda_1=\lambda_2=1)$

$$= C(2)\int_a^b \int_0^{\frac{1}{4}v^2} g^{n_1-2}(1-v+g)^{n_2-2}(v^2-4g)^{\frac{1}{2}}dg\,dv.$$

Therefore, we have the following theorem:

Theorem 5. Let

$$T_1(x)=\begin{cases} \sum_{r=0}^{n_2-2} \frac{\binom{n_2-2}{r}(n_1+r-2)!}{2^{n_1+r-1}\,3.5\ldots(2(n_1+r)-1)} B_x(2n_1+2r,n_2-r-1) & \text{if } 0\le x\le 1, \\ \sum_{r=0}^{n_2-2} \frac{\binom{n_2-2}{r}(n_1+r-2)!}{2^{n_1+r-1}\,3.5\ldots(2(n_1+r)-1)} B_1(2n_1+2r,\ n_2-r-1) + \\ 2\sum_{r=0}^{n_1-2}(-1)^{n_1-r-2} \frac{\binom{n_1-2}{r}2^{n_2+r}(n_2+r-2)!}{3.5\ldots(2(n_2+r)-1)} \sum_{r=0}^{n_1-r-2}(-2)^j\binom{n_1-r-2}{j} \\ \qquad [B_{\frac{x}{2}}(j+1,2(n_2+r))-B_{\frac{1}{2}}(j+1,2(n_2+r))] & \text{if } 1 \le x \le 2, \end{cases}$$

and

$$T_2(x) = \begin{cases} \sum_{r=0}^{n_2-2} \frac{\binom{n_2-2}{r}(n_1+r-2)!}{2^{n_1+r-1}\,3.5\ldots(2(n_1+r)-1)} B_x(2n_1+2r+1,n_2-r-1) & \text{if } 0\le x\le 1, \\ \sum_{r=0}^{n_2-2} \frac{\binom{n_2-2}{r}(n_1+r-2)!}{2^{n_1+r-1}\,3.5\ldots(2(n_1+r)-1)} B_1(2n_1+2r+1,n_2-r-1) + \\ 4\sum_{r=0}^{n_1-2}(-1)^{n_1-r-2} \frac{\binom{n_1-2}{r}2^{n_2+r}(n_2+r-2)!}{3.5\ldots(2(n_2+r)-1)} \sum_{j=0}^{n_1-r-2}(-2)^j\binom{n_1-r-2}{j} \end{cases}$$

$$\times\ [B_{\frac{x}{2}}(j+2,2(n_2+r))-B_{\frac{1}{2}}(j+2,2(n_2+r))] \quad \text{if } 1 \le x \le 2.$$

Then the acceptance region based on $V^{(2)}=[c_1/(1+c_1)]+[c_2/(1+c_2)]$ with $\ell$up and $\alpha$ level is given by $a \le V^{(2)} \le b$ where a and b are so chosen as to satisfy

(i) $C(2)[T_1(b)-T_1(a)] = 1-\alpha$ and

(ii) $C(2)\{2n_1[T_1(b)-T_1(a)]-(n_1+n_2)[T_2(b)-T_2(a)]\} = 0.$

4) Wilks' criterion $W^{(2)} = [(1+c_1)(1+c_2)]^{-1}$. Again from Theorem 2, we obtain $\left.\frac{\partial P(a \le W^{(2)} \le b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$ by replacing $L_2^{(2)}$ in the limits of the integrals in (4.1) by $W^{(2)}$. Now transform $\ell_1=c_1/(1+c_1)$ and $\ell_2=c_2/(1+c_2)$.

Then $\left.\frac{\partial P(a \le W^{(2)} \le b|\lambda_1=\lambda_2=\lambda)}{\partial\lambda^{-1}}\right|_{\lambda=1}$

$$= C(2)[\int_{a\le(1-\ell_1)(1-\ell_2)\le b} g_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2 - \int_{a\le(1-\ell_1)(1-\ell_2)\le b} h_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2].$$

The density function of $W^{(2)}$ has been found by Pillai and Jouris [30] as follows:

$$f(w) = \frac{\tilde{\Gamma}_2(n_1+n_2)}{\tilde{\Gamma}_2(n_2)\Gamma(2n_1)} \sum_{k=0}^{\infty} \frac{(n_1)_k(n_1-1)_k}{(2n_1)_k k!} w^{n_2-2}(1-w)^{2n_1+k-1}.$$

Thus, the distribution function of $W^{(2)}$ is

$$-2\sum_{j=0}^{\infty}\frac{(n_1+j-2)!}{4xj!(j-\frac{1}{2})\cdots(n_1+j-\frac{1}{2})}\sum_k{}'(-1)^k\binom{-2j+1}{k}B_{w^{\frac{1}{2}}}(2(n_2+k-1),2(n_1+2j)-1)$$

where $\sum_k{}' = \sum_{k=0}^{1}$ if $j = 0$ and $\sum_k{}' = \sum_{k=0}^{\infty}$ if $j > 0$. Therefore, we have the following:

Theorem 6. Let

$$T_1(w) = -2\sum_{j=0}^{\infty}\frac{(n_1+j-2)!}{4xj!(j-\frac{1}{2})\cdots(n_1+j-\frac{1}{2})}\sum_k{}'(-1)^k\binom{-2j+1}{k}B_{w^{\frac{1}{2}}}(2(n_2+k-1),\; 2(n_1+2j)-1),$$

$$T_2(w) = -2\sum_{j=0}^{\infty}\frac{(n_1+j-2)!}{4xj!(j-\frac{1}{2})\cdots(n_1+j-\frac{1}{2})}\sum_k{}'(-1)^k\binom{-2j+1}{k}B_{w^{\frac{1}{2}}}(2(n_2+k),\; 2(n_1+2j)-1),$$

and

$$T_3(w) = -2\sum_{j=0}^{\infty}\frac{(n_1+j-1)!}{4xj!(j-\frac{1}{2})\cdots(n_1+j+\frac{1}{2})}\sum_k{}'(-1)^k\binom{-2j+1}{k}B_{w^{\frac{1}{2}}}(2(n_2+k-1),\; 2(n_1+2j)+1).$$

Then the acceptance region based on $W^{(2)} = [(1+c_1)(1+c_2)]^{-1}$ with $\ell$up and $\alpha$ level is given by $a \leq W^{(2)} \leq b$ where a and b are so chosen as to satisfy

(i) $C(2)[T_1(b)-T_1(a)] = 1-\alpha$ and

(ii) $C(2)[(n_1-n_2)\{T_1(b)-T_1(a)\}+(n_1+n_2)\{T_2(b)-T_2(a)\}-(n_1+n_2)\{T_3(b)-T_3(a)\}] = 0.$

5) Roy's largest-smallest roots, $LS^{(2)} = c_2/(1+c_2),\ c_1/(1+c_1)$.

From Theorem 2 in the previous section, we have

$$(4.11)\quad \left.\frac{\partial P(a \le LS^{(2)} \le b|\lambda_1=\lambda_2=\lambda)}{\partial \lambda^{-1}}\right|_{\lambda=1}$$

$$= C(2)[\int_{a\le c_1/(1+c_1)\le c_2/(1+c_2)\le b} g_2(c_1,c_2;n_1,n_2)dc_1dc_2 - \int_{a\le c_1/(1+c_1)\le c_2/(1+c_2)\le b} h_2(c_1,c_2;n_1,n_2)dc_1dc_2].$$

Now transform $\ell_1 = c_1/(1+c_1)$ and $\ell_2 = c_2/(1+c_2)$. Then (4.11) becomes

$$C(2)[\int_a^b \int_a^{\ell_2} g_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2 - \int_a^b \int_a^{\ell_2} h_3(\ell_1,\ell_2;n_1,n_2)d\ell_1 d\ell_2].$$

Further, note that by making the same transformation,

$$P(a \le LS^{(2)} \le b|\lambda_1=\lambda_2=1)$$

$$= C(2)\int_a^b \int_a^{\ell_2} (\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2.$$

Use the technique as in Khatri [9], we have

$$\int_x^y \int_x^{\ell_2} (\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$$

$$= B_{x,y}(n_1-1,n_2-1)B_{x,y}(n_1+1,n_2-1)-B_{x,y}(n_1,n_2-1)B_{x,y}(n_1,n_2-1),$$

where $B_{x,y}(r,s) = \int_x^y t^{r-1}(1-t)^{s-1}dt$.

Now $\int_x^y \int_x^{\ell_2} (\ell_1+\ell_2)(\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$

$$= \int_x^y \int_x^{\ell_2} [1-(1-\ell_1)(1-\ell_2)+\ell_1\ell_2](\ell_1\ell_2)^{n_1-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_1-\ell_2)^2 d\ell_1 d\ell_2$$

$$= B_{x,y}(n_1-1,n_2-1)B_{x,y}(n_1+1,n_2-1)-B_{x,y}(n_1,n_2-1)B_{x,y}(n_1,n_2-1)$$
$$- B_{x,y}(n_1-1,n_2)B_{x,y}(n_1+1,n_2)+B_{x,y}(n_1,n_2)B_{x,y}(n_1,n_2)$$
$$+ B_{x,y}(n_1,n_2-1)B_{x,y}(n_1+2,n_2-1)-B_{x,y}(n_1+1,n_2-1)B_{x,y}(n_1+1,n_2-1).$$

Thus, we have proved the following:

Theorem 7. Let $T_1(x,y) = B_{x,y}(n_1-1,n_2-1)B_{x,y}(n_1+1,n_2-1)-B_{x,y}(n_1,n_2-1)$ $B_{x,y}(n_1,n_2-1)$ and $T_2(x,y) = T_1(x,y)-B_{x,y}(n_1-1,n_2)B_{x,y}(n_1+1,n_2)+$ $B_{x,y}(n_1,n_2)B_{x,y}(n_1,n_2) +B_{x,y}(n_1,n_2-1)B_{x,y}(n_1+2,n_2-1)-B^2_{x,y}(n_1+1,n_2-1)$. Then the acceptance region based on Roy's largest-smallest roots, $LS^{(2)}=c_2/(1+c_2)$, $c_1/(1+c_1)$ with $\ell$up and $\alpha$ level is given by $a \leq LS^{(2)} \leq b$ where a and b are so chosen as to satisfy

(i) $C(2)\ T_1(a,b) = 1-\alpha$

(ii) $C(2)[2n_1T_1(a,b)-(n_1+n_2)T_2(a,b)] = 0.$

## 5. $P(a \leq [c_1/(1+c_1)] \leq [c_2/(1+c_2)] \leq b)$ in the non-null case

The non-null distribution of $c_1,\ldots,c_p$ was obtained by Khatri [13] in the form

$$(5.1)\quad C(p)|\underset{\sim}{\Lambda}|^{-n_1}|\underset{\sim}{C}|^{n_1-p}|\underset{\sim}{I}+\underset{\sim}{C}|^{-(n_1+n_2)}\ {}_1\tilde{F}_0(n_1+n_2;\underset{\sim}{I}-\underset{\sim}{\Lambda}^{-1},\underset{\sim}{C}(\underset{\sim}{I}+\underset{\sim}{C})^{-1})$$
$$\prod_{i>j}(c_i-c_j)^2,$$
$$0 < c_1 \leq\ldots\leq c_p < \infty,$$

where the complex hypergeometric function of a matrix argument is defined by James [8]:

$$ {}_s\tilde{F}_t(a_1,\ldots,a_s;b_1,\ldots,b_t;\underset{\sim}{A},\underset{\sim}{B}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a_1]_\kappa \cdots [a_s]_\kappa}{[b_1]_\kappa \cdots [b_t]_\kappa} \frac{\tilde{C}_\kappa(\underset{\sim}{A})\tilde{C}_\kappa(\underset{\sim}{B})}{k!\tilde{C}_\kappa(I)}, $$

where $a_1,\ldots,a_s,b_1,\ldots,b_t$ are real or complex constants and the coefficient $[a]_\kappa$ is defined by

$$ [a]_\kappa = \prod_{i=1}^{p} (a-i+1)_{k_i} $$

where $(a)_k = a(a+1)\ldots(a+k-1)$ and $\kappa$ of $k$ is a partition of $k$, $\kappa = (k_1,\ldots,k_p)$, $k_1 \geq \ldots \geq k_p \geq 0$ such that $k_1+\ldots+k_p = k$ and the complex zonal polynomials $\tilde{C}_\kappa(\underset{\sim}{A})$ are expressible in terms of elementary symmetric functions of the characteristic roots of the hermitian matrix $\underset{\sim}{A}$. Now putting $p = 2$, in (5.1), the joint density of $c_1$, $c_2$ is given by

$$ (5.2) \quad C(2)(\lambda_1\lambda_2)^{-n_1}(c_1c_2)^{n_1-2}[(1+c_1)(1+c_2)]^{-(n_1+n_2)}(c_2-c_1)^2 $$

$$ \times \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n_1+n_2]_\kappa}{k!} \frac{\tilde{C}_\kappa\begin{pmatrix} 1-1/\lambda_1 & 0 \\ 0 & 1-1/\lambda_2 \end{pmatrix} \tilde{C}_\kappa\begin{pmatrix} c_1/(1+c_1) & 0 \\ 0 & c_2/(1+c_2) \end{pmatrix}}{\tilde{C}_\kappa(\underset{\sim}{I})}. $$

Using the relation $\tilde{C}_\kappa\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \sum_{r+2s=k} \tilde{b}_\kappa(r,s)(a+b)^r(ab)^s$, where the $\tilde{b}_\kappa(r,s)$'s were given by Pillai and Hsu [27] up to $k = 6$, (5.2) becomes

$$ (5.3) \quad C(2)(\lambda_1\lambda_2)^{-n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n_1+n_2]_\kappa}{k!} \frac{\tilde{C}_\kappa\begin{pmatrix} 1-1/\lambda_1 & 0 \\ 0 & 1-1/\lambda_2 \end{pmatrix}}{\tilde{C}_\kappa(\underset{\sim}{I})} $$

$$ \times \sum_{r+2s=k} \tilde{b}_\kappa(r,s)[c/(1+c_1)+c_2/(1+c_2)]^r(c_1c_2)^{n_1+s-2} $$

$$ \times [(1+c_1)(1+c_2)]^{-(n_1+n_2+s)}(c_2-c_1)^2. $$

Now transform $\ell_1 = c_1/(1+c_1)$, $\ell_2 = c_2/(1+c_2)$, then the joint distribution of $\ell_1$, $\ell_2$ is

$$C(2)(\lambda_1\lambda_2)^{-n_1}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[n_1+n_2]_\kappa}{k!}\frac{\tilde{C}_\kappa\begin{pmatrix}1-1/\lambda_1 & 0\\ 0 & 1-1/\lambda_2\end{pmatrix}}{\tilde{C}_\kappa(I)}$$

$$\times\sum_{r+2s=k}\tilde{b}_\kappa(r,s)(\ell_1+\ell_2)^r(\ell_1\ell_2)^{n_1+s-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_2-\ell_1)^2.$$

Now let

$$h_{r,s}(a,b) = \int_a^b\int_a^{\ell_2}(\ell_1+\ell_2)^r(\ell_1\ell_2)^{n_1+s-2}[(1-\ell_1)(1-\ell_2)]^{n_2-2}(\ell_2-\ell_1)^2 d\ell_1 d\ell_2$$

$$= \sum_{i=0}^{r}\binom{r}{i}\sum_{j=0}^{n_2-2}(-1)^j\binom{n_2-2}{j}$$

$$\times\{[B_{a,b}(2n_1+2s+r+j,n_2-1)-a^{i+j+n_1+s-1}B_{a,b}(n_1+s+r-i+1,n_2-1)]/(i+j+n_1+s-1)$$

$$-2[B_{a,b}(2n_1+2s+r+j,n_2-1)-a^{i+j+n_1+s}B_{a,b}(n_1+s+r-i,n_2-1)]/(i+j+n_1+s)$$

$$+[B_{a,b}(2n_1+2s+r+j,n_2-1)-a^{i+j+n_1+s+1}B_{a,b}(n_1+s+r-i-1,n_2-1)]/(i+j+n_1+s+1)\},$$

where $B_{x,y}(r,s) = \int_x^y t^{r-1}(1-t)^{s-1}dt$.

Then we have, in the complex non-null case,

$$P(a \leq [c_1/(1+c_1)] \leq [c_2/(1+c_2)] \leq b]$$

$$=P(a \leq \ell_1 \leq \ell_2 \leq b)$$

$$= C(2)(\lambda_1\lambda_2)^{-n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[n_1+n_2]_\kappa}{k!} \frac{\tilde{C}_\kappa\begin{pmatrix} 1-1/\lambda_1 & 0 \\ 0 & 1-1/\lambda_2 \end{pmatrix}}{\tilde{C}_\kappa(\underset{\sim}{I})}$$

$$\times \sum_{r+2s=k} \tilde{b}_\kappa(r,s)h_{r,s}(a,b).$$

## 6. Numerical study of power

The results in the previous sections were used to obtain five percent points for the tests of $H_0$: $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ against $\underset{\sim}{\Sigma}_1 \neq \underset{\sim}{\Sigma}_2$ based on criteria 1) to 4) in the unbiased as well as equal tail areas cases and criterion 5) in the unbiased case for $p = 2$, values of $n_1 = 2$, 3, 4, 7 and $n_2 = 7$, 17, 32, 62, and are given in Table 7.

The next step was to compute the powers of the various tests using the percentage points evaluated and the non-null distributions. For tests 1) to 4), non-null distributions were available in Pillai and Hsu [27] and for test 5), it has been obtained in Section 5. Before computing the power for a specific value of $(\lambda_1,\lambda_2)$, the total probability in that case over the whole range of the respective statistic for all the terms included in the formula was calculated and the number of decimal places included in the tables was determined depending on the number of places of accuracy obtained in the total probability, at least as many decimal places as in the tables. Powers for tests 1) to 5) in the unbiased as well as equal tail areas cases for $p = 2$, for values of $n_1 = 2,3,4,7$, $n = 7,17,32,62$, and various $(\lambda_1,\lambda_2)$ are presented in Table 8 (to be continued in the Appendix D).

A few findings seem to emerge from tabulations of powers in Table 8, and they are in general agreement with those discussed before in the real case and are stated below for convenience.

1. $\lambda_1 \geq 1$, $\lambda_2 \geq 1$. It may be seen from Table 8 that equal tail areas tests based on 1) to 4) generally seem to perform better than corresponding unbiased ones except when very close to $H_0$ in which case bias is observed in some instances, mostly when $n_1$ is close to $n_2$.
2. $\lambda_1 < 1$, $\lambda_2 > 1$ or $\lambda_1 > 1$, $\lambda_2 < 1$. For tests 1) to 4), unbiased test is better than equal tails except when $\lambda_1 + \lambda_2 > 2$. When $\lambda_1+\lambda_2 \leq 2$, bias is observed.
3. $\lambda_1 < 1$, $\lambda_2 < 1$. For tests 1) to 4), unbiased test seems to be better than equal tails. There exists some bias when close to $H_0$.
4. $\lambda_1 \geq 1$, $\lambda_2 \geq 1$. In regard to comparative performance of the criteria, findings in the equal tail areas case are as in the one-sided case for 1) to 4) described by Pillai and Hsu [27], in the unbiased case when $\lambda_1$ and $\lambda_2$ are far apart but both greater than unity, in terms of power, $U^{(2)} > W^{(2)} > V^{(2)} > L_2^{(2)} > LS^{(2)}$, but with only one large positive deviation, $L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)} > LS^{(2)}$. But if $\lambda_1$ and $\lambda_2$ are close, then $V^{(2)} > W^{(2)} > U^{(2)} > L_2^{(2)} > LS^{(2)}$.
5. $\lambda_1 < 1$, $\lambda_2 > 1$ or $\lambda_1 > 1$, $\lambda_2 < 1$. In the unbiased case, $U^{(2)} > W^{(2)} > L_2^{(2)} > V^{(2)} > LS^{(2)}$ when $\lambda_1 + \lambda_2 < 2$, $LS^{(2)} > L_2^{(2)} > U^{(2)}$ $W^{(2)} > V^{(2)}$ when $\lambda_1 + \lambda_2 = 2$, $L_2^{(2)} > U^{(2)} > W^{(2)} > LS^{(2)} > V^{(2)}$ when $\lambda_1 + \lambda_2 > 2$. In the equal tail areas case, $L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$.

6. $\lambda_1 < 1$, $\lambda_2 < 1$. In the unbiased case, $V^{(2)} > W^{(2)} > U^{(2)} > L_2^{(2)} > LS^{(2)}$, and in the equal tail areas case, $L_2^{(2)} > U^{(2)} > W^{(2)} > V^{(2)}$.

7. $L_2^{(2)}$ seems to be least biased, then $U^{(2)}$, then $W^{(2)}$ and lastly $V^{(2)}$.

8. If a single test has to be recommended on an overall basis over the whole parameter space, Roy's largest root seems to be the proper candidate. In the two-sided case as well as when both $\lambda_1$ and $\lambda_2$ are less than unity, among tests 1) to 4), largest root performs best in the equal tail areas case. Since the largest root is the least biased, even equal tail areas could be adequate. However, for the two-sided case, the unbiased largest root test compares favorably with $LS^{(2)}$ when $\lambda_1+\lambda_2=2$ and is even the best when $\lambda_1+\lambda_2 > 2$.

## TABLE 7

Percentage points of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $(W^{(2)})^{\frac{1}{2}}$ and $LS^{(2)}$ in the complex case

| $n_1$ | $n_2$ | | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $(W^{(2)})^{\frac{1}{2}}$ | $LS^{(2)}$ |
|---|---|---|---|---|---|---|---|
| | | | 5% points with ℓup | | | | |
| 2 | 7 | a | .12821869 | .17322768 | .15462404 | .52095645 | .00290803 |
| | | b | .68638534 | 2.33623275 | .83099169 | .92137257 | .73009510 |
| 2 | 17 | a | .05691831 | .07123695 | .06758791 | .76177015 | .00128268 |
| | | b | .38418795 | .68369464 | .44339986 | .96588346 | .42569680 |
| 2 | 32 | a | .03103056 | .03779785 | .03671862 | .86442150 | .00069522 |
| | | b | .22888365 | .32840755 | .26031058 | .98154025 | .25820667 |
| 2 | 62 | a | .01624908 | .01949062 | .01919565 | .92720307 | .00036260 |
| | | b | .12623932 | .16072592 | .14244743 | .99037409 | .14399519 |
| 3 | 7 | a | .20698277 | .33252292 | .27375573 | .44734424 | .02184729 |
| | | b | .73839249 | 3.11217282 | .99247727 | .85980401 | .76040658 |
| 3 | 17 | a | .09711506 | .13788433 | .12599620 | .70797238 | .00999849 |
| | | b | .43787605 | .89741927 | .54943215 | .93618932 | .46268266 |
| 3 | 32 | a | .05408345 | .07336740 | .06979313 | .82951671 | .00549699 |
| | | b | .26816592 | .42912028 | .32887181 | .96484313 | .28695772 |
| 3 | 62 | a | .02867536 | .03789078 | .03690387 | .90703078 | .00289081 |
| | | b | .15054372 | .20951297 | .18230195 | .98147333 | .16241415 |
| 4 | 7 | a | .27478561 | .50547305 | .38512321 | .39285041 | .05126577 |
| | | b | .77506206 | 3.87520188 | 1.11249648 | .80184315 | .78854720 |
| 4 | 17 | a | .13544033 | .21125777 | .18593691 | .66252198 | .02433498 |
| | | b | .48139225 | 1.10500451 | .63966431 | .90562858 | .49910319 |
| 4 | 32 | a | .07697518 | .11271701 | .10497306 | .79836759 | .01357882 |
| | | b | .30201897 | .52638535 | .39023355 | .94705376 | .31634268 |
| 4 | 62 | a | .04131685 | .05829979 | .05614305 | .88842129 | .00720264 |
| | | b | .17230585 | .25644795 | .21910635 | .97179362 | .18169966 |
| 7 | 7 | a | .42584076 | 1.05120667 | .65768820 | .28904005 | .15435958 |
| | | b | .84088837 | 6.13942238 | 1.34231221 | .65985927 | .84564042 |
| 7 | 17 | a | .23589279 | .45197490 | .35630405 | .55821793 | .08073340 |
| | | b | .57591970 | 1.71342571 | .84851380 | .81860922 | .58506668 |
| 7 | 32 | a | .14009359 | .23800115 | .21172106 | .72029908 | .04709359 |
| | | b | .38037284 | .79449537 | .54458620 | .89298639 | .39197944 |
| 7 | 62 | a | .07740531 | .12309289 | .11462655 | .84151811 | .02600531 |
| | | b | .22509263 | .38389322 | .31205213 | .94233896 | .23702259 |

TABLE 7 (continued)

| $n_1$ | $n_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $(W^{(2)})^{\frac{1}{2}}$ |
|---|---|---|---|---|---|
| | | Lower 2.5% points | | | |
| 2 | 7 | .12468851 | .16691032 | .14724334 | .52979838 |
| 2 | 7 | .05308474 | .06585292 | .06255707 | .77222609 |
| 2 | 32 | .02853982 | .03453679 | .03360826 | .87189960 |
| 2 | 62 | .01483010 | .01770404 | .01745680 | .93172574 |
| 3 | 7 | .20706925 | .33060577 | .26740121 | .45134357 |
| 3 | 17 | .09382076 | .13191976 | .12055789 | .71553703 |
| 3 | 32 | .05159252 | .06945693 | .06617373 | .83550310 |
| 3 | 62 | .02715750 | .03568282 | .03479595 | .91085069 |
| 4 | 7 | .27784817 | .51006342 | .38047654 | .39432275 |
| 4 | 17 | .13294046 | .20539781 | .18065307 | .66826463 |
| 4 | 32 | .07468683 | .10850647 | .10118622 | .80338722 |
| 4 | 62 | .03981586 | .05585176 | .05384671 | .89179196 |
| 7 | 7 | .43382403 | 1.08844933 | .65768751 | .28750623 |
| 7 | 17 | .23562338 | .44697472 | .35240554 | .56095351 |
| 7 | 32 | .14009359 | .23800115 | .20825615 | .72352607 |
| 7 | 62 | .07740531 | .12309289 | .11462655 | .84151811 |
| | | Upper 2.5% points | | | |
| 2 | 7 | .68003853 | 2.25922247 | .81403456 | .92464846 |
| 2 | 17 | .37027612 | .64272268 | .42488231 | .96841446 |
| 2 | 32 | .21743266 | .30687762 | .24642746 | .98310554 |
| 2 | 62 | .11879574 | .14978666 | .13376147 | .99124679 |
| 3 | 7 | .73849492 | 3.09457433 | .98167352 | .86208147 |
| 3 | 17 | .42971616 | .86447864 | .53541519 | .93890283 |
| 3 | 32 | .26015821 | .41061170 | .31758986 | .96666453 |
| 3 | 62 | .14488374 | .19988504 | .17491998 | .98253133 |
| 4 | 7 | .77786858 | 3.91163088 | 1.10600989 | .80292555 |
| 4 | 17 | .47647854 | 1.07785551 | .62861711 | .90823270 |
| 4 | 32 | .29608354 | .50983566 | .38066189 | .94895748 |
| 4 | 62 | .16773388 | .24763762 | .21256117 | .97294580 |
| 7 | 7 | .84582037 | 6.39841090 | 1.34231227 | .65817298 |
| 7 | 17 | .57555746 | 1.69511974 | .84269368 | .82041870 |
| 7 | 32 | .38037284 | .79449537 | .53832400 | .89473045 |
| 7 | 62 | .22509268 | .38389348 | .31205224 | .94233896 |

TABLE 8

Powers of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and $LS^{(2)}$ in the unbiased and equal tail areas cases for testing $\lambda_1 = 1$, $\lambda_2 = 1$ against different simple two-sided alternative hypotheses in the complex case, $\alpha = .05$

$n_1 = 2, \quad n_2 = 7$

| | | With local unbiased property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050007 | .050009 | .050014 | .050011 |
| 1 | 1.1 | .050671 | .050690 | .050619 | .050686 | .050343 | .051346 | .051523 | .051931 | .051779 |
| 1.05 | 1.05 | .050537 | .050576 | .050629 | .050610 | .050214 | .051211 | .051410 | .051955 | .051709 |
| 1 | 1.5 | .0656 | .0659 | .0640 | .0657 | .0595 | .0688 | .0698 | .0700 | .0708 |
| 1.25 | 1.25 | .0621 | .0630 | .0641 | .0638 | .0557 | .0653 | .0670 | .0705 | .0691 |
| 1 | 2 | .1177 | .1183 | .1123 | .1175 | .1021 | .1231 | .1250 | .1225 | .1262 |
| 1.333 | 1.5 | .0853 | .0875 | .0898 | .0893 | .0714 | .0903 | .0937 | .0997 | .0976 |
| 1 | 3 | .288 | .289 | .279 | .287 | .264 | .295 | .297 | .291 | .298 |
| 2.5 | 1.5 | .276 | .280 | .282 | .283 | .247 | .283 | .289 | .297 | .295 |
| 2 | 2 | .272 | .278 | .284 | .283 | .242 | .280 | .287 | .299 | .295 |
| 1.00001 | 0.9 | .050747 | .050773 | .050717 | .050773 | .050373 | .050047 | .049908 | .049333 | .049633 |
| 1.00001 | 0.8 | .052792 | .052916 | .052687 | .052919 | .051121 | .051365 | .051150 | .049844 | .050591 |
| 1.01 | 0.99 | .050006 | .050005 | .050000 | .050004 | .050006 | .050006 | .050005 | .050000 | .050003 |
| 1.1 | 0.9 | .050586 | .050518 | .050044 | .050369 | .050561 | .050587 | .050507 | .049978 | .050340 |
| 1.1 | 0.8 | .051863 | .051777 | .050799 | .051491 | .051407 | .051165 | .050888 | .049278 | .050292 |
| 1.05 | 0.95 | .050146 | .050129 | .050010 | .050092 | .050140 | .050147 | .050127 | .049994 | .050085 |
| 1.2 | 0.99 | .0524 | .0524 | .0521 | .0524 | .0513 | .0537 | .0540 | .0545 | .0544 |
| 1.2 | 0.8 | .0522 | .0519 | .0500 | .0513 | .0521 | .0522 | .0519 | .0498 | .0512 |
| 2 | 0.9 | .110 | .109 | .100 | .107 | .097 | .115 | .116 | .109 | .115 |
| 2 | 0.7 | .098 | .096 | .079 | .091 | .090 | .102 | .101 | .086 | .097 |
| 3 | 0.9 | .27 | .27 | .26 | .27 | .25 | .28 | .28 | .27 | .28 |
| 0.99999 | 0.9 | .050748 | .050773 | .050717 | .050773 | .050373 | .050047 | .049908 | .049333 | .049633 |
| 0.9 | 0.999 | .050756 | .050783 | .050731 | .050784 | .050374 | .050049 | .049909 | .049333 | .049633 |
| 0.9 | 0.9 | .052444 | .052610 | .052839 | .052749 | .050870 | .051012 | .050849 | .050057 | .050443 |
| 0.9 | 0.8 | .0547 | .0551 | .0555 | .0554 | .0507 | .0525 | .0524 | .0513 | .0519 |
| 0.9 | 0.76 | .0525 | .0531 | .0534 | .0535 | .0506 | .0504 | .0502 | .0488 | .0497 |
| 0.85 | 0.9 | .0538 | .0540 | .0544 | .0543 | .0512 | .0520 | .0518 | .0509 | .0513 |
| 0.85 | 0.8 | .0551 | .0557 | .0565 | .0562 | .0522 | .0525 | .0526 | .0516 | .0522 |
| 0.81 | 0.9 | .0547 | .0550 | .0554 | .0553 | .0510 | .0526 | .0524 | .0513 | .0519 |
| 0.8 | 0.8 | .0517 | .0527 | .0539 | .0536 | .0529 | .0499 | .0493 | .0486 | .0491 |

TABLE 8 (continued)

$n_1 = 2, \ n_2 = 17$

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property: $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas: $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050021 | .050024 | .050025 | .050024 |
| 1 | 1.1 | .050973 | .050983 | .050931 | .050969 | .050479 | .053057 | .053329 | .053386 | .053388 |
| 1.05 | 1.05 | .050766 | .050822 | .050836 | .050835 | .050310 | .052839 | .053170 | .053300 | .053259 |
| 1 | 1.5 | .0724 | .0724 | .0710 | .0720 | .0630 | .0826 | .0838 | .0829 | .0836 |
| 1.25 | 1.25 | .0674 | .0688 | .0691 | .0690 | .0583 | .0775 | .0803 | .0812 | .0810 |
| 1 | 2 | .1361 | .1358 | .1319 | .1346 | .1127 | .1538 | .1554 | .1523 | .1547 |
| 1.333 | 1.5 | .0983 | .1014 | .1020 | .1020 | .0782 | .1144 | .1199 | .1212 | .1210 |
| 1 | 3 | .318 | .317 | .310 | .315 | .282 | .339 | .341 | .335 | .339 |
| 2.5 | 1.5 | .306 | .312 | .312 | .313 | .265 | .330 | .340 | .341 | .341 |
| 2 | 2 | .304 | .312 | .314 | .314 | .260 | .328 | .340 | .343 | .343 |
| 1.00001 | 0.9 | .051052 | .051071 | .051027 | .051055 | .050493 | .048959 | .048701 | .048532 | .048606 |
| 1.00001 | 0.8 | .054106 | .054207 | .054032 | .054150 | .051639 | .049917 | .049448 | .049005 | .049225 |
| 1.01 | 0.99 | .050009 | .050007 | .050005 | .050006 | .050007 | .050009 | .050007 | .050004 | .050006 |
| 1.1 | 0.9 | .050876 | .050707 | .050459 | .050606 | .050709 | .050915 | .050698 | .050416 | .050581 |
| 1.1 | 0.8 | .052779 | .052501 | .051983 | .052293 | .051828 | .050770 | .050116 | .049399 | .049796 |
| 1.05 | 0.95 | .050219 | .050176 | .050114 | .050151 | .050177 | .050229 | .050174 | .050104 | .050145 |
| 1.2 | 0.99 | .0535 | .0535 | .0533 | .0534 | .0518 | .0575 | .0580 | .0579 | .0580 |
| 1.2 | 0.8 | .0534 | .0527 | .0517 | .0523 | .0527 | .0535 | .0527 | .0516 | .0522 |
| 2 | 0.9 | .127 | .125 | .120 | .123 | .107 | .143 | .143 | .138 | .141 |
| 2 | 0.7 | .114 | .109 | .100 | .105 | .100 | .128 | .122 | .114 | .119 |
| 3 | 0.9 | .30 | .30 | .29 | .29 | .27 | .32 | .32 | .31 | .32 |
| 0.99999 | 0.9 | .051052 | .051071 | .051028 | .051055 | .050493 | .048959 | .048701 | .048532 | .048605 |
| 0.9 | 0.999 | .051064 | .051085 | .051043 | .051070 | .050495 | .048950 | .048692 | .048523 | .048596 |
| 0.9 | 0.9 | .053392 | .053619 | .053677 | .053655 | .051204 | .049145 | .048851 | .048677 | .048737 |
| 0.9 | 0.8 | .0572 | .0577 | .0578 | .0578 | .0516 | .0508 | .0506 | .0503 | .0504 |
| 0.9 | 0.76 | .0566 | .0574 | .0573 | .0574 | .0514 | .0497 | .0496 | .0492 | .0494 |
| 0.85 | 0.9 | .0553 | .0557 | .0557 | .0557 | .0517 | .0500 | .0497 | .0495 | .0496 |
| 0.85 | 0.8 | .0587 | .0595 | .0597 | .0597 | .0536 | .0513 | .0513 | .0511 | .0512 |
| 0.81 | 0.9 | .0569 | .0574 | .0575 | .0575 | .0518 | .0508 | .0505 | .0502 | .0503 |
| 0.8 | 0.8 | .0574 | .0588 | .0591 | .0591 | .0544 | .0499 | .0499 | .0497 | .0498 |

TABLE 8 (continued)

$n_1 = 2, \ n_2 = 32$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050027 | .050030 | .050030 | .050030 |
| 1 | 1.1 | .051109 | .051107 | .051073 | .051097 | .050542 | .053803 | .054071 | .054070 | .054083 |
| 1.05 | 1.05 | .050867 | .050927 | .050932 | .050935 | .050354 | .053544 | .053891 | .053933 | .053922 |
| 1 | 1.5 | .0755 | .0752 | .0743 | .0749 | .0648 | .0888 | .0897 | .0889 | .0894 |
| 1.25 | 1.25 | .0698 | .0713 | .0714 | .0714 | .0596 | .0830 | .0860 | .0863 | .0862 |
| 1 | 2 | .1447 | .1436 | .1412 | .1426 | .1181 | .1677 | .1685 | .1663 | .1677 |
| 1.333 | 1.5 | .1042 | .1076 | .1077 | .1078 | .0817 | .1252 | .1310 | .1314 | .1314 |
| 1 | 3 | .331 | .329 | .325 | .328 | .291 | .359 | .359 | .356 | .358 |
| 2.5 | 1.5 | .320 | .327 | .326 | .327 | .274 | .351 | .361 | .361 | .361 |
| 2 | 2 | .318 | .327 | .328 | .328 | .270 | .350 | .363 | .364 | .363 |
| 1.00001 | 0.9 | .051183 | .051192 | .051164 | .051177 | .050545 | .048511 | .048229 | .048158 | .048188 |
| 1.00001 | 0.8 | .054664 | .054725 | .054613 | .054672 | .051860 | .049350 | .048810 | .048600 | .048700 |
| 1.01 | 0.99 | .050010 | .050008 | .050006 | .050007 | .050008 | .050011 | .050008 | .050006 | .050007 |
| 1.1 | 0.9 | .051006 | .050780 | .050638 | .050717 | .050773 | .051074 | .050780 | .050616 | .050706 |
| 1.1 | 0.8 | .053173 | .052783 | .052484 | .052647 | .052003 | .050645 | .049823 | .049435 | .049640 |
| 1.05 | 0.95 | .050251 | .050195 | .050159 | .050179 | .050193 | .050269 | .050195 | .050154 | .050176 |
| 1.2 | 0.99 | .0540 | .0540 | .0538 | .0539 | .0521 | .0592 | .0596 | .0595 | .0596 |
| 1.2 | 0.8 | .0539 | .0530 | .0525 | .0528 | .0530 | .0542 | .0530 | .0524 | .0527 |
| 2 | 0.9 | .135 | .132 | .129 | .130 | .112 | .156 | .154 | .151 | .153 |
| 2 | 0.7 | .122 | .114 | .109 | .112 | .105 | .139 | .132 | .126 | .129 |
| 3 | 0.9 | .31 | .31 | .30 | .31 | .28 | .34 | .34 | .33 | .34 |
| 0.99999 | 0.9 | .051184 | .051193 | .051165 | .051177 | .050545 | .048510 | .048229 | .048158 | .048188 |
| 0.9 | 0.999 | .051197 | .051209 | .051181 | .051193 | .050548 | .048497 | .048215 | .048145 | .048175 |
| 0.9 | 0.9 | .053798 | .054032 | .054052 | .054037 | .051349 | .048392 | .048095 | .048041 | .048055 |
| 0.9 | 0.8 | .0583 | .0588 | .0588 | .0588 | .0520 | .0502 | .0500 | .0498 | .0499 |
| 0.9 | 0.76 | .0584 | .0591 | .0590 | .0591 | .0516 | .0496 | .0495 | .0493 | .0494 |
| 0.85 | 0.9 | .0560 | .0563 | .0563 | .0563 | .0520 | .0492 | .0489 | .0488 | .0489 |
| 0.85 | 0.8 | .0602 | .0611 | .0611 | .0611 | .0542 | .0509 | .0509 | .0508 | .0508 |
| 0.81 | 0.9 | .0579 | .0584 | .0584 | .0584 | .0522 | .0501 | .0498 | .0497 | .0497 |
| 0.8 | 0.8 | .0599 | .0613 | .0614 | .0614 | .0556 | .0504 | .0502 | .0502 | .0502 |

TABLE 8 (continued)

$n_1 = 2, \ n_2 = 62$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050030 | .050033 | .050033 | .050033 |
| 1 | 1.1 | .051194 | .051182 | .051162 | .051177 | .050583 | .054262 | .054511 | .054500 | .054509 |
| 1.05 | 1.05 | .050931 | .050991 | .050992 | .050995 | .050382 | .053977 | .054318 | .054330 | .054328 |
| 1 | 1.5 | .0775 | .0769 | .0764 | .0767 | .0660 | .0927 | .0932 | .0928 | .0930 |
| 1.25 | 1.25 | .0714 | .0728 | .0729 | .0729 | .0605 | .0865 | .0894 | .0895 | .0895 |
| 1 | 2 | .1501 | .1483 | .1470 | .1477 | .1217 | .1764 | .1763 | .1750 | .1757 |
| 1.333 | 1.5 | .1080 | .1114 | .1114 | .1114 | .0839 | .1319 | .1377 | .1378 | .1378 |
| 1 | 3 | .340 | .337 | .335 | .336 | .298 | .372 | .371 | .369 | .370 |
| 2.5 | 1.5 | .329 | .335 | .335 | .335 | .280 | .364 | .374 | .374 | .374 |
| 2 | 2 | .327 | .336 | .337 | .337 | .276 | .364 | .376 | .376 | .376 |
| 1.00001 | 0.9 | .051265 | .051265 | .051248 | .051254 | .050577 | .048242 | .047956 | .047926 | .047939 |
| 1.00001 | 0.8 | .055003 | .055030 | .054966 | .054994 | .051995 | .049018 | .048447 | .048348 | .048395 |
| 1.01 | 0.99 | .050011 | .050008 | .050007 | .050008 | .050008 | .050012 | .050008 | .050007 | .050008 |
| 1.1 | 0.9 | .051087 | .050824 | .050746 | .050787 | .050813 | .051176 | .050830 | .050741 | .050788 |
| 1.1 | 0.8 | .053415 | .052946 | .052785 | .052867 | .052110 | .050581 | .049657 | .049454 | .049558 |
| 1.05 | 0.95 | .050272 | .050206 | .050186 | .050197 | .050203 | .050294 | .050207 | .050185 | .050197 |
| 1.2 | 0.99 | .0544 | .0542 | .0542 | .0542 | .0522 | .0602 | .0606 | .0605 | .0606 |
| 1.2 | 0.8 | .0542 | .0532 | .0529 | .0531 | .0532 | .0546 | .0532 | .0529 | .0531 |
| 2 | 0.9 | .140 | .136 | .134 | .135 | .115 | .164 | .161 | .160 | .161 |
| 2 | 0.7 | .127 | .117 | .115 | .116 | .108 | .147 | .137 | .134 | .136 |
| 3 | 0.9 | .32 | .32 | .31 | .31 | .28 | .35 | .35 | .35 | .35 |
| 0.99999 | 0.9 | .051265 | .051265 | .051249 | .051254 | .050577 | .048241 | .047956 | .047926 | .047939 |
| 0.9 | 0.999 | .051279 | .051282 | .051266 | .051271 | .050580 | .048225 | .047940 | .047910 | .047923 |
| 0.9 | 0.9 | .054046 | .054279 | .054284 | .054275 | .051438 | .047944 | .047665 | .047650 | .047652 |
| 0.9 | 0.8 | .0589 | .0594 | .0594 | .0594 | .0523 | .0498 | .0496 | .0496 | .0496 |
| 0.9 | 0.76 | .0594 | .0601 | .0601 | .0601 | .0518 | .0496 | .0494 | .0493 | .0494 |
| 0.85 | 0.9 | .0564 | .0567 | .0567 | .0567 | .0521 | .0487 | .0485 | .0484 | .0484 |
| 0.85 | 0.8 | .0611 | .0620 | .0620 | .0620 | .0546 | .0507 | .0507 | .0506 | .0506 |
| 0.81 | 0.9 | .0584 | .0589 | .0589 | .0589 | .0524 | .0497 | .0494 | .0494 | .0494 |
| 0.8 | 0.8 | .0613 | .0628 | .0628 | .0628 | .0564 | .0505 | .0504 | .0504 | .0504 |

# CHAPTER III

# MONOTONICITY OF THE POWER FUNCTIONS OF TESTS BASED ON TRACES OF MULTIVARIATE COMPLEX BETA MATRIX AND CANONICAL CORRELATION MATRIX

## 1. Introduction

For the MANOVA problem in the real case, Perlman [18] has shown that the power function of the test based on Pillai's trace of a multivariate beta matrix is monotonically increasing in each noncentrality parameter provided that the cutoff point is not too large. This result has also been proved true for the problem of testing independence of two sets of real variates. In this chapter, both of these results are extended to the complex case. An illustrative table is also given, of the smallest error degrees of freedom less the number of variables in MANOVA guaranteeing the above monotonicity property given other sample arguments and significance level.

## 2. Invariant tests for the MANOVA problem

Let $\underset{\sim}{Z}_1(p\times r)$ and $\underset{\sim}{Z}_2(p\times n)$ be independent complex matrix variates. The columns of $\underset{\sim}{Z}_1$ and $\underset{\sim}{Z}_2$ are mutually independent and complex normally distributed with common nonsingular covariance matrix $\underset{\sim}{\Sigma}$, and $E\underset{\sim}{Z}_1 = \underset{\sim}{\theta}$, $E\underset{\sim}{Z}_2 = \underset{\sim}{0}$. The joint density function of $\underset{\sim}{Z}_1$ and $\underset{\sim}{Z}_2$ is given by

$$(2.1)\quad \pi^{-p(r+n)}|\underset{\sim}{\Sigma}|^{-(r+n)}\exp[-\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\{(\underset{\sim}{Z}_1-\underset{\sim}{\theta})(\bar{\underset{\sim}{Z}}_1-\bar{\underset{\sim}{\theta}})'+\underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_2'\}].$$

The problem is to test

$$\underset{\sim}{\theta} = \underset{\sim}{0} \quad \text{against} \quad \underset{\sim}{\theta} \neq \underset{\sim}{0}.$$

This problem is invariant under all transformations of the form:

$$(2.2)\qquad (\underset{\sim}{Z}_1,\underset{\sim}{Z}_2) \longrightarrow (\underset{\sim}{B}\underset{\sim}{Z}_1\underset{\sim}{F}_1,\underset{\sim}{B}\underset{\sim}{Z}_2\underset{\sim}{F}_2),$$

where $\underset{\sim}{B}(p\text{x}p)$ is nonsingular and $\underset{\sim}{F}_1(r\text{x}r)$ and $\underset{\sim}{F}_2(n\text{x}n)$ are unitary. Here, we assume that $p < n + r$. Let $t = \min\{p,r\}$. A maximal invariant statistic is $(\ell_1,\ldots,\ell_t)$, where $1 \geq \ell_1 \geq\ldots\geq \ell_t \geq 0$ are the ordered t largest characteristic roots of the multivariate complex beta matrix $\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1'(\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1'+\underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_2')^{-1}$. An invariant parameter is $(\omega_1,\ldots,\omega_t)$, where $\omega_1 \geq\ldots\geq \omega_t \geq 0$ are the ordered t largest characteristic roots of $\underset{\sim}{\theta}\bar{\underset{\sim}{\theta}}'\underset{\sim}{\Sigma}^{-1}$.

Start with equation (2.1) and put $\underset{\sim}{\theta}\bar{\underset{\sim}{\theta}}' = \binom{\underset{\sim}{\mu}_1}{\underset{\sim}{\mu}_2}\underset{\sim}{D}_{\omega}(t\text{x}t)(\bar{\underset{\sim}{\mu}}_1',\bar{\underset{\sim}{\mu}}_2')$ and $\underset{\sim}{\Sigma}_1 = \begin{pmatrix}\underset{\sim}{\mu}_1 & \underset{\sim}{\mu}_3\\ \underset{\sim}{\mu}_2 & \underset{\sim}{\mu}_4\end{pmatrix}\begin{pmatrix}\bar{\underset{\sim}{\mu}}_1' & \bar{\underset{\sim}{\mu}}_2'\\ \bar{\underset{\sim}{\mu}}_3' & \bar{\underset{\sim}{\mu}}_4'\end{pmatrix} = \underset{\sim}{\mu}\bar{\underset{\sim}{\mu}}'$ where $\underset{\sim}{\mu}_1((p-t)\text{x}t)$, $\underset{\sim}{\mu}_2(t\text{x}t)$, $\underset{\sim}{\mu}_3((p-t)\text{x}(p-t))$ and $\underset{\sim}{\mu}_4(t\text{x}(p-t))$; and $\underset{\sim}{\mu}_2$ and $\underset{\sim}{\mu}_3$ are nonsingular; and $\underset{\sim}{D}_{\omega}$ denotes the diagonal matrix with characteristic roots $\omega_1 \geq\ldots\geq \omega_t$ of $\underset{\sim}{\theta}\bar{\underset{\sim}{\theta}}'\underset{\sim}{\Sigma}^{-1}$ as its diagonal elements. Put $\underset{\sim}{\theta} = \binom{\underset{\sim}{\theta}_1}{\underset{\sim}{\theta}_2} = \binom{\underset{\sim}{\mu}_1}{\underset{\sim}{\mu}_2}\underset{\sim}{D}_{\sqrt{\omega}}\underset{\sim}{\phi}(t\text{x}r)$ where $\underset{\sim}{\phi}$ is determined by $\underset{\sim}{\phi} = \underset{\sim}{D}_{\omega^{-\frac{1}{2}}}\underset{\sim}{\mu}_2^{-1}\underset{\sim}{\theta}_2$ and

$\underset{\sim}{\phi}\bar{\underset{\sim}{\phi}}' = \underset{\sim}{I}$ and complete $\bar{\underset{\sim}{\phi}}'(r\times t)$ into a unitary matrix $\bar{\underset{\sim}{\psi}}'(r\times r)$. Finally transform

$$\underset{\sim}{V} = \underset{\sim}{\mu}^{-1}\underset{\sim}{Z}_1\bar{\underset{\sim}{\psi}}', \quad \underset{\sim}{W} = \underset{\sim}{\mu}^{-1}\underset{\sim}{Z}_2.$$

Then the joint density of $\underset{\sim}{V}$ and $\underset{\sim}{W}$ is

$$\pi^{-p(r+n)}\exp[-\mathrm{tr}(\underset{\sim}{W}\bar{\underset{\sim}{W}}'+\underset{\sim}{V}\bar{\underset{\sim}{V}}'-2\mathrm{Re}\underset{\sim}{V}\underset{\sim}{D}^{**}_{\sqrt{\omega}}+\underset{\sim}{D}^{*}_{\omega})],$$

where $\underset{\sim}{D}^{*}_{\omega}(p\times p) = \begin{pmatrix} \underset{\sim}{D}_{\omega} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{pmatrix}$ and $\underset{\sim}{D}^{**}_{\sqrt{\omega}}\ (r\times p) = \begin{pmatrix} \underset{\sim}{D}_{\sqrt{\omega}} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0}_1 \end{pmatrix}$ and $\underset{\sim}{0}_1$ is $(r-t)\times(p-t)$ zero matrix. Note that the characteristic roots of $\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1'(\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1' + \underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_2')^{-1}$ are the same as those of $\underset{\sim}{V}\bar{\underset{\sim}{V}}'(\underset{\sim}{V}\bar{\underset{\sim}{V}}'+\underset{\sim}{W}\bar{\underset{\sim}{W}}')^{-1}$.

Now, for any region $Q \subseteq C^{p(r+n)}$ invariant under all transformations (2.2), define

$$\varphi_Q(\omega_1,\ldots,\omega_t) = P_{\underset{\sim}{\theta},\underset{\sim}{\Sigma}}\{(\underset{\sim}{Z}_1,\underset{\sim}{Z}_2) \notin Q\} = P_{\underset{\sim}{D}^{**'}_{\sqrt{\omega}},\underset{\sim}{I}}\{(\underset{\sim}{V},\underset{\sim}{W}) \notin Q\},$$

so $\varphi_Q$ is the power function of the test with acceptance region Q. For each $i = 1,\ldots,r$, denote the $v_i$-section of Q for fixed $v_j$, $j \neq i$ and fixed $\underset{\sim}{W}$ by

$$Q^{(i)}(\tilde{\underset{\sim}{V}}_i,\underset{\sim}{W}) = \{v_i|(\underset{\sim}{V},\underset{\sim}{W}) \in Q\} \subseteq C^p,$$

where $\tilde{\underset{\sim}{V}}_i(p\times(r-1)) \equiv (v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_r)$.

Pillai and Li [31] have proved the following theorem. They assumed that $p \leq n$, but their proof is also valid if $p < r + n$.

Theorem 1. Let $Q \subseteq C^{p(r+n)}$ be univariant under all transformations (2.2). Suppose that for each $i = 1,..,r$, $Q^{(i)}(\tilde{\underset{\sim}{V}}_i,\underset{\sim}{W})$ is convex, then

$\varphi_Q(\omega_1,\ldots,\omega_t)$ increases monotonically in each $\omega_j$.

Now consider the following acceptance regions:

(1) Roy's largest root test:

$$Q_1 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \ell_1 \leq k_1\}, \qquad 0 < k_1 < 1,$$

(2) Hotelling's trace test:

$$Q_2 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \sum_{i=1}^{t} \ell_i(1-\ell_i)^{-1} \leq k_2\}, \quad 0 < k_2,$$

(3) Likelihood ratio test:

$$Q_3 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \prod_{i=1}^{t} (1-\ell_i) \geq k_3\}, \qquad 0 < k_3 < 1,$$

(4) Pillai's trace test:

$$Q_4 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \sum_{i=1}^{t} \ell_i \leq k_4\}, \qquad 0 < k_4 < t.$$

The tests are defined for $p \leq n$; but the last will be defined for $p < n + r$ in the sequel. Pillai and Li [31] have shown that $Q_1$, $Q_2$ and $Q_3$ satisfy the conditions of Theorem 1 and hence their power functions are monotonically increasing in each population root. However, the monotonicity property has not yet been established for test $Q_4$ based on the trace statistic $\sum_{i=1}^{t} \ell_i$. In this chapter, we show that $\varphi_{Q_4}(\omega_1,\ldots,\omega_t)$ is monotonically increasing in each $\omega_j$ provided that the cutoff point $k_4$ is not too large. In order to show this, we need the following lemma:

Lemma 1. Let $\xi' = (z_1,\ldots,z_p)$ and $\eta' = (x_1,y_1,\ldots,x_p,y_p)$ where $z_j = x_j + iy_j$, $j = 1,\ldots,p$, and let T be a one-one transformation between $\xi$ and $\eta$ such that $T(\xi) = \eta$ with the following properties:

(1) $T(\xi_1 + \xi_2) + T(\xi_1) + T(\xi_2)$ and

(2) $T(a\xi) = aT(\xi)$ where a is a real number. Let Q be a subset of $\xi$'s in p-dimensional complex sample $C^p$, and Q* be its corresponding subset of $\eta$'s in the 2p-dimensional real sample space $R^{2p}$. If Q is convex in $C^p$ and symmetric in $\xi$, then Q* is convex in $R^{2p}$ and symmetric in $\eta$ and conversely.
The proof is given by Pillai and Li [31].

For $0 < \alpha < 1$ and $p < r + n$, define $k_4(\alpha,p,r,n)$ to be the size $\alpha$ cutoff point, i.e.

$$P(\sum_{i=1}^{t} \ell_i > k_4(\alpha,p,r,n)|\underset{\sim}{\theta} = \underset{\sim}{0}) = \alpha.$$

<u>Theorem 2</u>. The invariant acceptance region $Q_4$ satisfies the conditions of Theorem 1 if and only if

$$k_4(\alpha,p,r,n) \leq \max\{1,p-n\}.$$

<u>Proof</u>. $Q_4 = \{(\underset{\sim}{V},\underset{\sim}{W}) | \sum_{i=1}^{t} \ell_i \leq k_4\}$

$$= \{(\underset{\sim}{V},\underset{\sim}{W}) | tr[\underset{\sim}{V}\underset{\sim}{\bar{V}}'(\underset{\sim}{V}\underset{\sim}{\bar{V}}'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1}] \leq k_4\},$$

where $0 < k_4 < t$. Since $Q_4$ is symmetric in the columns $v_1,\ldots,v_r$ of $\underset{\sim}{V}$, it suffices to prove that $Q_4^{(1)}(\underset{\sim}{\tilde{V}}_1,\underset{\sim}{W})$ is convex for almost all $(\underset{\sim}{\tilde{V}}_1,\underset{\sim}{W})$ if and only if $k_4 \leq \max\{1,p-n\}$. Now

$$tr[\underset{\sim}{V}\underset{\sim}{\bar{V}}'(\underset{\sim}{V}\underset{\sim}{\bar{V}}'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1}] = p-tr[\underset{\sim}{W}\underset{\sim}{\bar{W}}'(\underset{\sim}{V}\underset{\sim}{\bar{V}}'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1}]$$

and $(\underset{\sim}{V}\underset{\sim}{\bar{V}}'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1} = (v_1\bar{v}_1'+\underset{\sim}{\tilde{V}}_1\underset{\sim}{\bar{\tilde{V}}}_1'+\underset{\sim}{W}\underset{\sim}{\bar{W}}')^{-1} = \underset{\sim}{U}^{-1} - \dfrac{\underset{\sim}{U}^{-1}v_1\bar{v}_1'\underset{\sim}{U}^{-1}}{1+\bar{v}_1'\underset{\sim}{U}^{-1}v_1},$

where $\underset{\sim}{U} = \tilde{\underset{\sim}{V}}_1\bar{\tilde{\underset{\sim}{V}}}_1' + \underset{\sim}{W}\bar{\underset{\sim}{W}}'$ is nonsingular for almost all $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ since $p \leq n+r-1$.

Hence, excluding a null set of $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ values,

$$Q_4^{(1)}(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \{v_1 \left| \frac{\bar{v}_1'\underset{\sim}{U}^{-1}\underset{\sim}{W}\bar{\underset{\sim}{W}}'\underset{\sim}{U}^{-1}v_1}{1+\bar{v}_1'\underset{\sim}{U}^{-1}v_1} \leq \text{tr } \underset{\sim}{W}\bar{\underset{\sim}{W}}'\underset{\sim}{U}^{-1}-p+k_4\right.\}.$$

Let $\underset{\sim}{\Lambda} = \underset{\sim}{U}^{-\frac{1}{2}}\,\underset{\sim}{W}\bar{\underset{\sim}{W}}'\underset{\sim}{U}^{-\frac{1}{2}}$, let $y = \underset{\sim}{U}^{-\frac{1}{2}}\,v_1$, and define the region $M = M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) \subseteq C^p$ by

$$M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \{y \left| \frac{\bar{y}'\Lambda\, y}{1+\bar{y}'y} \leq \text{tr}\Lambda-p+k_4\right.\}.$$

Thus $Q_4^{(1)}(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \underset{\sim}{U}^{\frac{1}{2}}[M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})]$,

where $\underset{\sim}{U}^{\frac{1}{2}}[M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})] = \{\underset{\sim}{U}^{\frac{1}{2}}y \mid y \in M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})\}$. Choose $\underset{\sim}{\psi}$ to be a pxp unitary matrix such that

$$\underset{\sim}{\Lambda} = \underset{\sim}{\psi}\underset{\sim}{D}_{\lambda_i}(pxp)\bar{\underset{\sim}{\psi}}',$$

where $\lambda_1 \geq \cdots \geq \lambda_p$ are the ordered characteristic roots of $\underset{\sim}{\Lambda}$. Let $z = \bar{\underset{\sim}{\psi}}'y = (z_1,\ldots,z_p)'$ and define the region $H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) \subseteq C^p$ by

$$\text{(2.3)} \quad H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \{z \mid \sum_{j=1}^{p} z_j\bar{z}_j(\lambda_j - \sum_{m=1}^{p}\lambda_m + p - k_4) \leq \sum_{m=1}^{p}\lambda_m - p + k_4\}.$$

Then $M(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \underset{\sim}{\psi}[H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})]$, so that except for a null set of $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ values,

$$\text{(2.4)} \qquad Q_4^{(1)}(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \underset{\sim}{U}^{\frac{1}{2}}\underset{\sim}{\psi}[H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})].$$

Now assume that $k_4 \leq \max\{1,p-n\}$. In view of (2.4) and the linearity of the operator $\underset{\sim}{U}^{\frac{1}{2}}\underset{\sim}{\psi}$, to verify that $Q_4$ satisfies the conditions of Theorem 1, it suffices to show that $H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ is convex for all $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$. Now define the region $H^* \subseteq R^{2p}$ by

$$(2.5)\quad H^* = \{(x_1,y_1,\ldots,x_p,y_p)' \mid \sum_{j=1}^{p}(x_j^2+y_j^2)(\lambda_j-\sum_{m=1}^{p}\lambda_m+p-k_4)\leq\sum_{m=1}^{p}\lambda_m-p+k_4\},$$

where $x_j+iy_j=z_j$, $j = 1,\ldots,p$.

Since $\underset{\sim}{\Lambda}$ and $\underset{\sim}{I}-\underset{\sim}{\Lambda}$ are positive semi-definite and and rank $(\underset{\sim}{\Lambda})$ = rank $(\underset{\sim}{W}) \leq \min\{p,n\} \equiv s$, say, we have that

$$1 \geq \lambda_1 \geq\ldots\geq \lambda_s \geq 0 = \lambda_{s+1} =\ldots= \lambda_p.$$

Hence for each $j = 1,\ldots,p$,

$$\begin{aligned}\lambda_j-\sum_{m=1}^{p}\lambda_m+p-k_4 &\geq -\min\{p-1,s\} + p-k_4\\ &= -\min\{p-1,n\} + p-k_4\\ &= \max\{1,p-n\} - k_4\\ &\geq 0.\end{aligned}$$

Therefore from (2.5), $H^*$ is an ellipsoid (possibly degenerate or empty) in $R^{2p}$ and hence is convex in $R^{2p}$. By Lemma 1, we have that $H(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ is convex for all $(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W})$ in $C^p$. So $Q_4$ satisfies the conditions of Theorem 1.

Conversely, suppose that $k_4 > \max\{1,p-n\}$. Since $k_4 < t = \min\{p,r\}$, this requires that $r > 1$. Let $\delta = \delta(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \sum_{m=1}^{p}\lambda_m-p+k_4$ and $\beta_j=\beta_j(\tilde{\underset{\sim}{V}}_1,\underset{\sim}{W}) = \lambda_j-\delta$, so that

$$H^* = \{(x_1,y_1,\ldots,x_p,y_p)' \mid \sum_{j=1}^{p}(x_j^2+y_j^2)\beta_j \leq \delta\}.$$

We shall show later that there exists $(\tilde{V}_1, W)$ such that

$$(2.6) \qquad \beta_1(\tilde{V}_1, W) > 0 > \beta_p(\tilde{V}_1, W).$$

Since $\beta_j(\tilde{V}_1, W)$ is a continuous function of $(\tilde{V}_1, W)$, there must exist an open set $\Delta \subseteq C^{p(n+r-1)}$ such that (2.6) holds for all $(\tilde{V}_1, W) \in \Delta$. Thus H* fails to be a convex set whenever $(\tilde{V}_1, W) \in \Delta$, which is a non-null set, so by Lemma 1, $H(\tilde{V}_1, W)$ also fails to be a convex set whenever $(\tilde{V}_1, W) \in \Delta$, so $Q_4$ cannot satisfy the conditions of Theorem 1. Back to the existence of $(\tilde{V}_1, W)$ satisfying (2.6), which can be rewritten as

$$(2.7) \qquad \sum_{j=2}^{p} \lambda_j < p-k_4 < \sum_{j=1}^{p-1} \lambda_j.$$

By assumption, $\max\{1,p-n\} < k_4 < t = \min\{p,r\}$, or

$$(2.8) \qquad \max\{0,p-r\} < p-k_4 < \min\{p-1,n\}$$

Case (i). $p \leq n$, $p < r$.

Choose $(\tilde{V}_1, W)$ such that $W\bar{W}' = I$ and $\tilde{V}_1\bar{\tilde{V}}_1' = D_{d_i}$, where $0 \leq d_1 \leq \cdots \leq d_p$ are defined below. For such $(\tilde{V}_1, W)$ we have

$$(\lambda_1, \ldots, \lambda_p) = ((1+d_1)^{-1}, \ldots, (1+d_p)^{-1}),$$

so (2.7) becomes

$$(2.9) \qquad \sum_{j=2}^{p} (1+d_j)^{-1} < p-k_4 < \sum_{j=1}^{p-1} (1+d_j)^{-1}.$$

Also (2.8) reduces to $0 < p-k_4 < p-1$, so $0 \leq a \leq p-2$ where $a \equiv [p-k_4]$. If $a = 0$ choose the $d_j$ such that $(1+d_1)^{-1} = 1$ and $(1+d_j)^{-1} < (p-k_4)/(p-1)$ for $2 \leq j \leq p$. If $1 \leq a \leq p-2$ select

$\theta, \varepsilon$ such that $(p-k_4) - a < \theta < 1$ and $0 < \varepsilon < 1-\theta$, and choose the $d_j$ such that $(1+d_1)^{-1} = \ldots = (1+d_a)^{-1} = 1, (1+d_{a+1})^{-1} = \theta$, and $\sum_{j=a+2}^{p} (1+d_j)^{-1} = \varepsilon$. Then it is easy to see that for all $0 \leq a \leq p-2$, (2.9) is satisfied. Similarly for the other cases (ii) $p \leq n$, $p \geq r$, (iii) $p > n$, $p < r$, and (iv) $p > n$, $p \geq r$.

Since $k_4(\alpha,p,r,n)$ is decreasing in $\alpha$ and n, while increasing in p,r. The power function of Pillai's trace test for the MANOVA problem has the monotonicity property with respective to each population root provided that $\alpha$ and n are not too small and p and r are not too large.

Approximate values of $k_4(\alpha,p,m+p,s+p)$, where $m = r-p$, $s = n-p$, has been tabulated by Krishnaiah and Schuurmann [15]. In Table 1 and 2, we give the values of $S^*(\alpha,p,m)$ for $\alpha = 0.05$ and $\alpha = 0.01$, where $S^*(\alpha,p,m)$ is the smallest value of $s \geq 0$ such that $k_4(\alpha,p,m+p,s+p) \leq 1$. Thus, by Theorem 2, Pillai's trace test has the monotonicity property if $s \geq S^*(\alpha,p,m)$.

TABLE 1

Values of $S^*(0.05,p,m)$

| p \ m | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| 2 | 6 | 9 | 12 | 14 | 17 |
| 4 | 22 | 29 | 36 | 42 | 49 |
| 6 | 45 | 56 | 69 | 79 | 90 |
| 8 | 77 | 92 | 109 | 126 | 140 |
| 10 | 119 | 138 | 155 | 177 | 194 |

TABLE 2

Values of S*(0.01,p,m)

| p \ m | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| 2 | 8 | 11 | 14 | 17 | 20 |
| 4 | 25 | 33 | 40 | 48 | 55 |
| 6 | 50 | 62 | 74 | 85 | 97 |
| 8 | 83 | 99 | 118 | 134 | 147 |
| 10 | 127 | 145 | 165 | 185 | 203 |

## 3. Invariant tests for independence of variates

Let $\underset{\sim}{Z}((p_1+p_2)xm)$ be a complex random matrix whose columns are independent and complex normally distributed with mean $\underset{\sim}{0}$ and common nonsingular hermitian covariance matrix $\underset{\sim}{\Sigma}$. Let

$$\underset{\sim}{Z} = \begin{pmatrix} \underset{\sim}{Z}_1 \\ \underset{\sim}{Z}_2 \end{pmatrix} \text{ and } \underset{\sim}{\Sigma} = \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{pmatrix},$$

where $\underset{\sim}{Z}_1(p_1xm)$, $\underset{\sim}{Z}_2(p_2xm)$, $\underset{\sim}{\Sigma}_{11}(p_1xp_1)$, $\underset{\sim}{\Sigma}_{21}(p_2xp_1)$, $\underset{\sim}{\Sigma}_{22}(p_2xp_2)$.

The problem of testing independence of two sets of variates is to test

$$\underset{\sim}{\Sigma}_{12} = \underset{\sim}{0} \quad \text{against } \underset{\sim}{\Sigma}_{12} \neq \underset{\sim}{0}.$$

This problem is invariant under all transformations of the form:

$$\begin{pmatrix} \underset{\sim}{Z}_1 \\ \underset{\sim}{Z}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \underset{\sim}{B}_1 & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{B}_2 \end{pmatrix} \begin{pmatrix} \underset{\sim}{Z}_1 \\ \underset{\sim}{Z}_2 \end{pmatrix} \underset{\sim}{F},$$

where $\underset{\sim}{B}_1(p_1xp_1)$ and $\underset{\sim}{B}_2(p_2xp_2)$ are nonsingular and $\underset{\sim}{F}(mxm)$ is unitary.

A maximal invariant statistic is $(r_1^2,\ldots,r_t^2)$, where $t = \min\{p_1,p_2\}$ and $1 \geq r_1^2 \geq\cdots\geq r_t^2 \geq 0$ are the ordered t largest characteristic roots of

$$(\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_1')^{-1}(\underset{\sim}{Z}_1\bar{\underset{\sim}{Z}}_2')(\underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_2')^{-1}(\underset{\sim}{Z}_2\bar{\underset{\sim}{Z}}_1') = \underset{\sim}{S}_{11}^{-1}\ \underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{21},$$

where $\underset{\sim}{S} = \underset{\sim}{Z}\bar{\underset{\sim}{Z}}'$ is partitioned as $\underset{\sim}{\Sigma}$. Here we assume that $\max\{p_1,p_2\}<m$ to ensure the nonsingularity of $\underset{\sim}{S}_{11}$ and $\underset{\sim}{S}_{22}$. An invariant parameter is the vector of canonical correlations $(\rho_1,\ldots,\rho_t)$, where $\rho_1^2 \geq\cdots\geq \rho_t^2 \geq 0$ are the ordered t largest characteristic roots of $\underset{\sim}{\Sigma}_{11}^{-1}\ \underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}_{22}^{-1}\underset{\sim}{\Sigma}_{21}$. The power function of any invariant test is a function of $(\rho_1,\ldots,\rho_t)$.

The conditional distribution given $\underset{\sim}{S}_{22}$ of the matrix

$$\underset{\sim}{S}_{11}^{-1}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{21} = (\underset{\sim}{S}_{11.2}+\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{21})^{-1}\underset{\sim}{S}_{12}\underset{\sim}{S}_{22}^{-1}\underset{\sim}{S}_{21}$$

is of the same form as the distribution of the matrix $(\underset{\sim}{V}\bar{\underset{\sim}{V}}'+\underset{\sim}{W}\bar{\underset{\sim}{W}}')^{-1}\underset{\sim}{V}\bar{\underset{\sim}{V}}'$ in the MANOVA problem if we take $(p,r,n) = (p_1,p_2,m-p_2)$, and their unconditional null distributions are the same. (See Pillai and Li [31]). Therefore, we have the following:

Theorem 3. If the power function of an invariant test for the MANOVA problem, which accepts the hypothesis $\underset{\sim}{\theta} = \underset{\sim}{0}$ if and only if $(\ell_1,\ldots,\ell_t) \in Q$, increases monotonically in each noncentrality parameter $\omega_j$, then the power function of the invariant test for the independence problem, which accepts the hypothesis $\underset{\sim}{\Sigma}_{12} = \underset{\sim}{0}$ if and only if $(r_1^2,\ldots,r_t^2) \in Q$, increases monotonically in each canonical correlation $\rho_j$.

For $p_1+p_2 \leq m$, Pillai and Li [31] have shown that the power function of Roy's largest root test based on $r_1^2$, Hotelling's trace test based on $\sum_{i=1}^{t} r_i^2/(1-r_i^2)^{-1}$, and the likelihood ratio test based on $\prod_{i=1}^{t}(1-r_i^2)$ all have the monotonicity property.

Now consider Pillai's trace that for testing the independence. This test accepts $\underset{\sim}{\Sigma}_{12} = \underset{\sim}{0}$ if and only if

$$\sum_{i=1}^{t} r_i^2 \leq k_4(\alpha, p_1, p_2, m-p_2),$$

where $k_4$ was defined in section 2.

Theorem 1, 2 and 3 imply the following:

<u>Theorem 4</u>. The power function of Pillai's trace test for independence increases in each canonical correlation $\rho_j$ if $k_4(\alpha, p_1, p_2, m-p_2) \leq \max\{1, p_1+p_2-m\}$.

Since $k_4(\alpha, p_1, p_2, m-p_2)$ is decreasing in $\alpha$ and m, while increasing in $p_1$ and $p_2$. The power function of Pillai's trace test for the independence problem has the monotonicity property provided that $\alpha$ and m are not too small and $p_1$ and $p_2$ are not too large. Table 1, 2 can also be used in this case.

# CHAPTER IV

# ON THE EXACT DISTRIBUTION OF WILKS' CRITERION

## 1. Introduction

The joint distribution of p non-null characteristic roots of a matrix in multivariate analysis given by Fisher [5], Hsu [7] and Roy [35] can be expressed in the form:

$$(1.1)\quad C(p,m,n)\prod_{i=1}^{p} b_i^n(1-b_i)^m \prod_{i<j}(b_i-b_j), \qquad 0 < b_p \leq \cdots \leq b_1 < 1,$$

where $C(p,m,n) = \{\pi^{\frac{1}{2}p}\prod_{i=1}^{p}\Gamma[\frac{1}{2}(2m+2n+p+i+2)]\}$

$$/ \{\prod_{i=1}^{p}\Gamma[\tfrac{1}{2}(2m+i+1)]\Gamma[\tfrac{1}{2}(2n+i+1)]\Gamma(\tfrac{1}{2}\, i)\},$$

and the parameters m and n are defined differently for various situations as described by Pillai [20, 21]. Now Wilks' criterion $W^{(p)}$ may be defined as

$$(1.2)\qquad W^{(p)} = \prod_{i=1}^{p} b_i .$$

The moments of $W^{(p)}$ are obtained readily from (1.1) and (1.2) since

$$E[(W^{(p)})^h] = E[\prod_{i=1}^{p} b_i^h]$$

$$= C(p,m,n)\int_{R^*}\cdots\int \prod_{i=1}^{p} b_i^{n+h}(1-b_i)^m \prod_{i<j}(b_i-b_j)\prod_{i=1}^{p} db_i,$$

where $R^* = \{(b_1,\ldots,b_p)|0 < b_p \leq\ldots\leq b_1 < 1\}$. Thus

(1.3) $$E[(W^{(p)})^h] = C(p,m,n)/C(p,m,n+h).$$

Using (1.3), Wilks [43] showed that $W^{(p)}$ can be expressed as a product of p independent beta variables and obtained the density of $W^{(2)}$, [44]. Using the same idea, Pillai and Gupta [26] have found the exact distribution of $-\log W^{(p)}$ for p = 3 to 6 by successive convolutions. Schatzoff [39] has used the convolution method to compute exact percentage points for p up to 10 and values of n such that $p(2n+p+1) \leq 70$ through computer algorithm, but has not given the distribution explicitly for any p. Lee [16] has also obtained the exact distribution of $(W^{(p)})^{\frac{1}{2}}$ for odd p in the form of an integral which could not be evaluated except numerically. In this chapter, an attempt is made to obtain the exact distribution of $W^{(p)}$ by variable transformation. Explicit expressions for the distributions are given here for p = 3 and 4.

## 2. Method of derivation of the density of $W^{(p)}$

Consider the transformation

$$q_1=b_1, q_i=b_i/b_{i-1},\ i=2,\ldots,p-2,\ q_{p-1}=b_{p-1}/b_{p-2},\ q_p=b_p/b_{p-2}.$$

Then from (1.1) the joint density of $q_1,\ldots,q_p$ is given by

$$(2.1)\quad C(p,m,n)\Big[\prod_{j=1}^{p-2} q_j^{(p-j+1)n+(p-j)+\frac{1}{2}(p-j+1)(p-j)}\Big(1-\prod_{k=1}^{j} q_k\Big)^m\Big](q_{p-1}q_p)^n$$

$$\times\Big\{\Big[1-\Big(\prod_{j=1}^{p-2} q_j\Big)q_{p-1}\Big]\Big[1-\Big(\prod_{j=1}^{p-2} q_j\Big)q_p\Big]\Big\}^m\Big[\prod_{j=2}^{p-2}\prod_{i=2}^{p-j}\Big(1-\prod_{k=i}^{i+j-2} q_k\Big)\Big]$$

$$\times\Big\{\prod_{j=2}^{p-2}\Big[1-\Big(\prod_{i=j}^{p-2} q_i\Big)(q_{p-1}+q_p)+\Big(\prod_{i=j}^{p-2} q_i^2\Big)q_{p-1}q_p\Big]\Big\}(1-q_{p-1})(1-q_p)$$

$$(q_{p-1}-q_p),$$

where $0 < q_i < 1$, $i = 1,\dots,p-2$, $0 < q_p \le q_{p-1} < 1$. Further, transform $t = (1-q_{p-1})(1-q_p)$, $g = q_{p-1}q_p$. Then the joint density of $q_1,\dots,q_{p-2}$, $g$, $t$ is obtained as

$$(2.2)\quad C(p,m,n)\Big[\prod_{j=1}^{p-2} q_j^{(p-j+1)n+(p-j)+\frac{1}{2}(p-j+1)(p-j)}\Big(1-\prod_{k=1}^{j} q_k\Big)^m\Big]g^n$$

$$\times\,[(1-a_1)(1-a_1g)+a_1t]^m\Big[\prod_{j=2}^{p-2}\prod_{i=2}^{p-j}\Big(1-\prod_{k=i}^{i+j-2} q_k\Big)\Big]$$

$$\times\Big\{\prod_{j=2}^{p-2}[(1-a_j)(1-a_jg)+a_jt]\Big\}t,$$

where $0 < q_i < 1$, $i = 1,\dots,p-2$, $g^{\frac{1}{2}}+t^{\frac{1}{2}} \le 1$ and $a_j = \prod_{i=j}^{p-2} q_i$, $j = 1,\dots,p-2$. Now integrate out $t$ to get the joint density of $q_1,\dots,q_{p-2}$, $g$ as follows:

$$(2.3)\quad C(p,m,n)\Big[\prod_{j=1}^{p-2} q_j^{(p-j+1)n+(p-j)+\frac{1}{2}(p-j+1)(p-j)}\Big(1-\prod_{k=1}^{j} q_k\Big)^m\Big]g^n$$

$$\times\Big[\prod_{j=2}^{p-2}\prod_{i=2}^{p-j}\Big(1-\prod_{k=i}^{i+j-2} q_k\Big)\Big]\times\Big[\sum_{h=0}^{m}\sum_{\ell=0}^{p-3}(h+\ell+2)^{-1}d_h c_\ell (1-g^{\frac{1}{2}})^{2(h+\ell+2)}\Big],$$

where $d_h = \binom{m}{h}[(1-a_1)(1-a_1 g)]^{m-h} a_1^h$, $h = 0,\ldots,m$,

$$c_\ell = \sum_{\substack{j_1=2 \\ j_x>j_y \text{ if } x>y}}^{p-2} \cdots \sum_{j_\ell=2}^{p-2} \left(\prod_{i=1}^{\ell} a_{j_i}\right)\left[\prod_{\substack{k=2 \\ k\neq j_1,\ldots,j_\ell}}^{p-2} (1-a_k)(1-a_k g)\right], \quad \ell = 0,\ldots,p-3.$$

Further, transform $w = \left(\prod_{j=1}^{p-2} q_j^{p-j+1}\right) g$ which is in fact Wilks' criterion $W^{(p)}$. Then the joint density of $q_1,\ldots,q_{p-2}$, $w$ is given by

$$(2.4)\quad C(p,m,n)\left[\prod_{j=1}^{p-2} q_j^{\frac{1}{2}(p-j+1)(p-j)-1}\left(1-\prod_{k=1}^{j} q_k\right)^m\right] w^n \left[\prod_{j=2}^{p-2}\prod_{i=2}^{p-j}\left(1-\prod_{k=i}^{i+j-2} q_k\right)\right]$$

$$\times \left[\sum_{h=0}^{m}\sum_{\ell=0}^{p-3}(h+\ell+2)^{-1} d_h c_\ell \left(1-w^{\frac{1}{2}}\prod_{j=1}^{p-2} q_j^{-\frac{1}{2}(p-j+1)}\right)^{2(h+\ell+2)}\right],$$

where $d_h = \binom{m}{h}\left[(1-a_1)\left(1-a_1 w\prod_{j=1}^{p-2} q_j^{-p+j-1}\right)\right]^{m-h} a_1^h$, $h = 0,\ldots,m$,

$$c_\ell = \sum_{\substack{j_1=2 \\ j_x>j_y \text{ if } x>y}}^{p-2} \cdots \sum_{j_\ell=2}^{p-2}\left(\prod_{i=1}^{\ell} a_{j_i}\right)\left[\prod_{\substack{k=2 \\ k\neq j_1,\ldots,j}}^{p-2}(1-a_k)\left(1-a_k w\prod_{j=1}^{p-2} q_j^{-p+j-1}\right)\right], \quad \ell = 0,\ldots,p-3,$$

and $w^{\frac{1}{3}}\prod_{j=1}^{p-3} q_j^{-\frac{1}{3}(p-j+1)} < q_{p-2} < 1$,

$w^{\frac{1}{4}}\prod_{j=1}^{p-4} q_j^{-\frac{1}{4}(p-j+1)} < q_{p-3} < 1$,

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$w^{\frac{1}{p-1}} q_1^{-\frac{p}{p-1}} < q_2 < 1$,

$w^{\frac{1}{p}} < q_1 < 1$,

$0 < w < 1$,

where integration is carried out in the order $q_{p-2}, q_{p-3}, \ldots, q_2, q_1$. From (2.4), the density function of Wilks' criterion $W^{(p)}$ can be obtained by integrating out $q_{p-2}, q_{p-3}, \ldots, q_2, q_1$ successively. We may illustrate the method by considering p = 3 and 4.

## 3. Exact distribution of $W^{(3)}$

Putting p = 3 in (2.2), we get the joint density of $q_1, g, t$ as

$$C(3,m,n)q_1^{3n+5}(1-q_1)^m g^n[(1-q_1)(1-q_1g)+q_1t]^m t,$$

where $g^{\frac{1}{2}} + t^{\frac{1}{2}} \leq 1$. Now integrate out t to get the joint density of $q_1, g$:

$$C(3,m,n)\sum_{i=0}^{m}\frac{\binom{m}{i}}{(m-i+2)}q_1^{3n+m+5-i}(1-q_1)^{m+i}g^n(1-q_1g)^i(1-g^{\frac{1}{2}})^{2m-2i+4}.$$

Further, transform $w = q_1^3 g$, the joint density of $q_1$, w becomes

$$C(3,m,n)\sum{}^{*}K^{*}w^{n+k+\frac{1}{2}\ell}q_1^{m+2-i+j-2k-\frac{3}{2}\ell},$$

where $0 < w < q_1^3 < 1$,

$$\sum{}^{*}=\sum_{i=0}^{m}\sum_{j=0}^{m+i}\sum_{k=0}^{i}\sum_{\ell=0}^{2m-2i+4}, \quad K^{*}=\frac{1}{m-i+2}\binom{m}{i}\binom{m+i}{j}\binom{i}{k}\binom{2m-2i+4}{\ell}(-1)^{j+k+\ell}.$$

Now integrate out $q_1$, we get the density of $W^{(3)}$ as follows:

$$(3.1)\quad C(3,m,n)\Big[\sum_{a\neq 0}{}^{*}K^{*}a^{-1}w^{n+k+\frac{1}{2}\ell}(1-w^{\frac{1}{3}a})-\sum_{a=0}{}^{*}K^{*}\frac{1}{3}w^{n+k+\frac{1}{2}\ell}\ln w\Big],$$

where $a = m+3-i+j-2k-\frac{3}{2}\ell$.

<u>m = 0</u>: Put m = 0 in (3.1) and we have the density

(3.2) $C(3,0,n)[\frac{1}{6} w^n - \frac{4}{3} w^{n+\frac{1}{2}} + \frac{4}{3} w^{n+\frac{3}{2}} - \frac{1}{6} w^{n+2} - w^{n+1} \ln w]$,

and the exact cdf

(3.3) $$P(W^{(3)} < w) = C(3,0,n)[\frac{1}{6(n+1)} w^{n+1} - \frac{8}{3(2n+3)} w^{n+\frac{3}{2}} + \frac{8}{3(2n+5)} w^{n+\frac{5}{2}} - \frac{1}{6(n+3)} w^{n+3} - \frac{1}{(n+2)} w^{n+2} \ln w + \frac{1}{(n+2)^2} w^{n+2}].$$

m = 1: Put m = 1 in (3.1) and we have the density

(3.4) $$\frac{1}{2} C(3,1,n)[\frac{1}{15} w^n - \frac{16}{15} w^{n+\frac{1}{2}} - \frac{13}{3} w^{n+1} + \frac{32}{3} w^{n+\frac{3}{2}} - \frac{13}{3} w^{n+2} - \frac{16}{15} w^{n+\frac{5}{2}} + \frac{1}{15} w^{n+3} - 2w^{n+1} \ln w + 2w^{n+2} \ln w],$$

and the exact cdf

(3.5) $$P(W^{(3)} < w) = \frac{1}{2} C(3,1,n)[\frac{1}{15(n+1)} w^{n+1} - \frac{32}{15(2n+3)} w^{n+\frac{3}{2}} - \frac{13}{3(n+2)} w^{n+2} + \frac{64}{3(2n+5)} w^{n+\frac{5}{2}} - \frac{13}{3(n+3)} w^{n+3} - \frac{32}{15(2n+7)} w^{n+\frac{7}{2}} + \frac{1}{15(n+4)} w^{n+4} - \frac{2}{n+2} w^{n+2} \ln w + \frac{2}{(n+2)^2} w^{n+2} + \frac{2}{n+3} w^{n+3} \ln w - \frac{2}{(n+3)^2} w^{n+3}].$$

m = 2: Put m = 2 in (3.1) and we have the density

(3.6) $$\frac{1}{3} C(3,2,n)[\frac{1}{28} w^n - \frac{32}{35} w^{n+\frac{1}{2}} - \frac{51}{5} w^{n+1} + 32w^{n+\frac{3}{2}} - 32w^{n+\frac{5}{2}} + \frac{51}{5} w^{n+3} + \frac{32}{35} w^{n+\frac{7}{2}} - \frac{1}{28} w^{n+4} - 3w^{n+1} \ln w + 15w^{n+2} \ln w - 3w^{n+3} \ln w],$$

and the exact cdf

$$(3.7)\quad P(W^{(3)}<w) = \frac{1}{3}\, C(3,2,n)\Big[\frac{1}{28(n+1)}\, w^{n+1} - \frac{64}{35(2n+3)}\, w^{n+\frac{3}{2}} - \frac{51}{5(n+2)}\, w^{n+2}$$

$$+ \frac{64}{2n+5}\, w^{n+\frac{5}{2}} - \frac{64}{2n+7}\, w^{n+\frac{7}{2}} + \frac{51}{5(n+4)}\, w^{n+4}$$

$$+ \frac{64}{35(2n+9)}\, w^{n+\frac{9}{2}} - \frac{1}{28(n+5)}\, w^{n+5}$$

$$- \frac{3}{n+2}\, w^{n+2} \ln w + \frac{3}{(n+2)^2}\, w^{n+2} - \frac{15}{n+3}\, w^{n+3} \ln w$$

$$- \frac{15}{(n+3)^2}\, w^{n+3} - \frac{3}{n+4}\, w^{n+4} \ln w + \frac{3}{(n+4)^2}\, w^{n+4}\Big].$$

## 4. Exact distribution of $W^{(4)}$

Putting $p = 4$ in (2.2), we get the joint density of $q_1, q_2$, $g, t$ as

$$C(4,m,n) q_1^{4n+9} q_2^{3n+5} (1-q_1)^m (1-q_1q_2)^m g^n [(1-q_1q_2)(1-q_1q_2g)+q_1q_2t]^m$$

$$\cdot\ (1-q_2)[(1-q_2)(1-q_2g)+q_2t]t,$$

where $g^{\frac{1}{2}} + t^{\frac{1}{2}} \le 1$. Now integrate out $t$ to get the joint density of $q_1, q_2, g$:

$$C(4,m,n)\Big[\sum_{i=0}^{m} \frac{\binom{m}{i}}{(m-i+2)}\, q_1^{4n+9+m-i} q_2^{3n+5+m-i} g^n (1-q_1)^m (1-q_2)^2 (1-q_1q_2)^{m+i}$$

$$\times\ (1-q_1q_2g)^i (1-q_2g)(1-g^{\frac{1}{2}})^{2m-2i+4}$$

$$+ \sum_{i=0}^{m} \frac{\binom{m}{i}}{(m-i+3)}\, q_1^{4n+9+m-i} q_2^{3n+6+m-i} g^n (1-q_1)^m (1-q_2)(1-q_1q_2)^{m+i}$$

$$\times\ (1-q_1q_2g)^i (1-g^{\frac{1}{2}})^{2m-2i+6}\Big].$$

Further, transform $w = q_1^4 q_2^3 g$, the joint density of $q_1$, $q_2$, $w$ becomes

$$C(4,m,n)[\sum{}^{*}K^{*}w^{c}q_1^{a+5}q_2^{b+2}(1-q_2)^2(1-wq_1^{-4}q_2^{-2}) + \sum{}^{**}K^{**}w^{c}q_1^{a+5}q_2^{b+3}(1-q_2)],$$

where $w^{\frac{1}{3}} q_1^{-\frac{4}{3}} < q_2 < 1$, $w^{\frac{1}{4}} < q_1 < 1$, $0 < w < 1$,

$$\sum{}^{*} = \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m+i}\sum_{\ell=0}^{i}\sum_{h=0}^{2m-2i+4}, \quad K^{*} = \frac{1}{(m-i+2)}\binom{m}{i}\binom{m}{j}\binom{m+i}{k}\binom{i}{\ell}\binom{2m-2i+4}{h}(-1)^{j+k+\ell+h},$$

$$\sum{}^{**} = \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m+i}\sum_{\ell=0}^{i}\sum_{h=0}^{2m-2i+6}, \quad K^{**} = \frac{1}{(m-i+3)}\binom{m}{i}\binom{m}{j}\binom{m+i}{k}\binom{i}{\ell}\binom{2m-2i+6}{h}(-1)^{j+k+\ell+h},$$

$a = m-i+j+k-3\ell-2h$, $b = m-i+k-2\ell-\frac{3}{2}h$, $c = n+\ell+\frac{1}{2}h$.

Now integrate out $q_2$ to get the joint density of $q_1$,$w$:

$$C(4,m,n)[\sum{}_1 w^{c}q_1^{a+5} + \sum{}_2 w^{c}(\ell n\, w)q_1^{a+5} - \sum{}_2 4w^{c}q_1^{a+5}(\ell n\, q_1)$$
$$+ \sum{}_3 w^{c+1}q_1^{a+1} + \sum{}_4 w^{c+1}(\ell n\, w)q_1^{a+1} - \sum{}_4 4w^{c+1}q_1^{a+1}(\ell n\, q_1)$$
$$- \sum_{b+3\neq 0}{}^{*} \frac{K^{*}}{b+3} w^{r}q_1^{d} + \sum_{b+4\neq 0}{}^{*} \frac{2K^{*}}{b+4} w^{r+\frac{1}{3}}q_1^{d-\frac{4}{3}}$$
$$- \sum_{b+5\neq 0}{}^{*} \frac{K^{*}}{b+5} w^{r+\frac{2}{3}}q_1^{d-\frac{8}{3}} + \sum_{b+1\neq 0}{}^{*} \frac{K^{*}}{b+1} w^{s}q_1^{e}$$
$$- \sum_{b+2\neq 0}{}^{*} \frac{2K^{*}}{b+2} w^{s+\frac{1}{3}}q_1^{e-\frac{4}{3}} + \sum_{b+3\neq 0}{}^{*} \frac{K^{*}}{b+3} w^{s+\frac{2}{3}}q_1^{e-\frac{8}{3}}$$
$$- \sum_{b+4\neq 0}{}^{**} \frac{K^{**}}{b+4} w^{r+\frac{1}{3}}q_1^{d-\frac{4}{3}} + \sum_{b+5\neq 0}{}^{**} \frac{K^{**}}{b+5} w^{r+\frac{2}{3}}q_1^{d-\frac{8}{3}}],$$

where $\sum_1 = \sum^*_{b+3\neq 0} \frac{K^*}{b+3} - \sum^*_{b+4\neq 0} \frac{2K^*}{b+4} + \sum^*_{b+5\neq 0} \frac{K^*}{b+5} + \sum^{**}_{b+4\neq 0} \frac{K^{**}}{b+4} - \sum^{**}_{b+5\neq 0} \frac{K^{**}}{b+5}$,

$$\sum_2 = -\sum^*_{b+3=0} \frac{K^*}{3} + \sum^*_{b+4=0} \frac{2K^*}{3} - \sum^*_{b+5=0} \frac{K^*}{3} - \sum^{**}_{b+4=0} \frac{K^{**}}{3} + \sum^{**}_{b+5=0} \frac{K^{**}}{3},$$

$$\sum_3 = -\sum^*_{b+1\neq 0} \frac{K^*}{b+1} + \sum^*_{b+2\neq 0} \frac{2K^*}{b+2} - \sum^*_{b+3\neq 0} \frac{K^*}{b+3},$$

$$\sum_4 = \sum^*_{b+1=0} \frac{K^*}{3} - \sum^*_{b+2=0} \frac{2K^*}{3} + \sum^*_{b+3=0} \frac{K}{3},$$

$r = c+ \frac{1}{3}(b+3)$, $s = c+1+ \frac{1}{3}(b+1)$, $d = a+5- \frac{4}{3}(b+3)$, $e = a+1- \frac{4}{3}(b+1)$.

Finally integrate out $q_1$, we get the density of $W^{(4)}$ as follows:

$$(4.1)\quad C(4,m,n)\{[\sum_{1\,a+6\neq 0} \frac{1}{a+6} + \sum_{2\,a+6\neq 0} \frac{4}{(a+6)^2}][w^c - w^{c+\frac{1}{4}(a+b)}] + \sum_{2\,a+6\neq 0} \frac{1}{a+6} w^c \ell n\, w$$

$$- \sum_{1\,a+6=0} \frac{1}{4} w^c \ell n\, w - \sum_{2\,a+6=0} \frac{1}{8} w^c (\ell n\, w)^2$$

$$+ [\sum_{3\,a+2\neq 0} \frac{1}{a+2} + \sum_{4\,a+2\neq 0} \frac{4}{(a+2)^2}][w^{c+1} - w^{c+1+\frac{1}{4}(a+2)}] + \sum_{4\,a+2\neq 0} \frac{1}{a+2} w^{c+1} \ell n\, w$$

$$- \sum_{3\,a+2=0} \frac{1}{4} w^{c+1} \ell n\, w - \sum_{4\,a+2=0} \frac{1}{8} w^{c+1} (\ell n\, w)^2$$

$$- \sum^*_{\substack{b+3\neq 0\\ d+1\neq 0}} \frac{K^*}{(b+3)(d+1)} w^r (1 - w^{\frac{1}{4}(d+1)})$$

$$+ [\sum^*_{\substack{b+4\neq 0\\ d-\frac{1}{3}\neq 0}} \frac{2K^*}{(b+4)(d-\frac{1}{3})} - \sum^{**}_{\substack{b+4\neq 0\\ d-\frac{1}{3}\neq 0}} \frac{K^{**}}{(b+4)(d-\frac{1}{3})}] w^{r+\frac{1}{3}} (1 - w^{\frac{1}{4}(d-\frac{1}{3})})$$

$$- [\sum_{\substack{b+5\neq 0\\ d-\frac{5}{3}\neq 0}}^{*} \frac{K^*}{(b+5)(d-\frac{5}{3})} - \sum_{\substack{b+5\neq 0\\ d-\frac{5}{3}\neq 0}}^{**} \frac{K^{**}}{(b+5)(d-\frac{5}{3})}] w^{r+\frac{2}{3}}(1-w^{\frac{1}{4}(d-\frac{5}{3})})$$

$$+ \sum_{\substack{b+1\neq 0\\ e+1\neq 0}}^{*} \frac{K^*}{(b+1)(e+1)} w^{s}(1-w^{\frac{1}{4}(e+1)})$$

$$- \sum_{\substack{b+2\neq 0\\ e-\frac{1}{3}\neq 0}}^{*} \frac{2K^*}{(b+2)(e-\frac{1}{3})} w^{s+\frac{1}{3}}(1-w^{\frac{1}{4}(e-\frac{1}{3})})$$

$$+ \sum_{\substack{b+3\neq 0\\ e-\frac{5}{3}\neq 0}}^{*} \frac{K^*}{(b+3)(e-\frac{5}{3})} w^{s+\frac{2}{3}}(1-w^{\frac{1}{4}(e-\frac{5}{3})})$$

$$+ \sum_{\substack{b+3\neq 0\\ d+1=0}}^{*} \frac{K^*}{4(b+3)} w^{r} - (\sum_{\substack{b+4\neq 0\\ d-\frac{1}{3}=0}}^{*} \frac{2K^*}{4(b+4)} - \sum_{\substack{b+4\neq 0\\ d-\frac{1}{3}=0}}^{**} \frac{K^{**}}{4(b+4)}) w^{r+\frac{1}{3}}$$

$$+ (\sum_{\substack{b+5\neq 0\\ d-\frac{5}{3}=0}}^{*} \frac{K^*}{4(b+5)} - \sum_{\substack{b+5\neq 0\\ d-\frac{5}{3}=0}}^{**} \frac{K^{**}}{4(b+5)}) w^{r+\frac{2}{3}} - \sum_{\substack{b+1\neq 0\\ e+1=0}}^{*} \frac{K^*}{4(b+1)} w^{s}$$

$$+ \sum_{\substack{b+2\neq 0\\ e-\frac{1}{3}=0}}^{*} \frac{2K^*}{4(b+2)} w^{s+\frac{1}{3}} - \sum_{\substack{b+3\neq 0\\ e-\frac{5}{3}=0}}^{*} \frac{K^*}{4(b+3)} w^{s+\frac{2}{3}}] \ln w\}.$$

<u>m = 0</u>: Put m = 0 in (4.1) and we have the density

$$(4.2)\quad \frac{1}{2} C(4,0,n)[\frac{1}{90} w^{n} - \frac{4}{15} w^{n+\frac{1}{2}} - \frac{25}{6} w^{n+1} + \frac{25}{6} w^{n+2} + \frac{4}{15} w^{n+\frac{5}{2}} - \frac{1}{90} w^{n+3} - w^{n+1}\ln w - \frac{8}{3} w^{n+\frac{3}{2}}\ln w - w^{n+2}\ln w],$$

and the exact cdf

$$(4.3)\quad P(W^{(4)}<w) = \frac{1}{2}C(4,0,n)\Big[\frac{1}{90(n+1)}w^{n+1} - \frac{8}{15(2n+3)}w^{n+\frac{3}{2}}$$

$$- \frac{25}{6(n+2)}w^{n+2} + \frac{25}{6(n+3)}w^{n+3} + \frac{8}{15(2n+7)}w^{n+\frac{7}{2}}$$

$$- \frac{1}{90(n+4)}w^{n+4} - \frac{1}{n+2}w^{n+2}\ell n\, w + \frac{1}{(n+2)^2}w^{n+2}$$

$$- \frac{16}{3(2n+5)}w^{n+\frac{5}{2}}\ell n\, w + \frac{32}{3(2n+5)^2}w^{n+\frac{5}{2}}$$

$$- \frac{1}{n+3}w^{n+3}\ell n\, w + \frac{1}{(n+3)^2}w^{n+3}\Big].$$

The special cases are exactly the same as those from the results of Pillai and Gupta [26]. However there are some additional terms in the general formulae derived here in comparison to theirs, but they cancel out under specialization.

# CHAPTER V

# THE NONCENTRAL DISTRIBUTIONS OF THE CHARACTERISTIC ROOTS OF THE WISHART MATRIX UNDER AN ASSUMPTION AND THE SPHERICITY CRITERION

## 1. Introduction

Let $\underset{\sim}{S}$ have a non-central Wishart distribution $W(p,n,\underset{\sim}{\Sigma},\underset{\sim}{\Omega})$ where $\underset{\sim}{\Sigma}$, $\underset{\sim}{\Omega}$ are unknown. In this chapter, the distribution of the characteristic roots $0 < c_1 \leq \cdots \leq c_p < \infty$ of $\underset{\sim}{S}$ is obtained, (a) when $\underset{\sim}{\Sigma}$ is partially random, and (b) when $\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1}$ is partially random (denote "random" hereafter). Here "random" implies diagonalization by an orthogonal transformation $\underset{\sim}{H}$ and integration over $\underset{\sim}{H}$, (see Pillai [23]). Under the same assumption, the distribution of the sphericity criterion $W = |\underset{\sim}{S}|/[\mathrm{tr}\ \underset{\sim}{S}/p]^p$ is derived for testing $\underset{\sim}{\Sigma} = \sigma^2\underset{\sim}{I}_p$, where $\sigma^2 > 0$ is unknown, against $\underset{\sim}{\Sigma} \neq \sigma^2\underset{\sim}{I}_p$, [Mauchly, 17]. For $p = 2$, the density functions of the characteristic roots of $\underset{\sim}{S}$ and $W$ have also been obtained in the general case, i.e. without using partial random approach. The above distributions of the sphericity criterion are useful for studying the exact robustness of the sphericity test.

## 2. The distribution of the characteristic roots of $\underset{\sim}{S}$ under an assumption

The joint distribution of $c_1,\ldots,c_p$ is derived in this section, (a) when $\underset{\sim}{\Sigma}$ is "random", and (b) when $\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1}$ is "random".

Theorem 1. Under the assumption that $\underset{\sim}{\Sigma}$ is "random", the joint density function of $c_1,\ldots,c_p$ is given by

$$(2.1)\quad C(p,n,\underset{\sim}{\Sigma})\exp(-\mathrm{tr}\underset{\sim}{\Omega})\exp(-\tfrac{1}{2}\,\mathrm{tr}\underset{\sim}{C})|\underset{\sim}{C}|^{\frac{1}{2}(n-p-1)}\prod_{i>j}(c_i-c_j)$$

$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k C_\kappa(\underset{\sim}{C})}{k!}\sum_{t=0}^{k}\sum_{\tau}\frac{a_{\kappa,\tau}C_\tau(-\underset{\sim}{\Sigma}^{-1})L_\tau^{\frac{1}{2}(n-p-1)}(\underset{\sim}{\Omega})}{(\frac{n}{2})_\tau C_\tau(\underset{\sim}{I}_p)C_\tau(\underset{\sim}{I}_p)},$$

where $\underset{\sim}{C}=\mathrm{diag}(c_1,\ldots,c_p)$ and

$$(2.2)\quad C(p,n,\underset{\sim}{\Sigma})=\pi^{\frac{1}{2}p^2}/\{2^{\frac{1}{2}pn}\Gamma_p(\tfrac{1}{2}n)\Gamma_p(\tfrac{1}{2}p)|\underset{\sim}{\Sigma}|^{\frac{1}{2}n}\}.$$

Proof. The density of $\underset{\sim}{S}$ is given by Constantine [3],

$$(2.3)\quad [\Gamma_p(\tfrac{1}{2}n)|2\underset{\sim}{\Sigma}|^{\frac{1}{2}n}]^{-1}\exp(-\mathrm{tr}\underset{\sim}{\Omega})\exp(-\tfrac{1}{2}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{S})|\underset{\sim}{S}|^{\frac{1}{2}(n-p-1)}$$

$$\times\,{}_0F_1(\tfrac{1}{2}n,\tfrac{1}{2}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\Omega}\underset{\sim}{S}).$$

Now use equation (29) of James [8] and write $\exp(-\frac{1}{2}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{S})$ as $\exp(-\frac{1}{2}\,\mathrm{tr}\underset{\sim}{S})\exp(\frac{1}{2}\,\mathrm{tr}(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1})\underset{\sim}{S})$. Then (2.3) becomes

$$(2.4)\quad [\Gamma_p(\tfrac{1}{2}n)|2\underset{\sim}{\Sigma}|^{\frac{1}{2}n}]^{-1}\exp(-\mathrm{tr}\underset{\sim}{\Omega})\exp(-\tfrac{1}{2}\,\mathrm{tr}\underset{\sim}{S})|\underset{\sim}{S}|^{\frac{1}{2}(n-p-1)}$$

$$\times K\int_{R(\underset{\sim}{T})=\underset{\sim}{X}_0>\underset{\sim}{0}}\exp(\mathrm{tr}\underset{\sim}{T})|\underset{\sim}{T}|^{-\frac{1}{2}n}\exp\{\tfrac{1}{2}\,\mathrm{tr}[(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1}+\underset{\sim}{T}^{-1}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\Omega})\underset{\sim}{S}]\}d\underset{\sim}{T},$$

where $K = 2^{\frac{1}{2}p(p-1)}\Gamma_p(\frac{1}{2}n)/(2\pi i)^{\frac{1}{2}p(p+1)}$. $\underset{\sim}{S}$ can be diagonalized by an orthogonal transformation $\underset{\sim}{H}$ such that $\underset{\sim}{H}\underset{\sim}{S}\underset{\sim}{H}' = \underset{\sim}{C} = \operatorname{diag}(c_1,\ldots,c_p)$ where $c_1,\ldots,c_p$ are the characteristic roots of $\underset{\sim}{S}$. For uniqueness, we assume that the elements in the first row of $\underset{\sim}{H}$ are positive and the roots are arranged in the order $0 < c_1 \leq \ldots \leq c_p < \infty$. The volume element $d\underset{\sim}{S}$ becomes (see Constantine [3])

$$d\underset{\sim}{S} = \prod_{i>j}(c_i-c_j)\prod_{i=1}^{p} dc_i d\underset{\sim}{H}.$$

And also

$$\int_{O(p)} d\underset{\sim}{H} = 2^p\pi^{\frac{1}{2}p^2}/\Gamma_p(\tfrac{1}{2}p).$$

Substituting $\underset{\sim}{C}$ in (2.4) and integrating out $\underset{\sim}{H}$, we have the joint density of $c_1,\ldots,c_p$ in the form

$$(2.5)\quad [\Gamma_p(\tfrac{1}{2}n)|2\underset{\sim}{\Sigma}|^{\frac{1}{2}n}]^{-1}\exp(-\operatorname{tr}\underset{\sim}{\Omega})\exp(-\tfrac{1}{2}\operatorname{tr}\underset{\sim}{C})|\underset{\sim}{C}|^{\frac{1}{2}(n-p-1)}\prod_{i>j}(c_i-c_j)$$

$$\times K\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k}{k!}\int_{R(T)=\underset{\sim}{X}>\underset{\sim}{0}}\exp(\operatorname{tr}\underset{\sim}{T})|\underset{\sim}{T}|^{-\frac{1}{2}n}$$

$$\times 2^{-p}\int_{O(p)} C_\kappa[(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{I}-\underset{\sim}{T}^{-1}\underset{\sim}{\Omega}))\underset{\sim}{H}'\underset{\sim}{C}\underset{\sim}{H}]d\underset{\sim}{H}d\underset{\sim}{T}.$$

The factor $2^{-p}$ arises from the restriction that the first row elements of $\underset{\sim}{H}$ are positive. Now use equation (23) of James [8] and equation (19) of Constantine [4], (2.5) becomes

$$(2.6)\quad C(p,n,\underset{\sim}{\Sigma})\exp(-\operatorname{tr}\underset{\sim}{\Omega})\exp(-\tfrac{1}{2}\operatorname{tr}\underset{\sim}{C})|\underset{\sim}{C}|^{\frac{1}{2}(n-p-1)}\prod_{i>j}(c_i-c_j)$$

$$\times K\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k C_\kappa(\underset{\sim}{C})}{k!}\sum_{t=0}^{k}\sum_{\tau}\frac{a_{\kappa,\tau}}{C_\tau(\underset{\sim}{I}_p)}$$

$$\times \int_{R(\underset{\sim}{T})=\underset{\sim}{X}_0>0} \exp(\text{tr}\underset{\sim}{T})|\underset{\sim}{T}|^{-\frac{1}{2}n} C_\tau(-\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{I}-\underset{\sim}{T}^{-1}\underset{\sim}{\Omega}))d\underset{\sim}{T}.$$

Now assume $\underset{\sim}{\Sigma}$ "random" and transform $\underset{\sim}{\Sigma}^{-1} \to \underset{\sim}{H}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{H}'$ by an orthogonal transformation $\underset{\sim}{H}$ and integrate over $\underset{\sim}{H}$ using equation (23) of James [8]. Then (2.6) becomes

$$(2.7) \quad C(p,n,\underset{\sim}{\Sigma})\exp(-\text{tr}\underset{\sim}{\Omega})\exp(-\frac{1}{2}\text{tr}\underset{\sim}{C})|\underset{\sim}{C}|^{\frac{1}{2}(n-p-1)} \prod_{i>j}(c_i-c_j)$$

$$\times K \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\frac{1}{2})^k C_\kappa(\underset{\sim}{C})}{k!} \sum_{t=0}^{k}\sum_{\tau} \frac{a_{\kappa,\tau} C_\tau(-\underset{\sim}{\Sigma}^{-1})}{C_\tau(\underset{\sim}{I}_p)C_\tau(\underset{\sim}{I}_p)}$$

$$\times \int_{R(\underset{\sim}{T})=\underset{\sim}{X}_0>\underset{\sim}{0}} \exp(\text{tr}\,\underset{\sim}{T})|\underset{\sim}{T}|^{-\frac{1}{2}n} C_\tau(\underset{\sim}{I}-\underset{\sim}{T}^{-1}\underset{\sim}{\Omega})d\underset{\sim}{T}.$$

Finally, by using equation (17) of Constantine [4], we have the result given in (2.1).

For $\underset{\sim}{\Omega} = \underset{\sim}{0}$, and use $L_\tau^{\frac{1}{2}(n-p-1)}(\underset{\sim}{0}) = (\frac{n}{2})_\tau C_\tau(\underset{\sim}{I}_p)$, the expression (2.1) gives the result stated in (2.1) of Pillai, Al-Ani and Jouris [25].

<u>Theorem 2</u>. Under the assumption that $\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1}$ is "random", the joint density function of $c_1,\ldots,c_p$ is given by

$$(2.8) \quad C(p,n,\underset{\sim}{\Sigma})\exp(-\text{tr}\underset{\sim}{\Omega})\exp(-\frac{1}{2}\text{tr}\underset{\sim}{C})|\underset{\sim}{C}|^{\frac{1}{2}(n-p-1)} \prod_{i>j}(c_i-c_j)$$

$$\times \sum_{k=0}^{\infty}\sum_{\kappa} \frac{(\frac{1}{2})^k C_\kappa(\underset{\sim}{C})C_\kappa(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1})L_\kappa^{\frac{1}{2}(n-p-1)}[-\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\Omega}(\underset{\sim}{I}-\underset{\sim}{\Sigma}^{-1})^{-1}]}{k!(\frac{n}{2})_\kappa\, C_\kappa(\underset{\sim}{I}_p)C_\kappa(\underset{\sim}{I}_p)},$$

where $C(p,n,\underset{\sim}{\Sigma})$ is defined in (2.2).

Proof. Starting from (2.5) and using equation (23) of James [8], the joint density of $c_1,\ldots,c_p$ becomes

$$(2.9)\quad C(p,n,\Sigma)\exp(-\mathrm{tr}\Omega)\exp(-\tfrac{1}{2}\,\mathrm{tr}C)|C|^{\frac{1}{2}(n-p-1)}\prod_{i>j}(c_i-c_j)$$

$$\times\, K\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k C_\kappa(C)}{k!\,C_\kappa(I_p)}$$

$$\times\int_{R(T)=X_0>0}\exp(\mathrm{tr}\;T)|T|^{-\frac{1}{2}n}C_\kappa[(I+T^{-1}\Omega^*)(I-\Sigma^{-1})]dT,$$

where $\Omega^* = \Sigma^{-1}\Omega(I-\Sigma^{-1})^{-1}$.

Now assume $I-\Sigma^{-1}$ "random" and transform $I-\Sigma^{-1}\rightarrow H(I-\Sigma^{-1})H'$ by an orthogonal transformation $H$ and integrate over $H$.

Then (2.9) becomes

$$(2.10)\quad C(p,n,\Sigma)\exp(-\mathrm{tr}\Omega)\exp(-\tfrac{1}{2}\,\mathrm{tr}C)|C|^{\frac{1}{2}(n-p-1)}\prod_{i>j}(c_i-c_j)$$

$$K\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k C_\kappa(C)C_\kappa(I-\Sigma^{-1})}{k!\,C_\kappa(I_p)C_\kappa(I_p)}$$

$$\times\int_{R(T)=X_0>0}\exp(\mathrm{tr}T)|T|^{-\frac{1}{2}n}C_\kappa(I+T^{-1}\Omega^*)dT.$$

Finally, by using equation (17) of Constantine [4], we have the result given in (2.8).

For $\Omega = 0$, the expression (2.8) gives the result stated in (2.1) of Pillai, Al-Ani and Jouris [25].

## 3. The distribution of sphericity criterion under violation

Using (2.1), the hth moment of the sphericity criterion $W = |\underset{\sim}{C}|/[\mathrm{tr}\ \underset{\sim}{C}/p]^p$ is given by

$$(3.1)\quad [\Gamma_p(\tfrac{1}{2}n)|2\underset{\sim}{\Sigma}|^{\frac{1}{2}n}]^{-1}\exp(-\mathrm{tr}\underset{\sim}{\Omega})\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k}{k!}\sum_{t=0}^{k}\sum_{\tau}$$

$$\frac{a_{\kappa,\tau}C_\tau(-\underset{\sim}{\Sigma}^{-1})L_\tau^{\frac{1}{2}(n-p-1)}(\underset{\sim}{\Omega})}{(\frac{n}{2})_\tau C_\tau(\underset{\sim}{I}_p)C_\tau(\underset{\sim}{I}_p)}$$

$$\times\ p^{ph}\int_{\underset{\sim}{C}>\underset{\sim}{0}}\exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{C})|\underset{\sim}{C}|^{\frac{1}{2}(n-p-1)+h}(\mathrm{tr}\underset{\sim}{C})^{-ph}C_\kappa(\underset{\sim}{C})d\underset{\sim}{C}.$$

Now making use of Lemma 7 of Khatri [11], we have

$$(3.2)\quad E(W^h) = \{p^{ph}/[\Gamma_p(\tfrac{1}{2}n)|\underset{\sim}{\Sigma}|^{\frac{1}{2}n}]\}\exp(-\mathrm{tr}\underset{\sim}{\Omega})$$

$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\underset{\sim}{I}_p)\Gamma_p(\frac{1}{2}n+h,\kappa)\Gamma(\frac{1}{2}np+k)}{k!\ \Gamma(\frac{1}{2}np+hp+k)}$$

$$\times\sum_{t=0}^{k}\sum_{\tau}\frac{a_{\kappa,\tau}C_\tau(-\underset{\sim}{\Sigma}^{-1})L_\tau^{\frac{1}{2}(n-p-1)}(\underset{\sim}{\Omega})}{(\frac{n}{2})_\tau C_\tau(\underset{\sim}{I}_p)C_\tau(\underset{\sim}{I}_p)}.$$

Further, we prove the following theorem:

<u>Theorem 3</u>. For any finite p, the density function of W is given by

$$(3.3)\quad f(w)=C_1(p,n,\underset{\sim}{\Sigma})\exp(-\mathrm{tr}\underset{\sim}{\Omega})\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\underset{\sim}{I}_p)}{k!}p^{\frac{1}{2}-\frac{1}{2}np-k}\Gamma(\tfrac{1}{2}np+k)$$

$$\sum_{t=0}^{k}\sum_{\tau}\frac{a_{\kappa,\tau}C_\tau(-\underset{\sim}{\Sigma}^{-1})L_\tau^{\frac{1}{2}(n-p-1)}(\underset{\sim}{\Omega})}{(\frac{n}{2})_\tau C_\tau(\underset{\sim}{I}_p)C_\tau(\underset{\sim}{I}_p)}$$

$$\times\ w^{\frac{1}{2}(n-p-1)} G^{p,0}_{p,p}\left(w\middle|{a_1,\ldots,a_p \atop b_1,\ldots,b_p}\right),$$

where

(3.4) $$C_1(p,n,\underset{\sim}{\Sigma}) = \pi^{\frac{1}{4}p(p-1)}(2\pi)^{\frac{1}{2}(p-1)}/[\Gamma_p(\frac{1}{2}n)|\underset{\sim}{\Sigma}|^{\frac{1}{2}n}],$$

(3.5) $$a_j = (k+j-1)/p + \frac{1}{2}(p-1),\ b_j = k_j + \frac{1}{2}(p-j),$$

and Meijer's G-function is defined as

$$G^{m,n}_{p,q}\left(x\middle|{a_1,\ldots,a_p \atop b_1,\ldots,b_q}\right) = (2\pi i)^{-1}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\, x^s ds,$$

where C is a curve separating the singularities of $\prod_{j=1}^{m}\Gamma(b_j-s)$ from those of $\prod_{j=1}^{n}\Gamma(1-a_j+s)$, $1 \le q$, $0 \le n \le p \le q$, $0 \le m \le q$; $x \ne 0$ and $|x| < 1$ if $q = p$; $x \ne 0$ if $p < q$.

<u>Proof</u>. Applying Gauss and Legendre's multiplication formula on $\Gamma[p(\frac{1}{2}n+h+k/p)]$, we have from (3.2)

(3.6) $$E(W^h)=C_1(p,n,\underset{\sim}{\Sigma})\exp(-\mathrm{tr}\underset{\sim}{\Omega})\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\underset{\sim}{I}_p)}{k!}p^{\frac{1}{2}-\frac{1}{2}np-k}\Gamma(\frac{1}{2}np+k)$$

$$\times\sum_{t=0}^{k}\sum_{\tau}\frac{a_{\kappa,\tau}C_\tau(-\underset{\sim}{\Sigma}^{-1})L_\tau^{\frac{1}{2}(n-p-1)}(\underset{\sim}{\Omega})}{(\frac{n}{2})_\tau C_\tau(I_p)C_\tau(\underset{\sim}{I}_p)}\prod_{j=1}^{p}\frac{\Gamma[\frac{1}{2}n+h+k_j-\frac{1}{2}(j-1)]}{[\frac{1}{2}n+h+(k+j-1)/p]},$$

where $C_1(p,n,\underset{\sim}{\Sigma})$ is defined in (3.4).

Further using inverse Mellin transform, the density of W has the form

$$(3.7)\quad f(w) = C_1(p,n,\underset{\sim}{\Sigma})\exp(-\mathrm{tr}\underset{\sim}{\Omega})\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\underset{\sim}{I}_p)}{k!}\,p^{\frac{1}{2}-\frac{1}{2}np-k}\Gamma(\tfrac{1}{2}np+k)$$

$$\times\sum_{t=0}^{k}\sum_{\kappa}\frac{a_{\kappa,\tau}C_\tau(-\underset{\sim}{\Sigma}^{-1})L_\tau^{\frac{1}{2}(n-p-1)}(\underset{\sim}{\Omega})}{(\frac{n}{2})_\tau C_\tau(\underset{\sim}{I}_p)C_\tau(\underset{\sim}{I}_p)}$$

$$w^{\frac{1}{2}(n-p-1)}(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} w^{-r}\frac{\prod_{i=1}^{p}\Gamma(r+b_i)}{\prod_{i=1}^{p}\Gamma(r+a_i)}\,dr,$$

where $r = \frac{1}{2}n+h-\frac{1}{2}(p-1)$ and $a_j$, $b_j$ are defined in (3.5). Noting that the integral in (3.7) is in the form of Meijer's G-function, we can write the density of W as in (3.3).

Remark. Putting $\underset{\sim}{\Omega} = \underset{\sim}{0}$ in (3.3), we can easily deduced the result of Pillai and Nagarsenker [32].

## 4. The non-central distribution of the characteristic roots of $\underset{\sim}{S}$ for $p = 2$

In this section, we deal with the density of the characteristic roots of the non-central Wishart $\underset{\sim}{S}$ for $p = 2$ in the general case, i.e. without using partial random approach.

The density of $\underset{\sim}{S}$ for $p = 2$ is given by

$$(4.1)\quad [\Gamma_2(\tfrac{1}{2}n)|2\underset{\sim}{\Sigma}|^{\frac{1}{2}n}]^{-1}\exp(-\mathrm{tr}\underset{\sim}{\Omega})\exp(-\tfrac{1}{2}\mathrm{tr}\underset{\sim}{\Sigma}^{-1}\underset{\sim}{S})|\underset{\sim}{S}|^{\frac{1}{2}(n-3)}$$

$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k}{(\frac{1}{2}n)_\kappa k!}C_\kappa(\underset{\sim}{\Sigma}^{-1}\underset{\sim}{\Omega}\underset{\sim}{S}).$$

Let $\underset{\sim}{\Omega} = \underset{\sim}{H}_1\underset{\sim}{\Omega}_1\underset{\sim}{H}_1'$ and $\underset{\sim}{\Sigma}^{-1} = \underset{\sim}{H}_2'\underset{\sim}{\Lambda}^{-1}\underset{\sim}{H}_2$ where $\underset{\sim}{H}_1$, $\underset{\sim}{H}_2$ are orthogonal matrices and $\underset{\sim}{\Omega}_1 = \mathrm{diag}(\omega_1,\omega_2)$ with $\omega_1,\omega_2$ being the characteristic roots of $\underset{\sim}{\Omega}$,

$\Lambda^{-1} = \text{diag}(\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2$ being the characteristic roots of $\Sigma^{-1}$. Then (4.1) becomes

$$(4.2)\quad [\Gamma_2(\tfrac{1}{2}n)|2\Sigma|^{n}]^{-1}\exp(-\text{tr}\Omega_1)\exp(-\tfrac{1}{2}\text{tr}S)\exp[\tfrac{1}{2}\text{tr}(I-\Lambda^{-1})H_2SH_2']$$

$$\times |S|^{\frac{1}{2}(n-3)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k}{(\frac{1}{2}n)_\kappa k!}C_\kappa(H_2'\Lambda^{-1}H_2H_1\Omega_1H_1'S).$$

Now transform $H_2SH_2' = HCH'$ where $C = \text{diag}(c_1, c_2)$ with $c_1 < c_2$ being the characteristic roots of $S$ and $H = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$. Then the joint density of $c_1$, $c_2$, $\theta$ is of the form

$$(4.3)\quad [\Gamma_2(\tfrac{1}{2}n)|2\Sigma|^{\frac{1}{2}n}]^{-1}\exp(-\text{tr}\Omega_1)\exp(-\tfrac{1}{2}\text{tr}C)\exp[\tfrac{1}{2}\text{tr}(I-\Lambda^{-1})HCH']$$

$$\times |C|^{\frac{1}{2}(n-3)}(c_2-c_1)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k}{(\frac{1}{2}n)_\kappa k!}C_\kappa(\Lambda^{-1}H_2H_1\Omega_1H_1'H_2'HCH').$$

Now let $H_2H_1 = \begin{pmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}$ and $a_1$, $a_2$ be the sum and product of the two characteristic roots of $\Lambda^{-1}H_2H_1\Omega_1H_1'H_2'HCH'$. Then

$$a_1=(d_1c_1+d_2c_2)+(d_1-d_2)(c_2-c_1)\sin^2\theta+d_3(c_2-c_1)\cos\theta\sin\theta,$$

$$a_2=\lambda_1\lambda_2\omega_1\omega_2c_1c_2, \text{ where}$$

$$(4.4)\quad d_1=\lambda_1(h_{11}^2\omega_1+h_{12}^2\omega_2),\ d_2=\lambda_2(h_{21}^2\omega_1+h_{22}^2\omega_2),$$

$$d_3=(\lambda_1+\lambda_2)(h_{11}h_{21}\omega_1+h_{12}h_{22}\omega_2).$$

Then the joint density of $c_1$, $c_2$, $\theta$ becomes

$$(4.5)\quad [\Gamma_2(\tfrac{1}{2}n)|2\Sigma|^{\frac{1}{2}n}]^{-1}\exp[-(\omega_1+\omega_2)]\exp[-\tfrac{1}{2}(\lambda_1c_1+\lambda_2c_2)](c_1c_2)^{\frac{1}{2}(n-3)}$$

$$(c_2-c_1)\exp[\tfrac{1}{2}(c_2-c_1)(\lambda_2-\lambda_1)\sin^2\theta]$$

$$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k}{(\frac{1}{2}n)_\kappa k!}\sum_{r+2s=k} b_{k_1,k_2}(r,s)a_1^r\, a_2^s,$$

where $\kappa = (k_1,k_2)$, $k_1 \geq k_2$, $k_1+k_2=k$ and

$$(4.6)\quad b_{k_1,k_2}(r,s) = \frac{2^{2k_1}(2(k_1-k_2)+1)}{2k_1+1}\prod_{i=1}^{k_1}\frac{k_2+i}{k_1+i}\prod_{i=1}^{r}(1-\frac{1}{2i})\prod_{i=1}^{s-k_2}(1+\frac{r-\frac{1}{2}}{i})(-1)^{s-k_2}.$$

Now expanding $\exp[\frac{1}{2}(c_2-c_1)(\lambda_2-\lambda_1)\sin^2\theta]$ and $a_1^r$ into series and integrating out $\theta$, we get the joint density of $c_1,c_2$ as follows:

$$(4.7)\quad [\Gamma_2(\tfrac{1}{2}n)|2\Sigma|^{\frac{1}{2}n}]^{-1}\exp[-(\omega_1+\omega_2)]\exp[-\tfrac{1}{2}(\lambda_1c_1+\lambda_2c_2)]$$

$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k}{(\frac{1}{2}n)_\kappa k!}\sum_{r+2s=k} b_{k_1,k_2}(r,s)(\lambda_1\lambda_2\omega_1\omega_2)^s$$

$$\times\sum_{m=0}^{\infty}\frac{(\lambda_2-\lambda_1)^m}{2^m m!}\sum_{j=0}^{r}\sum_{\substack{\ell=0\\ \ell\text{ even}}}^{j}\binom{r}{j}\binom{j}{\ell}C_2(m,j,\ell)$$

$$\times (d_1c_1+d_2c_2)^{r-j}(c_2-c_1)^{m+j+1}(d_1-d_2)^{j-\ell}d_3^{\ell}(c_1c_2)^{\frac{1}{2}(n-3)+5},$$

where $b_{k_1,k_2}(r,s)$ is defined in (4.6), $d_1$, $d_2$ and $d_3$ are defined in (4.4) and

$$(4.8)\quad C_2(m,j,\ell) = \frac{\ell-1}{2m+2j}\,\frac{\ell-3}{2m+2j-2}\cdots\frac{1}{2m+2j-(\ell-2)}\,\frac{2m+2j-\ell-1}{2m+2j-\ell}\,\frac{2m+2j-\ell-3}{2m+2j-\ell-2}\cdots\frac{1}{2}\,2\pi.$$

5. The density of the sphericity criterion for p = 2

Starting from (4.7), transform $y_1 = c_1/c_2$ and $y_2 = c_2$, then $\exp[-\frac{1}{2}(\lambda_1 c_1+\lambda_2 c_2)](c_1 c_2)^{\frac{1}{2}(n-3)+s}(c_2-c_1)^{m+j+1}(d_1 c_1+d_2 c_2)^{r-j}$ becomes $\exp[-\frac{1}{2}(\lambda_1 y_1+\lambda_2)y_2]y_2^{n-1+2s+m+r}y_1^{\frac{1}{2}(n-3)+s}(1-y_1)^{m+j+1}(d_1 y_1+d_2)^{r-j}$. Further, integrate out $y_2$ and transform $t = \sqrt{y_1}$, then the above becomes

$$\Gamma(n+2s+m+r)2^{n+2s+m+r+1}t^{n-2+2s}(1-t^2)^{m+j+1}(d_1 t^2+d_2)^{r-j}$$
$$\times (\lambda_1 t^2+\lambda_2)^{-r-n-2s-m}.$$

Now the sphericity criterion $U = (c_1 c_2)^{\frac{1}{2}}/[(c_1+c_2)/2] = 2t/(t^2+1)$. Thus, the density of U is given by

$$(5.1)\quad f(u) = [\Gamma_2(\tfrac{1}{2}n)|2\Sigma|^{\frac{1}{2}n}]^{-1}\exp[-(\omega_1+\omega_2)]$$
$$\times \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2})^k}{(\frac{1}{2}n)_\kappa k!}\sum_{r+2s=k} b_{k_1,k_2}(r,s)(\lambda_1\lambda_2\omega_1\omega_2)^s$$
$$\times \sum_{m=0}^{\infty}\frac{(\lambda_2-\lambda_1)^m}{2^m m!}\sum_{j=0}^{r}\sum_{\substack{\ell=0\\ \ell\ \text{even}}}^{j}\binom{r}{j}\binom{j}{\ell}C_2(m,j,\ell)$$
$$\times (d_1-d_2)^{j-\ell}d_3^{\ell}\Gamma(n+2s+m+r)2^{n+2s+2m+r+j+2}$$
$$\times u^{n-2+2s}(1-\sqrt{1-u^2})^{n+2s+m+j}(\sqrt{1-u^2})^{m+j}$$
$$\times [d_1(1-\sqrt{1-u^2})^2+d_2u^2]^{r-j}[\lambda_1(1-\sqrt{1-u^2})^2+\lambda_2u^2]^{-n-2s-m-r},$$

where $b_{k_1,k_2}(r,s)$ is defined in (4.6), $d_1, d_2$ and $d_3$ are defined in (4.4) and $C_2(m,j,\ell)$ is defined in (4.8).

Further, transform $F = (1-U^2)^{\frac{1}{2}}/[1-(1-U^2)^{\frac{1}{2}}] = \frac{1}{2}(c_2/c_1-1)$. Then the density of F is given by

$$(5.2) \quad g(f) = [\Gamma_2(\tfrac{1}{2}\, n)|2\underset{\sim}{\Sigma}|^{\frac{1}{2}n}]^{-1}\exp(-\ (\omega_1+\omega_2)]$$

$$\times \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2})^k}{(\frac{1}{2}n)_\kappa k!} \sum_{r+2s=k} b_{k_1,k_2}(r,s)(\lambda_1\lambda_2\omega_1\omega_2)^s$$

$$\times \sum_{m=0}^{\infty} \sum_{q=0}^{m} \frac{1}{2^m q!(m-q)!} \sum_{j=0}^{r} \sum_{\ell=0}^{j} \binom{r}{j}\binom{j}{\ell} C_2(m,j,2m-2q+\ell)$$

$$\times (d_1-d_2)^{j-\ell} d_3^{\ell}\Gamma(n+2s+m+r)2^{2+j}(\lambda_1+\lambda_2)^{-n-2s-m-r}$$

$$\times \textstyle\sum^* D^* f^{j+1+g+h+i+t}(1+f)^{-n-2s-m-r}$$

where

$$(5.3) \quad \textstyle\sum^* = \displaystyle\sum_{g=0}^{\frac{1}{2}(n-3)+s} \sum_{h=0}^{q} \sum_{i=0}^{m-q} \sum_{t=0}^{r-j} \text{ and } D^* = \binom{\frac{1}{2}(n-3)+5}{g}\binom{q}{h}\binom{m-q}{i}\binom{r-j}{t}$$

$$2^{g+h+i+t}(\lambda_1+\lambda_2)^{m-h-i}\lambda_1^i\lambda_2^h(d_1+d_2)^{r-j-t}d_2^t,\ 0 < f < \infty.$$

# CHAPTER VI

# SUMMARY AND SUGGESTIONS

The work reported in this thesis consists of power studies of tests of equality of covariance matrices of two p-variate normal populations against two-sided alternatives based on six criteria given below in the real case and the first five criteria in the complex case, some monotonicity properties of the power functions of tests based on Pillai's trace in the complex case for MANOVA and canonical correlation, the distribution of Wilks' criterion through variable transformation, and also the noncentral distributions of the characteristic roots of the Wishart matrix under an assumption and the sphericity criterion under violation. The six criteria are: 1) Roy's largest root, 2) Hotelling's trace, 3) Pillai's trace, 4) Wilks' criterion, 5) Roy's largest and smallest roots and 6) modified likelihood ratio.

The first three of the preceeding five chapters deal with power function studies of different tests and the last two on distribution problems of statistics. While in Chapter I, a general theorem has been proved in the real case establishing a local unbiasedness conditions connecting the two critical values for a class of tests of which tests 1) to 5) are special cases, in Chapter II, such a theorem has been proved in the complex case. In order to obtain the theorem in the complex case, some results on transformations and Jacobinas had to be proved first which are useful in complex normal distribution theory. In the real case, extensive unbiased power tabulations have been made for p=2. Equal tail areas approach has also been used further to compute powers of tests 1) to 4) for p=2 for studying the bias and facilitating comparisons with powers in the unbiased case. Comparisons of powers of test 1) to 5) have also been made with those of the modified likelihood ratio after obtaining the exact distribution of the latter in a special case involving unequal degrees of freedom. Again, a separate study has been made to compare the powers of the largest-smallest roots test with those of its three biased approximate approaches studied as well as the largest root. Since the largest root test was observed to have some advantage over the others, critical values were also obtained for this test in the unbiased as well as equal tail areas cases for p=3. In the complex case, similar power studies have been made for p=2 for tests 1) to 5) and the inferences have been found similar to those in the real case.

In Chapter III, the work of Perlman has been extended to the complex case showing that the power function of the test based on Pillai's trace of a multivariate complex beta matrix is monotonically increasing in each noncentrality parameter provided that the cutoff point is not too large. This result was also proved true for the problem of testing independence of two sets of complex variates.

In Chapter IV, the distribution of Wilks' criterion $W^{(p)}$ has been studied based on variable transformation. Exact distributions of $W^{(p)}$ were given explicitly for p=3 and 4.

The noncentral distribution of the characteristic roots of the Wishart matrix has been obtained in Chapter V, when (a) the covariance matrix $\underset{\sim}{\Sigma}$ is partially random, and (b) $I-\underset{\sim}{\Sigma}^{-1}$ is partially random. Under the same assumption, the distribution of the sphericity criterion has been derived for testing $\underset{\sim}{\Sigma} = \sigma^2 \underset{\sim}{I}_p$, where $\sigma^2 > 0$ is unknown. For p=2, the density functions of the characteristic roots and the sphericity criterion were also obtained without using the partial random approach. The above distributions of the sphericity criterion are useful for studying the exact robustness of the sphericity test.

The following are some suggestions for future work:

1) In Chapter I and II, for p=2, the numerical results for powers seem to suggest that the largest root has some advantage compared to others. This fact needs to be explored for larger p.

2) The forms for p=3 and 4 of the distribution of Wilks' criterion, (as seen from special cases worked out for specific degrees of freedom) involve extra terms which should cancel making the forms much simpler. This needs investigation. Obtaining simpler results would further enable to derive the distribution explicitly for higher values of p than 4.

3) In order to study the robustness of the sphericity test against nonnormality, tabulations of powers needs to be carried out using the results of Chapter V.

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APPENDICES

# APPENDIX A

The coefficients $c''_{ij}$ for the non-central distribution of the m$\ell$r and Roy's largest and smallest roots for p = 2 are given below in terms of the constants $A''_{ij}$ which are also provided here.

$c''_{00}=1$, $c''_{10}=A''_{11}$, $c''_{20}=3A''_{21}$, $c''_{01}=A''_{22}-4A''_{21}$, $c''_{30}-5A''_{31}$, $c''_{11}=A''_{32}-12A''_{31}$,

$c''_{40}=35A''_{41}$, $c''_{21}=3A''_{42}-120A''_{41}$, $c''_{02}=A''_{43}-4A''_{42}+48A''_{41}$, $c''_{50}=63A''_{51}$,

$c''_{31}=5A''_{52}-280A''_{51}$, $c''_{12}=A''_{53}-12A''_{52}+240A''_{51}$, $c''_{60}=231A''_{61}$,

$c''_{41}=35A''_{62}-1260A''_{61}$, $c''_{22}=3A''_{63}-120A''_{62}+1680A''_{61}$,

$c''_{03}=A''_{64}-4A''_{63}+48A''_{62}-320A''_{61}$,

where

$A''_{11} = \nu b_1/4$, $A''_{21} = \nu_{(1)}b_{21}/(8.4!)$, $A''_{22} = \nu(\nu-1)b_2/6$,

$A''_{31} = \nu_{(2)}b_{31}/(2^5.5!)$, $A''_{32} = \nu_{(1)}(\nu-1)b_1b_2/40$, $A''_{41} = 3\nu_{(3)}b_{41}/(2^7.8!)$,

$A''_{42}=\nu_{(2)}(\nu-1)b_{42}/(7.2^4.4!)$, $A''_{43}=\nu_{(1)}(\nu^2-1)b_2^2/120$, $A''_{51}=\nu_{(4)}b_{51}/(3.2^9.8!)$,

$A''_{52}=\nu_{(3)}(\nu-1)b_{52}/(3.2^5.6!)$, $A''_{53}=\nu_{(2)}(\nu^2-1)b_1b_2^2/(5.7.2^5)$,

$A''_{61}=\nu_{(5)}b_{61}/(33.2^{13}.8!)$, $A''_{62}=3\nu_{(4)}(\nu-1)b_{62}/(11.2^8.8!)$,

$A''_{63}=5\nu_{(3)}(\nu^2-1)b_{63}/(3.2^5.7!)$, $A''_{64}=\nu_{(2)}(\nu^2-1)(\nu+3)b_2^3/(5.7.9.2^4)$

where

$\nu = n_1+n_2$, $\nu_{(i)} = \nu(\nu+2)\ldots(\nu+2i)$

$b_1=2-[(1/\lambda_1)+(1/\lambda_2)]$, $b_2=[1-(1/\lambda_1)][1-(1/\lambda_2)]$,

$b_{21}-3b_1^2-4b_2$, $b_{22}=b_2$, $b_{31}=5b_1^3-12b_1b_2$, $b_{32}=b_1b_2$,

$b_{41}=35b_1^4-120b_1^2b_2+48b_2^2$, $b_{42}=b_2b_{21}$, $b_{43}=b_2^2$,

$b_{51}=63b_1^5-280b_1^3b_2+240b_1b_2^2$, $b_{52}=b_2b_{31}$, $b_{53}=b_1b_2^2$,

$b_{61}=231b_1^6-1260b_1^4b_2+1680b_1^2b_2^2-320b_2^3$, $b_{62}=b_2b_{41}$, $b_{63} = b_2^2b_{21}$,

$b_{64}=b_2^3$.

# APPENDIX B

Tabulated below are the functions $g_{ij}(z)$ appearing in the non-central density function of the m$\ell$r criterion $Z^{(2)}$ for testing the hypothesis $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$.

The following notations are used:

$$b-a = \frac{2}{\sqrt{3}} \cos(\frac{\theta}{3} + \frac{\pi}{6})$$

$$b^2-a^2 = \frac{4}{3\sqrt{3}} \cos(\frac{\theta}{3} + \frac{\pi}{6})[\cos(\frac{\theta}{3} - \frac{\pi}{3})+1] \text{ where } \cos=1- \frac{27}{2} z^{\frac{1}{n_1}}$$

$$g_{00}(z) = b-a, \; g_{10}(z) = 0.66666667(b-a)$$

$$g_{20}(z)=(0.1904762+1.37142858z^{\frac{1}{n_1}})(b-a)+(0.34285714-2.05714286z^{\frac{1}{n_1}})(b^2-a^2)$$

$$g_{01}(z)=(-0.33333333z^{\frac{-2}{n_1}})(b-a)+(0.33333333z^{\frac{-2}{n_1}})(b^2-a^2)$$

$$g_{30}(z)=(-0.51948052+3.94805195z^{\frac{1}{n_1}}+3.36623377z^{\frac{2}{n_1}})(b-a)$$

$$+(0.97662338-5.79740261z^{\frac{1}{n_1}})(b^2-a^2)$$

$$g_{11}(z)=(-0.47619047z^{\frac{-2}{n_1}}+0.57142858z^{\frac{-1}{n_1}})(b-a)$$

$$+(0.47619047z^{\frac{-2}{n_1}}-0.85714286z^{\frac{-1}{n_1}})(b^2-a^2)$$

$$g_{40}(z) = (-1.6207792+6.49696986z^{\frac{1}{n_1}}+13.46493514z^{\frac{2}{n_1}})(b-a)$$

$$+ (2.02712842-10.84675336z^{\frac{1}{n_1}})(b^2-a^2)$$

$$g_{21}(z) = (-0.7099567z^{\frac{-2}{n_1}}+0.63376626z^{\frac{-1}{n_1}}+1.3090909)(b-a)$$

$$+ (0.7099567z^{\frac{-2}{n_1}}-1.56883118z^{\frac{-1}{n_1}})(b^2-a^2)$$

$$g_{02}(z) = (-0.14285714z^{\frac{-4}{n_1}}-0.42857142z^{\frac{-3}{n_1}})(b-a)$$

$$+ (0.14285714z^{\frac{-4}{n_1}}+0.14285714z^{\frac{-3}{n_1}})(b^2-a^2)$$

$$g_{50}(z) = (-3.3902939+5.29940273z^{\frac{1}{n_1}}+33.8642416z^{\frac{2}{n_1}}+6.54166889z^{\frac{3}{n_1}})(b-a)$$

$$+ (3.75970227-15.10573663z^{\frac{1}{n_1}}-0.48456834z^{\frac{2}{n_1}}-9.81250329z^{\frac{3}{n_1}})(b^2-a^2)$$

$$g_{31}(z) = (-1.10129869z^{\frac{-2}{n_1}}-0.87965352z^{\frac{-1}{n_1}}+3.92727281)(b-a)$$

$$+ (1.10129869z^{\frac{-2}{n_1}}-1.46839838z^{\frac{-1}{n_1}})(b^2-a^2)$$

$$g_{12}(z) = (-0.23376623z^{\frac{-4}{n_1}}-1.13766232z^{\frac{-3}{n_1}}+0.10909091z^{\frac{-2}{n_1}})(b-a)$$

$$+ (0.23376623z^{\frac{-4}{n_1}}+0.48831168z^{\frac{-3}{n_1}})(b^2-a^2)$$

$$g_{60}(z) = (-6.32030669-10.47471613z^{\frac{1}{n_1}}+73.91738429z^{\frac{2}{n_1}}+35.12960155z^{\frac{3}{n_1}}$$

$$+ 17.16560576z^{\frac{4}{n_1}})(b-a)$$

$$+(7.75220811-19.74911937z^{\frac{1}{n_1}}+8.19005658z^{\frac{2}{n_1}}-58.05863903z^{\frac{3}{n_1}})(b^2-a^2)$$

$$g_{41}(z)=(-1.76951467z^{\frac{-2}{n_1}}-6.62254186z^{\frac{-1}{n_1}}+9.642863+7.54146909z^{\frac{1}{n_1}})(b-a)$$

$$+(1.76951467z^{\frac{-2}{n_1}}+1.27681404z^{\frac{-1}{n_1}}-1.89196093-5.31220359z^{\frac{1}{n_1}})(b^2-a^2)$$

$$g_{22}(z)=(-0.391134198z^{\frac{-4}{n_1}}-3.02453086z^{\frac{-3}{n_1}}+0.21818191z^{\frac{-2}{n_1}})(b-a)$$

$$+(0.39134198z^{\frac{-4}{n_1}}+1.6115439z^{\frac{-3}{n_1}})(b^2-a^2)$$

$$g_{03}(z)=(-0.09090909z^{\frac{-6}{n_1}}-0.9090909z^{\frac{-5}{n_1}}-0.09090909z^{\frac{-4}{n_1}})(b-a)$$

$$+(0.09090909z^{\frac{-6}{n_1}}+0.54545454z^{\frac{-5}{n_1}})(b^2-a^2).$$

# APPENDIX C

Powers of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and $LS^{(2)}$ in the unbiased and equal tail areas cases for testing $\lambda_1=1$, $\lambda_2=1$ against different simple two-sided alternative hypotheses, $\alpha = .05$, (supplementing Table 4 of Chapter I).

Table 4 (continued)

m = 1, n = 5

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050003 | .050006 | .050013 | .050010 |
| 1 | 1.1 | .050809 | .050843 | .050736 | .050820 | .050629 | .051092 | .051480 | .052000 | .051855 |
| 1.05 | 1.05 | .050621 | .050715 | .050798 | .050772 | .050374 | .050902 | .051354 | .052084 | .051817 |
| 1 | 1.5 | .0672 | .0676 | .0641 | .0667 | .0633 | .0686 | .0706 | .0699 | .0716 |
| 1.25 | 1.25 | .0632 | .0654 | .0673 | .0668 | .0585 | .0646 | .0685 | .0735 | .0718 |
| 1 | 2 | .1099 | .1103 | .0960 | .1079 | .0979 | .1125 | .1160 | .1064 | .1169 |
| 1.333 | 1.5 | .0835 | .0892 | .0933 | .0927 | .0729 | .0857 | .0942 | .1033 | .1009 |
| 1 | 4 | .384 | .383 | .342 | | .354 | .388 | .392 | .357 | |
| 1 | 5 | .508 | .506 | .454 | | .480 | .512 | .515 | .469 | |
| 2 | 4 | .489 | .517 | .520 | | .451 | .493 | .527 | .539 | |
| 3 | 3 | .486 | .523 | .541 | | .446 | .491 | .534 | .561 | |
| 1 | 8 | .740 | .739 | .701 | | .722 | .743 | .745 | .711 | |
| 4.5 | 4.5 | .779 | .803 | .812 | | .756 | .782 | .809 | .822 | |
| 1 | 11 | .849 | .848 | .822 | | .837 | .851 | .851 | .829 | |
| 6 | 6 | .903 | .915 | .919 | | .892 | .905 | .918 | .924 | |
| 1.00001 | 0.9 | .050895 | .050948 | .050867 | .050934 | .050713 | .050603 | .050283 | .049517 | .049846 |
| 1.00001 | 0.8 | .054072 | .054335 | .054124 | .054313 | .053393 | .053469 | .052957 | .051302 | .052053 |
| 1.01 | 0.99 | .050008 | .050006 | .050001 | .050003 | .050011 | .050008 | .050005 | .049997 | .050002 |
| 1.1 | 0.9 | .050792 | .050581 | .050084 | .050275 | .051132 | .050799 | .050572 | .049772 | .050237 |
| 1.1 | 0.8 | .052727 | .052434 | .051023 | .051868 | .053447 | .052445 | .051743 | .049523 | .050689 |
| 1.05 | 0.95 | .050198 | .050145 | .050021 | .050068 | .050281 | .050200 | .050142 | .049942 | .050058 |
| 1.2 | 0.99 | .0529 | .0530 | .0524 | .0528 | .0523 | .0535 | .0541 | .0547 | .0547 |
| 1.2 | 0.8 | .0527 | .0519 | .0504 | .0506 | .0541 | .0530 | .0518 | .0486 | .0505 |
| 2 | 0.9 | .103 | .101 | .080 | .095 | .095 | .106 | .106 | .089 | .103 |
| 2 | 0.7 | .119 | .112 | .088 | .102 | .119 | .120 | .116 | .093 | .107 |
| 3 | 0.9 | .22 | .22 | .17 | .22 | .20 | .23 | .23 | .18 | .23 |
| 5 | 0.9 | .49 | .48 | .41 | .49 | .46 | .49 | .49 | .43 | .51 |
| 0.99999 | 0.9 | .050895 | .050948 | .050867 | .050934 | .050715 | .050603 | .050283 | .049518 | .049846 |
| 0.99999 | 0.7 | .0842 | .0850 | .0864 | .0859 | .0850 | .0829 | .0822 | .0830 | .0814 |
| 0.999 | 0.9 | .050905 | .050961 | .050885 | .050949 | .050715 | .050681 | .050289 | .049523 | .049851 |
| 0.9 | 0.9 | .052897 | .053295 | .053641 | .053526 | .051650 | .052292 | .051939 | .050939 | .051326 |
| 0.9 | 0.8 | .058288 | .059138 | .059822 | .059575 | .055503 | .057339 | .057009 | .055588 | .056128 |
| 0.9 | 0.76 | .0656 | .0666 | .0677 | .0671 | .0625 | .0645 | .0640 | .0626 | .629 |
| 0.85 | 0.9 | .054819 | .055437 | .055943 | .055778 | .052833 | .054050 | .053714 | .052515 | .052986 |
| 0.85 | 0.8 | .0630 | .0640 | .0649 | .0646 | .0589 | .0618 | .0614 | .0599 | .0604 |
| 0.81 | 0.9 | .057340 | .058146 | .058785 | .058566 | .054707 | .056430 | .056105 | .054724 | .055260 |
| 0.8 | 0.8 | .0747 | .0755 | .0763 | .0758 | .0697 | .0732 | .0723 | .0701 | .0707 |

Table 4 (continued)

m = 1, n = 15

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050017 | .050023 | .050024 | .050024 |
| 1 | 1.1 | .051214 | .051212 | .051107 | .051172 | .050891 | .052949 | .053500 | .053544 | .053568 |
| 1.05 | 1.05 | .050901 | .051037 | .051058 | .051053 | .050536 | .052614 | .053326 | .053512 | .053458 |
| 1 | 1.5 | .0770 | .0763 | .0732 | .0752 | .0704 | .0857 | .0875 | .0851 | .0869 |
| 1.25 | 1.25 | .0698 | .0732 | .0737 | .0736 | .0628 | .0782 | .0845 | .0858 | .0855 |
| 1 | 2 | .1431 | .1394 | .1283 | .1377 | .1242 | .1595 | .1602 | .1503 | .1595 |
| 1.333 | 1.5 | .1011 | .1095 | .1103 | .1107 | .0853 | .1148 | .1280 | .1300 | .1301 |
| 1 | 4 | .475 | .463 | .431 | | .436 | .499 | .493 | .464 | |
| 1 | 5 | .597 | .585 | .553 | | .561 | .618 | .612 | .584 | |
| 2 | 4 | .581 | .615 | .612 | | .535 | .606 | .645 | .644 | |
| 3 | 3 | .581 | .627 | .630 | | .532 | .607 | .658 | .662 | |
| 1 | 8 | .802 | .794 | .771 | | .779 | .814 | .810 | .790 | |
| 4.5 | 4.5 | .837 | .864 | .865 | | .811 | .850 | .878 | .880 | |
| 1 | 11 | .888 | .883 | .869 | | .874 | .896 | .893 | .880 | |
| 6 | 6 | .932 | .945 | .945 | | .920 | .938 | .952 | .953 | |
| 1.00001 | 0.9 | .051296 | .051320 | .051230 | .051289 | .050955 | .049569 | .049005 | .048740 | .048852 |
| 1.00001 | 0.8 | .055576 | .055766 | .055478 | .055670 | .054249 | .052086 | .051048 | .050391 | .050691 |
| 1.01 | 0.99 | .050013 | .050008 | .050003 | .050006 | .050015 | .050013 | .050007 | .050002 | .050005 |
| 1.1 | 0.9 | .051270 | .050773 | .050303 | .050562 | .051513 | .051358 | .050764 | .050227 | .050523 |
| 1.1 | 0.8 | .053974 | .053157 | .052248 | .052757 | .054365 | .052408 | .050798 | .049565 | .050213 |
| 1.05 | 0.95 | .050318 | .050193 | .050075 | .050140 | .050376 | .050340 | .050190 | .050056 | .050130 |
| 1.2 | 0.99 | .0545 | .0543 | .0538 | .0541 | .0534 | .0578 | .0586 | .0584 | .0586 |
| 1.2 | 0.8 | .0546 | .0527 | .0508 | .0519 | .0557 | .0550 | .0527 | .0505 | .0518 |
| 2 | 0.9 | .136 | .127 | .112 | .123 | .121 | .151 | .146 | .132 | .143 |
| 2 | 0.7 | .142 | .126 | .109 | .118 | .138 | .153 | .139 | .121 | .131 |
| 3 | 0.9 | .30 | .28 | .25 | .30 | .27 | .33 | .31 | .28 | .33 |
| 5 | 0.9 | .58 | .56 | .52 | .63 | .55 | .60 | .59 | .56 | .66 |
| 0.99999 | 0.9 | .051296 | .051320 | .051230 | .051290 | .050955 | .049569 | .049005 | .048740 | .048852 |
| 0.99999 | 0.7 | .0803 | .0820 | .0842 | .0826 | .0805 | .0734 | .0727 | .0740 | .0728 |
| 0.999 | 0.9 | .051309 | .051339 | .051257 | .051310 | .050959 | .049564 | .049000 | .048737 | .048848 |
| 0.9 | 0.9 | .054065 | .054599 | .054693 | .054669 | .052243 | .050512 | .049931 | .049697 | .049772 |
| 0.9 | 0.8 | .060781 | .061865 | .062031 | .061975 | .056629 | .055287 | .054652 | .054303 | .054407 |
| 0.9 | 0.76 | .0677 | .0690 | .0694 | .0691 | .0629 | .0611 | .0603 | .0602 | .0600 |
| 0.85 | 0.9 | .056662 | .057470 | .057595 | .057566 | .053743 | .052176 | .051584 | .051295 | .051392 |
| 0.85 | 0.8 | .0659 | .0673 | .0675 | .0674 | .0601 | .0593 | .0586 | .0583 | .0584 |
| 0.81 | 0.9 | .059738 | .060769 | .060919 | .060876 | .055827 | .054460 | .053841 | .053498 | .053608 |
| 0.8 | 0.8 | .0771 | .0780 | .0782 | .0780 | .0699 | .0689 | .0674 | .0670 | .0669 |

Table 4 (continued)

m = 1, n = 30

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050024 | .050029 | .050030 | .050029 |
| 1 | 1.1 | .051400 | .051364 | .051294 | .051333 | .051014 | .053778 | .054312 | .054285 | .054316 |
| 1.05 | 1.05 | .051027 | .051170 | .051177 | .051176 | .050613 | .053370 | .054119 | .054179 | .054163 |
| 1 | 1.5 | .0816 | .0799 | .0779 | .0792 | .0739 | .0937 | .0945 | .0927 | .0939 |
| 1.25 | 1.25 | .0729 | .0765 | .0766 | .0766 | .0649 | .0845 | .0912 | .0916 | .0915 |
| 1 | 2 | .1582 | .1518 | .1440 | .1518 | .1368 | .1805 | .1784 | .1715 | .1788 |
| 1.333 | 1.5 | .1092 | .1181 | .1182 | .1188 | .0914 | .1281 | .1420 | .1425 | .1430 |
| 1 | 4 | .513 | .495 | .471 | | .472 | .542 | .531 | .512 | |
| 1 | 5 | .633 | .616 | .589 | | .596 | .658 | .647 | .627 | |
| 2 | 4 | .619 | .651 | .646 | | .571 | .649 | .685 | .682 | |
| 3 | 3 | .620 | .665 | .663 | | .570 | .652 | .700 | .700 | |
| 1 | 8 | .825 | .814 | .794 | | .803 | .840 | .832 | .818 | |
| 4.5 | 4.5 | .859 | .884 | .882 | | .834 | .874 | .900 | .900 | |
| 1 | 11 | .903 | .896 | .883 | | .889 | .912 | .907 | .898 | |
| 6 | 6 | .943 | .955 | .954 | | .931 | .949 | .961 | .961 | |
| | | | | | | | | | | |
| 1.00001 | 0.9 | .051471 | .051469 | .051412 | .051444 | .051057 | .049138 | .048517 | .048397 | .048450 |
| 1.00001 | 0.8 | .056216 | .056323 | .056150 | .056242 | .054589 | .051544 | .050348 | .050019 | .050168 |
| 1.01 | 0.99 | .050015 | .050008 | .050005 | .050007 | .050016 | .050016 | .050008 | .050005 | .050006 |
| 1.1 | 0.9 | .051491 | .050841 | .050555 | .050706 | .051679 | .051636 | .050839 | .050509 | .050684 |
| 1.1 | 0.8 | .054523 | .053417 | .052865 | .053163 | .054736 | .052463 | .050431 | .049719 | .050090 |
| 1.05 | 0.95 | .050373 | .050210 | .050138 | .050176 | .050418 | .050410 | .050209 | .050127 | .050170 |
| 1.2 | 0.99 | .0552 | .0549 | .0546 | .0547 | .0539 | .0598 | .0605 | .0602 | .0604 |
| 1.2 | 0.8 | .0556 | .0530 | .0518 | .0526 | .0564 | .0562 | .0530 | .0517 | .0525 |
| 2 | 0.9 | .151 | .138 | .130 | .102 | .133 | .172 | .162 | .153 | .110 |
| 2 | 0.7 | .153 | .132 | .121 | .127 | .147 | .169 | .148 | .137 | .143 |
| 3 | 0.9 | .33 | .31 | .29 | .34 | .30 | .37 | .35 | .33 | .37 |
| 5 | 0.9 | .62 | .60 | .56 | .70 | .58 | .65 | .63 | .60 | .73 |
| | | | | | | | | | | |
| 0.99999 | 0.9 | .051472 | .051470 | .051412 | .051445 | .051057 | .049138 | .048517 | .048396 | .048450 |
| 0.99999 | 0.7 | .0785 | .0806 | .0823 | .0809 | .0782 | .0695 | .0691 | .0702 | .0691 |
| 0.999 | 0.9 | .051487 | .051490 | .051434 | .051466 | .051061 | .049129 | .048509 | .048389 | .048442 |
| 0.9 | 0.9 | .054569 | .055126 | .055157 | .055149 | .052493 | .049780 | .049200 | .049127 | .049149 |
| 0.9 | 0.8 | .061829 | .062938 | .062992 | .062958 | .057070 | .054476 | .053844 | .053726 | .053744 |
| 0.9 | 0.76 | .0686 | .0699 | .0701 | .0699 | .0630 | .0598 | .0591 | .0591 | .0589 |
| 0.85 | 0.9 | .057449 | .058283 | .058318 | .058311 | .054121 | .051421 | .050833 | .050737 | .050767 |
| 0.85 | 0.8 | .0672 | .0685 | .0687 | .0686 | .0605 | .0583 | .0577 | .0576 | .0576 |
| 0.81 | 0.9 | .060750 | .061805 | .061849 | .061828 | .056275 | .053680 | .053062 | .052944 | .052971 |
| 0.8 | 0.8 | .0780 | .0789 | .0791 | .0788 | .0698 | .0672 | .0657 | .0656 | .0654 |

Table 4 (continued)

m = 1, n = 60

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050027 | .050033 | .050033 | .050033 |
| 1 | 1.1 | .051518 | .051455 | .051413 | .051435 | .051089 | .054294 | .054788 | .054757 | .054778 |
| 1.05 | 1.05 | .051106 | .051251 | .051253 | .051253 | .050658 | .053838 | .054583 | .054601 | .054596 |
| 1 | 1.5 | .0845 | .0822 | .0811 | .0818 | .0761 | .0986 | .0987 | .0976 | .0983 |
| 1.25 | 1.25 | .0749 | .0785 | .0786 | .0786 | .0663 | .0884 | .0952 | .0953 | .0953 |
| 1 | 2 | .1677 | .1592 | .1554 | .1607 | .1448 | .1933 | .1891 | .1855 | .1907 |
| 1.333 | 1.5 | .1144 | .1234 | .1234 | .1239 | .0953 | .1363 | .1503 | .1504 | .1510 |
| 1 | 4 | .534 | .513 | .504 | | .494 | .566 | .552 | .544 | |
| 1 | 5 | .653 | .634 | .625 | | .617 | .680 | .667 | .659 | |
| 2 | 4 | .641 | .671 | .669 | | .594 | .673 | .707 | .706 | |
| 3 | 3 | .643 | .686 | .686 | | .593 | .677 | .722 | .722 | |
| 1 | 8 | .838 | .826 | .820 | | .817 | .853 | .844 | .840 | |
| 4.5 | 4.5 | .872 | .895 | .895 | | .848 | .887 | .910 | .911 | |
| 1 | 11 | .911 | .904 | .900 | | .898 | .920 | .914 | .912 | |
| 6 | 6 | .949 | .960 | .960 | | .938 | .956 | .966 | .967 | |
| 1.00001 | 0.9 | .051580 | .051556 | .051522 | .051540 | .051122 | .048880 | .048239 | .048184 | .048210 |
| 1.00001 | 0.8 | .056609 | .056644 | .056538 | .056591 | .054798 | .051229 | .049956 | .049791 | .049866 |
| 1.01 | 0.99 | .050016 | .050009 | .050007 | .050008 | .050017 | .050018 | .050008 | .050006 | .050007 |
| 1.1 | 0.9 | .051631 | .050878 | .050719 | .050801 | .051781 | .051816 | .050882 | .050698 | .050793 |
| 1.1 | 0.8 | .054864 | .053561 | .053251 | .053418 | .054962 | .052518 | .050220 | .049833 | .050037 |
| 1.05 | 0.95 | .050408 | .050219 | .050179 | .050200 | .050443 | .050455 | .050220 | .050174 | .050198 |
| 1.2 | 0.99 | .0557 | .0553 | .0551 | .0552 | .0542 | .0610 | .0616 | .0614 | .0615 |
| 1.2 | 0.8 | .0562 | .0532 | .0525 | .0530 | .0569 | .0571 | .0532 | .0525 | .0530 |
| 2 | 0.9 | .160 | .145 | .140 | .138 | .141 | .184 | .172 | .167 | .131 |
| 2 | 0.7 | .160 | .135 | .130 | .132 | .152 | .179 | .154 | .148 | .150 |
| 3 | 0.9 | .35 | .33 | .32 | .36 | .32 | .39 | .37 | .36 | .41 |
| 5 | 0.9 | .64 | .61 | .61 | .70 | .61 | .67 | .65 | .64 | .71 |
| 0.99999 | 0.9 | .051581 | .051557 | .051523 | .051541 | .051122 | .048879 | .048239 | .048184 | .048209 |
| 0.99999 | 0.7 | .0775 | .0798 | .0804 | .0798 | .0767 | .0674 | .0671 | .0675 | .0670 |
| 0.999 | 0.9 | .051596 | .051579 | .051546 | .051563 | .051127 | .048867 | .048228 | .048174 | .048199 |
| 0.9 | 0.9 | .054878 | .055436 | .055445 | .055443 | .052653 | .049343 | .048794 | .048774 | .048780 |
| 0.9 | 0.8 | .062468 | .063562 | .063569 | .063552 | .057341 | .054007 | .053411 | .053365 | .053363 |
| 0.9 | 0.76 | .0691 | .0705 | .0705 | .0704 | .0631 | .0590 | .0584 | .0584 | .0582 |
| 0.85 | 0.9 | .057930 | .058758 | .058765 | .058764 | .054358 | .050976 | .050423 | .050392 | .050401 |
| 0.85 | 0.8 | .0680 | .0693 | .0693 | .0693 | .0608 | .0578 | .0572 | .0572 | .0571 |
| 0.81 | 0.8 | .061367 | .062408 | .062414 | .062404 | .056552 | .053226 | .052644 | .052600 | .052604 |
| 0.8 | 0.8 | .0786 | .0795 | .0795 | .0794 | .0697 | .0661 | .0648 | .0648 | .0646 |

Table 4 (continued)

m = 2, n = 5

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .049997 | .050001 | .050009 | .050006 |
| 1 | 1.1 | .050968 | .051036 | .050932 | .051015 | .050857 | .050652 | .051100 | .051805 | .051613 |
| 1.05 | 1.05 | .050760 | .050905 | .051035 | .050994 | .050516 | .050447 | .050969 | .051926 | .051598 |
| 1 | 1.5 | .0708 | .0716 | .0677 | .0707 | .0678 | .0693 | .0719 | .0716 | .0734 |
| 1.25 | 1.25 | .0660 | .0693 | .0724 | .0715 | .0615 | .0646 | .0696 | .0766 | .0744 |
| 1 | 2 | .1259 | .1271 | .1106 | .1250 | .1164 | .1231 | .1276 | .1173 | .1301 |
| 1.333 | 1.5 | .0907 | .0991 | .1060 | .1049 | .0808 | .0884 | .0996 | .1128 | .1096 |
| 1 | 4 | .499 | .499 | .458 | | .481 | .495 | .499 | .463 | |
| 1 | 5 | .645 | .644 | .605 | | .630 | .642 | .645 | .612 | |
| 2 | 4 | .620 | .648 | .656 | | .598 | .617 | .649 | .655 | |
| 3 | 3 | .622 | .657 | .679 | | .599 | .618 | .658 | .688 | |
| 1 | 8 | .864 | .864 | .844 | | .858 | .863 | .864 | .847 | |
| 4.5 | 4.5 | .397 | .910 | .918 | | .889 | .896 | .911 | .920 | |
| 1 | 11 | .938 | .938 | .929 | | .935 | .938 | .938 | .930 | |
| 6 | 6 | .970 | .974 | .976 | | .967 | .970 | .974 | .977 | |
| 1.00001 | 0.9 | .051071 | .051171 | .051111 | .051168 | .050998 | .051401 | .051103 | .050167 | .050532 |
| 1.00001 | 0.8 | .056131 | .056628 | .056651 | .056697 | .056080 | .056824 | .056485 | .054644 | .055351 |
| 1.01 | 0.99 | .050008 | .050006 | .050001 | .050002 | .050015 | .050008 | .050006 | .049996 | .050001 |
| 1.1 | 0.9 | .050859 | .050597 | .050097 | .050185 | .051551 | .050848 | .050596 | .049665 | .050159 |
| 1.1 | 0.8 | .053395 | .053112 | .051630 | .052403 | .055185 | .053709 | .053041 | .050521 | .051703 |
| 1.05 | 0.95 | .050215 | .050149 | .050022 | .050046 | .050385 | .050213 | .050149 | .049915 | .050039 |
| 1.2 | 0.99 | .0535 | .0536 | .0530 | .0534 | .0531 | .0529 | .0537 | .0546 | .0545 |
| 1.2 | 0.8 | .0509 | .0510 | .0508 | .0510 | .0537 | .0529 | .0499 | .0462 | .0489 |
| 2 | 0.9 | .117 | .115 | .090 | .107 | .112 | .115 | .115 | .096 | .111 |
| 2 | 0.7 | .155 | .147 | .121 | .136 | .161 | .150 | .148 | .124 | .135 |
| 3 | 0.9 | .29 | .30 | .23 | .28 | .27 | .28 | .28 | .24 | .29 |
| 5 | 0.9 | .62 | .62 | .57 | .63 | .61 | .62 | .62 | .57 | .64 |
| 0.99999 | 0.9 | .051071 | .051171 | .051111 | .051168 | .050998 | .051401 | .051103 | .050168 | .050532 |
| 0.99999 | 0.7 | .225 | .226 | .231 | .230 | .233 | .227 | .226 | .235 | .226 |
| 0.999 | 0.9 | .051084 | .051188 | .051135 | .051189 | .051001 | .051417 | .051120 | .050183 | .050547 |
| 0.9 | 0.9 | .053676 | .054299 | .054833 | .054661 | .052406 | .054365 | .054161 | .052938 | .053369 |
| 0.9 | 0.8 | .068291 | .069378 | .070446 | .069948 | .065952 | .069438 | .069147 | .067296 | .067795 |
| 0.9 | 0.76 | .1111 | .1121 | .1142 | .1125 | .1107 | .1134 | .1118 | .1098 | .1094 |
| 0.85 | 0.9 | .056841 | .057785 | .058562 | .058305 | .054881 | .057725 | .057607 | .056134 | .056650 |
| 0.85 | 0.8 | .0927 | .0931 | .0936 | .0932 | .0897 | .0942 | .0928 | .0895 | .0903 |
| 0.81 | 0.9 | .064301 | .065405 | .066389 | .065983 | .061961 | .065382 | .065187 | .063420 | .063956 |
| 0.8 | 0.8 | .1845 | .1805 | .1771 | .1774 | .1839 | .1869 | .1800 | .1705 | .1729 |

Table 4 (continued)

m = 2, n = 15

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050013 | .050019 | .050022 | .050021 |
| 1 | 1.1 | .051552 | .051574 | .051430 | .051519 | .051271 | .052801 | .053516 | .053590 | .053620 |
| 1.05 | 1.05 | .051168 | .051390 | .051429 | .051420 | .050768 | .052397 | .053333 | .053610 | .053532 |
| 1 | 1.5 | .0850 | .0842 | .0796 | .0826 | .0788 | .0914 | .0937 | .0900 | .0928 |
| 1.25 | 1.25 | .0755 | .0809 | .0819 | .0817 | .0681 | .0815 | .0905 | .0926 | .0922 |
| 1 | 2 | .1729 | .1680 | .1509 | .1667 | .1552 | .1846 | .1853 | .1698 | .1855 |
| 1.333 | 1.5 | .1158 | .1293 | .1310 | .1319 | .0997 | .1255 | .1449 | .1484 | .1488 |
| 1 | 4 | .594 | .581 | .560 | | .567 | .607 | .600 | .572 | |
| 1 | 5 | .725 | .714 | .681 | | .703 | .735 | .729 | .699 | |
| 2 | 4 | .707 | .742 | .740 | | .678 | .719 | .759 | .759 | |
| 3 | 3 | .710 | .756 | .760 | | .679 | .722 | .773 | .778 | |
| 1 | 8 | .902 | .897 | .881 | | .893 | .906 | .903 | .889 | |
| 4.5 | 4.5 | .929 | .945 | .946 | | .919 | .933 | .949 | .951 | |
| 1 | 11 | .957 | .954 | .947 | | .952 | .959 | .957 | .950 | |
| 6 | 6 | .980 | .985 | .985 | | .977 | .981 | .986 | .987 | |
| 1.00001 | 0.9 | .051644 | .051719 | .051611 | .051681 | .051387 | .050399 | .049742 | .049378 | .049526 |
| 1.00001 | 0.8 | .058015 | .058529 | .058315 | .058460 | .057274 | .055449 | .054414 | .053666 | .053962 |
| 1.01 | 0.99 | .050015 | .050008 | .050001 | .050005 | .050021 | .050015 | .050008 | .050000 | .050004 |
| 1.1 | 0.9 | .051510 | .050813 | .050149 | .050502 | .052170 | .051593 | .050805 | .050058 | .050457 |
| 1.1 | 0.8 | .054999 | .053989 | .052892 | .053444 | .056646 | .053896 | .051954 | .050348 | .051155 |
| 1.05 | 0.95 | .050378 | .050203 | .050037 | .050125 | .050539 | .050399 | .050201 | .050014 | .050114 |
| 1.2 | 0.99 | .0557 | .0556 | .0549 | .0553 | .0548 | .0582 | .0593 | .0589 | .0592 |
| 1.3 | 0.8 | .0538 | .0511 | .0502 | .0507 | .0563 | .0542 | .0511 | .0481 | .0506 |
| 2 | 0.9 | .164 | .151 | .128 | .145 | .150 | .175 | .166 | .145 | .162 |
| 2 | 0.7 | .184 | .163 | .140 | .152 | .186 | .190 | .173 | .149 | .160 |
| 3 | 0.9 | .38 | .36 | .32 | .38 | .36 | .40 | .38 | .34 | .40 |
| 5 | 0.9 | .71 | .69 | .65 | .75 | .69 | .72 | .71 | .67 | .77 |
| 0.99999 | 0.9 | .051644 | .051720 | .051612 | .051682 | .051387 | .050399 | .049742 | .049378 | .049526 |
| 0.99999 | 0.7 | .192 | .198 | .213 | .203 | .203 | .182 | .184 | .197 | .187 |
| 0.999 | 0.9 | .051663 | .051746 | .051645 | .051710 | .051392 | .050404 | .049748 | .049387 | .049533 |
| 0.9 | 0.9 | .055401 | .056286 | .056447 | .056405 | .053337 | .052810 | .052272 | .051957 | .052054 |
| 0.9 | 0.8 | .070673 | .072146 | .072611 | .072345 | .066687 | .066391 | .065517 | .065194 | .065158 |
| 0.9 | 0.76 | .1075 | .1083 | .1103 | .1087 | .1055 | .1014 | .0987 | .0996 | .0984 |
| 0.85 | 0.9 | .059454 | .060758 | .060984 | .060920 | .056228 | .056149 | .055644 | .055262 | .055376 |
| 0.85 | 0.8 | .0932 | .0934 | .0937 | .0934 | .0883 | .0875 | .0846 | .0840 | .0839 |
| 0.81 | 0.9 | .067021 | .068513 | .068877 | .068699 | .063072 | .062989 | .062272 | .061893 | .061933 |
| 0.8 | 0.8 | .1732 | .1658 | .1650 | .1645 | .1720 | .1640 | .1516 | .1492 | .1491 |

Table 4 (continued)

m = 2, n = 30

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050020 | .050027 | .050028 | .050028 |
| 1 | 1.1 | .051842 | .051818 | .051706 | .051766 | .051477 | .053855 | .054562 | .054523 | .054568 |
| 1.05 | 1.05 | .051367 | .051614 | .051622 | .051619 | .050895 | .053340 | .054358 | .054452 | .054428 |
| 1 | 1.5 | .0923 | .0900 | .0865 | .0888 | .0845 | .1026 | .1036 | .0997 | .1026 |
| 1.25 | 1.25 | .0803 | .0863 | .0866 | .0866 | .0716 | .0900 | .1001 | .1007 | .1007 |
| 1 | 2 | .1963 | .1868 | .1706 | .1882 | .1753 | .2149 | .2110 | .2037 | .2131 |
| 1.333 | 1.5 | .1285 | .1433 | .1433 | .1448 | .1096 | .1442 | .1654 | .1653 | .1674 |
| 1 | 4 | .636 | .617 | .600 | | .607 | .655 | .642 | .629 | |
| 1 | 5 | .759 | .743 | .689 | | .737 | .773 | .762 | .738 | |
| 2 | 4 | .745 | .778 | .767 | | .715 | .762 | .799 | .770 | |
| 3 | 3 | .748 | .792 | .792 | | .716 | .765 | .812 | .812 | |
| 1 | 8 | .917 | .910 | .880 | | .908 | .923 | .918 | .906 | |
| 4.5 | 4.5 | .942 | .956 | .954 | | .932 | .947 | .961 | .961 | |
| 1 | 11 | .964 | .961 | .945 | | .960 | .967 | .964 | .960 | |
| 6 | 6 | .984 | .988 | .988 | | .981 | .986 | .990 | .990 | |
| 1.00001 | 0.9 | .051915 | .051949 | .051875 | .051921 | .051560 | .049946 | .049189 | .049016 | .049091 |
| 1.00001 | 0.8 | .058864 | .059297 | .059276 | .059230 | .057746 | .054849 | .053598 | .053378 | .053372 |
| 1.01 | 0.99 | .050018 | .050009 | .050004 | .050007 | .050024 | .050020 | .050008 | .050003 | .050006 |
| 1.1 | 0.9 | .051840 | .050890 | .050460 | .050684 | .052455 | .052000 | .050886 | .050397 | .050652 |
| 1.1 | 0.8 | .055763 | .054309 | .053626 | .053956 | .057264 | .054078 | .051488 | .050607 | .051006 |
| 1.05 | 0.95 | .050461 | .050222 | .050114 | .050171 | .050610 | .050502 | .050221 | .050099 | .050163 |
| 1.2 | 0.99 | .0569 | .0565 | .0560 | .0562 | .0556 | .0608 | .0617 | .0613 | .0615 |
| 1.2 | 0.8 | .0552 | .0515 | .0502 | .0517 | .0570 | .0575 | .0516 | .0498 | .0517 |
| 2 | 0.9 | .187 | .167 | .141 | .165 | .170 | .204 | .189 | .220 | .188 |
| 2 | 0.7 | .199 | .170 | .155 | .162 | .199 | .211 | .184 | .167 | .176 |
| 3 | 0.9 | .43 | .40 | .38 | .43 | .40 | .45 | .43 | .42 | .46 |
| 5 | 0.9 | .75 | .72 | .70 | .82 | .72 | .76 | .74 | .71 | .84 |
| 0.99999 | 0.9 | .051916 | .051949 | .051876 | .051921 | .051560 | .049946 | .049189 | .049016 | .049091 |
| 0.99999 | 0.7 | .175 | .185 | .203 | .188 | .187 | .160 | .166 | .187 | .168 |
| 0.999 | 0.9 | .051937 | .051979 | .051908 | .051953 | .051566 | .049946 | .049192 | .049021 | .049094 |
| 0.9 | 0.9 | .056200 | .057130 | .057200 | .057184 | .053753 | .052112 | .051562 | .051463 | .051490 |
| 0.9 | 0.8 | .071670 | .073227 | .073722 | .073307 | .066852 | .064965 | .064108 | .064407 | .063973 |
| 0.9 | 0.76 | .1051 | .1062 | .1089 | .1065 | .1023 | .0955 | .0932 | .0961 | .0931 |
| 0.85 | 0.9 | .060647 | .062006 | .062119 | .062072 | .056808 | .055448 | .054934 | .054835 | .054839 |
| 0.85 | 0.8 | .0932 | .0933 | .0940 | .0933 | .0873 | .0843 | .0813 | .0818 | .0810 |
| 0.81 | 0.9 | .068206 | .069767 | .070102 | .069839 | .063459 | .061887 | .061170 | .061288 | .061042 |
| 0.8 | 0.8 | .1671 | .1588 | .1599 | .1581 | .1654 | .1527 | .1395 | .1410 | .1384 |

Table 4 (continued)

m = 2, n = 60

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property: $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas: $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050025 | .050033 | .050032 | .050032 |
| 1 | 1.1 | .052034 | .051961 | .051894 | .051929 | .051608 | .054541 | .055196 | .055146 | .055179 |
| 1.05 | 1.05 | .051496 | .051745 | .051748 | .051747 | .050975 | .053950 | .054980 | .055007 | .055000 |
| 1 | 1.5 | .0972 | .0937 | .0917 | .0930 | .0884 | .1100 | .1097 | .1078 | .1091 |
| 1.25 | 1.25 | .0835 | .0897 | .0898 | .0899 | .0738 | .0956 | .1059 | .1061 | .1061 |
| 1 | 2 | .2114 | .1984 | .1918 | .2023 | .1885 | .2342 | .2266 | .2203 | .2309 |
| 1.333 | 1.5 | .1368 | .1520 | .1520 | .1532 | .1161 | .1564 | .1779 | .1781 | .1793 |
| 1 | 4 | .661 | .637 | .625 | | .632 | .683 | .665 | .654 | |
| 1 | 5 | .779 | .761 | .750 | | .757 | .795 | .781 | .772 | |
| 2 | 4 | .768 | .798 | .797 | | .737 | .786 | .819 | .818 | |
| 3 | 3 | .771 | .813 | .813 | | .739 | .790 | .833 | .834 | |
| 1 | 8 | .927 | .918 | .914 | | .918 | .932 | .926 | .922 | |
| 4.5 | 4.5 | .949 | .963 | .963 | | .940 | .954 | .967 | .968 | |
| 1 | 11 | .969 | .965 | .963 | | .965 | .971 | .968 | .966 | |
| 6 | 6 | .987 | .991 | .991 | | .984 | .988 | .992 | .992 | |
| 1.00001 | 0.9 | .052090 | .052094 | .052047 | .052074 | .051672 | .049663 | .048868 | .048786 | .048823 |
| 1.00001 | 0.8 | .059400 | .059767 | .059670 | .059709 | .058034 | .054489 | .053135 | .052946 | .053017 |
| 1.01 | 0.99 | .050021 | .050009 | .050006 | .050008 | .050026 | .050022 | .050010 | .050006 | .050007 |
| 1.1 | 0.9 | .052058 | .050931 | .050684 | .050811 | .052638 | .052277 | .050933 | .050650 | .050795 |
| 1.1 | 0.8 | .056255 | .054496 | .054058 | .054291 | .057646 | .054227 | .051210 | .050660 | .050943 |
| 1.05 | 0.95 | .050516 | .050233 | .050171 | .050203 | .050657 | .050571 | .050234 | .050162 | .050198 |
| 1.2 | 0.99 | .0577 | .0571 | .0567 | .0569 | .0562 | .0625 | .0632 | .0629 | .0630 |
| 1.2 | 0.8 | .0561 | .0518 | .0508 | .0524 | .0581 | .0587 | .0519 | .0508 | .0525 |
| 2 | 0.9 | .202 | .177 | .169 | .178 | .183 | .223 | .203 | .195 | .205 |
| 2 | 0.7 | .210 | .175 | .167 | .170 | .208 | .226 | .191 | .182 | .200 |
| 3 | 0.9 | .46 | .42 | .41 | .46 | .43 | .48 | .45 | .44 | .50 |
| 5 | 0.9 | .77 | .74 | .73 | .86 | .75 | .78 | .76 | .75 | .80 |
| 0.99999 | 0.9 | .052090 | .052095 | .052047 | .052074 | .051672 | .049663 | .048868 | .048786 | .048823 |
| 0.99999 | 0.7 | .164 | .177 | .183 | .179 | .176 | .147 | .155 | .159 | .156 |
| 0.999 | 0.9 | .052113 | .052128 | .052081 | .052107 | .051679 | .049660 | .048869 | .048788 | .048824 |
| 0.9 | 0.9 | .056709 | .057657 | .057677 | .057673 | .054022 | .051680 | .051168 | .051139 | .051146 |
| 0.9 | 0.8 | .072283 | .073887 | .073993 | .073885 | .066924 | .064088 | .063323 | .063336 | .063247 |
| 0.9 | 0.76 | .1036 | .1050 | .1057 | .1050 | .1002 | .0918 | .0900 | .0905 | .0898 |
| 0.85 | 0.9 | .061401 | .062776 | .062800 | .062790 | .057176 | .055020 | .054553 | .054512 | .054518 |
| 0.85 | 0.8 | .0931 | .0933 | .0933 | .0932 | .0866 | .0822 | .0794 | .0794 | .0792 |
| 0.81 | 0.9 | .068944 | .070535 | .070602 | .070537 | .063683 | .061212 | .060565 | .060548 | .060501 |
| 0.8 | 0.8 | .1631 | .1545 | .1545 | .1540 | .1609 | .1456 | .1325 | .1323 | .1317 |

Table 4 (continued)

m = 5, n = 5

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .049985 | .049989 | .050000 | .049996 |
| 1 | 1.1 | .051264 | .051422 | .051074 | .051431 | .051339 | .049830 | .050335 | .051374 | .051111 |
| 1.05 | 1.05 | .051026 | .051290 | .051506 | .051465 | .050821 | .049611 | .050201 | .051539 | .051141 |
| 1 | 1.5 | .0789 | .0808 | .0785 | .0800 | .0781 | .0722 | .0758 | .0767 | .0786 |
| 1.25 | 1.25 | .0713 | .0773 | .0821 | .0815 | .0678 | .0650 | .0722 | .0831 | .0799 |
| 1 | 2 | .1804 | .1836 | .1551 | .1846 | .1764 | .1689 | .1752 | .1660 | .1822 |
| 1.333 | 1.5 | .1072 | .1218 | .1302 | .1332 | .0998 | .0972 | .1136 | .1352 | .1307 |
| 1 | 4 | .780 | .781 | .740 | | .777 | .774 | .776 | .770 | |
| 1 | 5 | .898 | .898 | .887 | | .896 | .895 | .896 | .889 | |
| 2 | 4 | .876 | .888 | .893 | | .869 | .872 | .885 | .895 | |
| 3 | 3 | .883 | .897 | .906 | | .876 | .879 | .894 | .909 | |
| 1 | 8 | .986 | .986 | .984 | | .986 | .986 | .986 | .985 | |
| 4.5 | 4.5 | .993 | .994 | .995 | | .992 | .993 | .994 | .995 | |
| 1 | 11 | .997 | .997 | .997 | | .997 | .997 | .997 | .997 | |
| 6 | 6 | .999 | .999 | .999 | | .999 | .999 | .999 | .999 | |
| 1.00001 | 0.9 | .051453 | .051690 | .051563 | .051743 | .051702 | .052962 | .052859 | .051743 | .052090 |
| 1.00001 | 0.8 | .132 | .133 | .136 | .134 | .135 | .137 | .137 | .136 | .135 |
| 1.01 | 0.99 | .050009 | .050006 | .050002 | .050000 | .050024 | .050008 | .050006 | .049995 | .050000 |
| 1.1 | 0.9 | .050917 | .050593 | .050132 | .050004 | .052444 | .050839 | .050605 | .049586 | .050021 |
| 1.1 | 0.8 | .0645 | .0645 | .0638 | .0639 | .0689 | .0660 | .0658 | .0638 | .0644 |
| 1.05 | 0.95 | .050234 | .050152 | .050034 | .050005 | .050607 | .050214 | .050155 | .049898 | .050008 |
| 1.2 | 0.99 | .0546 | .0549 | .0544 | .0548 | .0548 | .0519 | .0529 | .0544 | .0542 |
| 1.2 | 0.8 | .0782 | .0792 | .0818 | .0806 | .0812 | .0786 | .0795 | .0818 | .0807 |
| 2 | 0.9 | .167 | .164 | .142 | .155 | .167 | .156 | .156 | .136 | .153 |
| 2 | 0.7 | .318 | .312 | .292 | .301 | .331 | .313 | .309 | .292 | .300 |
| 3 | 0.9 | .51 | .51 | .48 | .51 | .51 | .50 | .50 | .47 | .51 |
| 5 | 0.9 | .89 | .89 | .88 | .89 | .89 | .88 | .88 | .87 | .89 |
| 0.99999 | 0.9 | .051453 | .051691 | .051563 | .051744 | .051703 | .052962 | .052860 | .051744 | .052090 |
| 0.99999 | 0.7 | | | | | | | | | |
| 0.999 | 0.9 | .051475 | .051720 | .051696 | .051779 | .051711 | .053001 | .052901 | .051783 | .052129 |
| 0.9 | 0.9 | .062828 | .063803 | .064427 | .064351 | .061819 | .066160 | .066342 | .064596 | .065092 |
| 0.9 | 0.8 | | | | | | | | | |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .150 | .150 | .149 | .149 | .150 | .156 | .154 | .149 | .150 |

Table 4 (continued)

m = 5, n = 15

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050002 | .050011 | .050015 | .050014 |
| 1 | 1.1 | .052313 | .052449 | .052248 | .052360 | .052206 | .052486 | .053531 | .053695 | .053729 |
| 1.05 | 1.05 | .051776 | .052256 | .052367 | .052326 | .051347 | .051944 | .053338 | .053837 | .053704 |
| 1 | 1.5 | .1048 | .1042 | .0961 | .1015 | .0999 | .1057 | .1094 | .1028 | .1080 |
| 1.25 | 1.25 | .0885 | .0999 | .1024 | .1019 | .0811 | .0894 | .1053 | .1096 | .1087 |
| 1 | 2 | .2570 | .2504 | .2227 | .2540 | .2437 | .2585 | .2591 | .2336 | .2655 |
| 1.333 | 1.5 | .1516 | .1793 | .1842 | .1865 | .1361 | .1528 | .1878 | .1956 | .1975 |
| 1 | 4 | .836 | .829 | .820 | | .828 | .836 | .832 | .822 | |
| 1 | 5 | .927 | .924 | .912 | | .923 | .927 | .925 | .914 | |
| 2 | 4 | .915 | .933 | .934 | | .907 | .916 | .935 | .936 | |
| 3 | 3 | .920 | .941 | .944 | | .912 | .921 | .943 | .947 | |
| 1 | 8 | .991 | .990 | .988 | | .990 | .991 | .990 | .989 | |
| 4.5 | 4.5 | .995 | .997 | .997 | | .995 | .995 | .997 | .997 | |
| 1 | 11 | .998 | .998 | .997 | | .998 | .998 | .998 | .997 | |
| 6 | 6 | .999 | .999 | .999 | | .999 | .999 | .999 | .999 | |
| 1.00001 | 0.9 | .052484 | .052753 | .052639 | .052725 | .052555 | .052312 | .051634 | .051104 | .051294 |
| 1.00001 | 0.8 | .126 | .129 | .132 | .130 | .129 | .126 | .125 | .127 | .126 |
| 1.01 | 0.09 | .050020 | .050008 | .050000 | .050003 | .050037 | .050020 | .050008 | .049997 | .050002 |
| 1.1 | 0.9 | .051966 | .050846 | .050175 | .050306 | .053813 | .051982 | .050841 | .049707 | .050262 |
| 1.1 | 0.8 | .0653 | .0647 | .0641 | .0644 | .0709 | .0651 | .0634 | .0621 | .0626 |
| 1.05 | 0.95 | .050496 | .050215 | .050156 | .050080 | .050946 | .050501 | .050214 | .049929 | .050068 |
| 1.2 | 0.99 | .0586 | .0586 | .0575 | .0581 | .0583 | .0590 | .0607 | .0601 | .0607 |
| 1.2 | 0.8 | .0750 | .0766 | .0798 | .0803 | .0805 | .0750 | .0762 | .0792 | .0799 |
| 2 | 0.9 | .242 | .221 | .185 | .213 | .234 | .243 | .229 | .194 | .223 |
| 2 | 0.7 | .349 | .326 | .302 | .313 | .362 | .350 | .329 | .305 | .307 |
| 3 | 0.9 | .60 | .58 | .54 | .61 | .59 | .61 | .59 | .55 | .62 |
| 5 | 0.9 | .92 | .91 | .90 | .94 | .91 | .92 | .91 | .90 | .93 |
| 0.99999 | 0.9 | .052485 | .052754 | .052640 | .052726 | .052555 | .052312 | .051635 | .051105 | .051295 |
| 0.99999 | 0.7 | | | | | | | | | |
| 0.999 | 0.9 | .052518 | .052802 | .052695 | .052778 | .052566 | .052344 | .051671 | .051144 | .051333 |
| 0.9 | 0.9 | .065542 | .067017 | .067328 | .067240 | .063303 | .065159 | .064606 | .064035 | .064185 |
| 0.9 | 0.8 | | | | | | | | | |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .147 | .146 | .138 | .145 | .146 | .147 | .141 | .134 | .140 |

Table 4 (continued)

m = 5, n = 30

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050011 | .050022 | .050025 | .050023 |
| 1 | 1.1 | .052886 | .052978 | .052873 | .052907 | .052720 | .054093 | .055191 | .055147 | .055204 |
| 1.05 | 1.05 | .052154 | .052760 | .052835 | .052808 | .051662 | .053325 | .054973 | .055199 | .055114 |
| 1 | 1.5 | .1209 | .1171 | .1137 | .1149 | .1138 | .1271 | .1279 | .1214 | .1261 |
| 1.25 | 1.25 | .0987 | .1126 | .1145 | .1136 | .0893 | .1044 | .1236 | .1252 | .1251 |
| 1 | 2 | .3032 | .2869 | .2539 | .2974 | .2858 | .3134 | .3044 | .2835 | .3165 |
| 1.333 | 1.5 | .1781 | .2169 | .2148 | .2165 | .1589 | .1873 | .2281 | .2355 | .2347 |
| 1 | 4 | .866 | .859 | .848 | | .857 | .870 | .860 | .847 | |
| 1 | 5 | .942 | .936 | .922 | | .938 | .944 | .939 | .923 | |
| 2 | 4 | .935 | .950 | .948 | | .927 | .937 | .954 | .955 | |
| 3 | 3 | .938 | .958 | .961 | | .931 | .941 | .961 | .964 | |
| 1 | 8 | .993 | .992 | .989 | | .992 | .993 | .992 | .991 | |
| 4.5 | 4.5 | .997 | .998 | .998 | | .996 | .997 | .998 | .998 | |
| 1 | 11 | .998 | .998 | .997 | | .998 | .998 | .998 | .998 | |
| 6 | 6 | .999 | .999 | .999 | | .999 | .999 | .999 | .999 | |
| | | | | | | | | | | |
| 1.00001 | 0.9 | .053123 | .053309 | .052990 | .053246 | .052984 | .051947 | .051062 | .050887 | .050895 |
| 1.00001 | 0.8 | .122 | .126 | .127 | .127 | .125 | .118 | .119 | .121 | .120 |
| 1.01 | 0.99 | .050026 | .050009 | .050002 | .050006 | .050045 | .050027 | .050010 | .050002 | .050005 |
| 1.1 | 0.9 | .052613 | .050937 | .050260 | .050546 | .054552 | .052758 | .050934 | .050300 | .050502 |
| 1.1 | 0.8 | .0656 | .0647 | .0641 | .0645 | .0716 | .0645 | .0621 | .0614 | .0617 |
| 1.05 | 0.95 | .050658 | .050237 | .050048 | .050140 | .051130 | .050695 | .050237 | .050033 | .050128 |
| 1.2 | 0.99 | .0611 | .0606 | .0607 | .0602 | .0604 | .0635 | .0648 | .0627 | .0645 |
| 1.2 | 0.8 | .0735 | .0749 | .0783 | .0799 | .0802 | .0735 | .0741 | .0718 | .0792 |
| 2 | 0.9 | .288 | .252 | .226 | .252 | .276 | .298 | .269 | .238 | .269 |
| 2 | 0.7 | .369 | .333 | .309 | .323 | .381 | .375 | .340 | .317 | .320 |
| 3 | 0.9 | .66 | .62 | .60 | .67 | .64 | .67 | .64 | .60 | .69 |
| 5 | 0.9 | .93 | .92 | .92 | .97 | .93 | .94 | .93 | .92 | .97 |
| | | | | | | | | | | |
| 0.99999 | 0.9 | .053123 | .053310 | .052992 | .053247 | .052984 | .051947 | .051062 | .050887 | .050896 |
| 0.99999 | 0.7 | | | | | | | | | |
| 0.999 | 0.9 | .053164 | .053368 | .053105 | .053307 | .052997 | .051975 | .051097 | .050805 | .050932 |
| 0.9 | 0.9 | .067117 | .068634 | .068723 | .068679 | .063985 | .064502 | .063819 | .063578 | .063667 |
| 0.9 | 0.8 | | | | | | | | | |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .145 | .143 | .132 | .143 | .143 | .141 | .134 | .126 | .134 |

Table 4 (continued)

m = 5, n = 60

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050018 | .050030 | .050030 | .050024 |
| 1 | 1.1 | .053156 | .053365 | .053259 | .053224 | .053060 | .055268 | .056293 | .056188 | .056256 |
| 1.05 | 1.05 | .052280 | .053133 | .053176 | .053067 | .051856 | .054323 | .056061 | .056121 | .056104 |
| 1 | 1.5 | .1320 | .1262 | .1215 | .1245 | .1241 | .1430 | .1405 | .1359 | .1395 |
| 1.25 | 1.25 | .1054 | .1215 | .1219 | .1215 | .0952 | .1155 | .1361 | .1367 | .1367 |
| 1 | 2 | .3347 | .3119 | .2966 | .3280 | .3159 | .3520 | .3346 | .3202 | .3526 |
| 1.333 | 1.5 | .1963 | .2326 | .2330 | .2368 | .1755 | .2122 | .2551 | .2556 | .2605 |
| 1 | 4 | .885 | .869 | .858 | | .877 | .891 | .877 | .868 | |
| 1 | 5 | .952 | .944 | .938 | | .948 | .954 | .948 | .943 | |
| 2 | 4 | .946 | .961 | .960 | | .939 | .949 | .964 | .963 | |
| 3 | 3 | .950 | .967 | .967 | | .943 | .953 | .970 | .970 | |
| 1 | 8 | .995 | .993 | .992 | | .994 | .994 | .994 | .993 | |
| 4.5 | 4.5 | .998 | .999 | .999 | | .997 | .997 | .998 | .998 | |
| 1 | 11 | .999 | .999 | .998 | | .999 | .999 | .998 | .998 | |
| 6 | 6 | .999 | .999 | .999 | | .999 | .999 | .999 | .999 | |
| 1.00001 | 0.9 | .053716 | .053644 | .053538 | .053688 | .053306 | .051696 | .050709 | .050568 | .050630 |
| 1.00001 | 0.8 | .119 | .124 | .125 | .125 | .122 | .113 | .115 | .116 | .115 |
| 1.01 | 0.99 | .050031 | .050010 | .050005 | .050008 | .050050 | .050033 | .050011 | .050005 | .050007 |
| 1.1 | 0.9 | .053070 | .050986 | .050497 | .050744 | .055067 | .053355 | .050987 | .050434 | .050713 |
| 1.1 | 0.8 | .0660 | .0646 | .0645 | .0646 | .0721 | .0641 | .0612 | .0607 | .0609 |
| 1.05 | 0.95 | .050773 | .050250 | .050126 | .050189 | .051257 | .050844 | .050251 | .050112 | .050181 |
| 1.2 | 0.99 | .0626 | .0620 | .0614 | .0615 | .0619 | .0667 | .0676 | .0669 | .0673 |
| 1.2 | 0.8 | .0728 | .0738 | .0790 | .0794 | .0802 | .0731 | .0727 | .0743 | .0785 |
| 2 | 0.9 | .319 | .274 | .253 | .280 | .306 | .336 | .295 | .277 | .303 |
| 2 | 0.7 | .584 | .338 | .323 | .331 | .395 | .395 | .348 | .332 | .350 |
| 3 | 0.9 | .69 | .65 | .63 | .72 | .68 | .71 | .67 | .65 | .74 |
| 5 | 0.9 | .94 | .93 | .93 | .99 | .95 | .95 | .94 | .93 | .98 |
| 0.99999 | 0.9 | .053716 | .053651 | .053539 | .053689 | .053306 | .051697 | .050710 | .050568 | .050631 |
| 0.99999 | 0.7 | | | | | | | | | |
| 0.999 | 0.9 | .053763 | .053715 | .053605 | .053755 | .053321 | .051721 | .050745 | .050606 | .050667 |
| 0.9 | 0.9 | .068517 | .069627 | .069596 | .069810 | .064487 | .064025 | .063356 | .063300 | .063306 |
| 0.9 | 0.8 | | | | | | | | | |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .144 | .141 | .141 | .141 | .142 | .136 | .129 | .129 | .129 |

# APPENDIX D

Powers of $L_2^{(2)}$, $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and $LS^{(2)}$ in the unbiased and equal tail areas cases for testing $\lambda_1=1$, $\lambda_2=1$ against different simple two-sided alternative hypotheses in the complex case, $\alpha = .05$, (supplementing Table 8 of Chapter II).

TABLE 8 (continued)

$n_1 = 3, \quad n_2 = 7$

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | With equal tail areas $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050000 | .050002 | .050010 | .050007 |
| 1 | 1.1 | .050846 | .050904 | .050824 | .050900 | .050603 | .050831 | .051086 | .051823 | .051554 |
| 1.05 | 1.05 | .050694 | .050784 | .050896 | .050853 | .050385 | .050678 | .050966 | .051913 | .051512 |
| 1 | 1.5 | .0736 | .0744 | .0720 | .0742 | .0691 | .0736 | .0752 | .0764 | .0771 |
| 1.25 | 1.25 | .0676 | .0694 | .0720 | .0711 | .0619 | .0675 | .0703 | .0767 | .0742 |
| 1 | 2 | .1702 | .1717 | .1655 | .1709 | .1603 | .1701 | .1730 | .1719 | .1754 |
| 1.333 | 1.5 | .1126 | .1166 | .1215 | .1200 | .1015 | .1125 | .1178 | .1286 | .1246 |
| 1 | 3 | .447 | .448 | .441 | .447 | .436 | .447 | .450 | .447 | .451 |
| 2.5 | 1.5 | .453 | .458 | .463 | .462 | .439 | .453 | .459 | .470 | .467 |
| 2 | 2 | .458 | .464 | .472 | .469 | .443 | .458 | .465 | .479 | .474 |
| 1.00001 | 0.9 | .050935 | .051000 | .050963 | .051018 | .050675 | .050951 | .050808 | .049891 | .050328 |
| 1.00001 | 0.8 | .051531 | .051874 | .051631 | .051934 | .050319 | .051563 | .051488 | .049447 | .050542 |
| 1.01 | 0.99 | .050007 | .050005 | .050000 | .050003 | .050010 | .050006 | .050005 | .049998 | .050002 |
| 1.1 | 0.9 | .050648 | .050542 | .050029 | .050270 | .050982 | .050648 | .050539 | .049755 | .050247 |
| 1.1 | 0.8 | .051486 | .051398 | .050463 | .050890 | .052039 | .051501 | .051204 | .048750 | .050161 |
| 1.05 | 0.95 | .050162 | .050135 | .050008 | .050067 | .050244 | .050162 | .050135 | .049938 | .050061 |
| 1.2 | 0.99 | .0530 | .0532 | .0527 | .0531 | .0523 | .0530 | .0535 | .0546 | .0543 |
| 1.2 | 0.8 | .0522 | .0518 | .0500 | .0507 | .0536 | .0522 | .0518 | .0486 | .0506 |
| 2 | 0.9 | .154 | .154 | .142 | .151 | .146 | .154 | .155 | .148 | .155 |
| 2 | 0.7 | .128 | .125 | .102 | .116 | .127 | .128 | .125 | .105 | .119 |
| 3 | 0.9 | .42 | .42 | .41 | .42 | .41 | .42 | .42 | .41 | .42 |
| 0.99999 | 0.9 | .050935 | .051000 | .050964 | .051019 | .050675 | .050951 | 050808 | .049891 | .050328 |
| 0.9 | 0.999 | .050948 | .051015 | .050984 | .051036 | .050678 | .050963 | .050821 | .049901 | .050339 |
| 0.9 | 0.9 | .053038 | .053405 | .053917 | .053724 | .051451 | .053070 | .053012 | .051766 | .052325 |
| 0.9 | 0.8 | | | | | | | | | |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .0536 | .0542 | .0550 | .0547 | .0510 | .0538 | .0537 | .0523 | .0530 |
| 0.85 | 0.8 | | | | | | | | | |
| 0.81 | 0.9 | .0505 | .0505 | .0515 | .0512 | .0505 | .0499 | .0499 | .0485 | .0493 |

TABLE 8 (continued)

$n_1 = 3, \quad n_2 = 17$

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050016 | .050020 | .050022 | .050022 |
| 1 | 1.1 | .051319 | .051358 | .051264 | .051332 | .050875 | .052872 | .053335 | .053457 | .053457 |
| 1.05 | 1.05 | .051054 | .051185 | .051221 | .051214 | .050571 | .052594 | .053164 | .053429 | .053347 |
| 1 | 1.5 | .0836 | .0839 | .0814 | .0831 | .0755 | .0911 | .0933 | .0916 | .0931 |
| 1.25 | 1.25 | .0755 | .0785 | .0793 | .0792 | .0666 | .0829 | .0882 | .0901 | .0895 |
| 1 | 2 | .1934 | .1935 | .1872 | .1915 | .1758 | .2050 | .2080 | .2029 | .2069 |
| 1.333 | 1.5 | .1300 | .1364 | .1377 | .1376 | .1124 | .1411 | .1508 | .1538 | .1532 |
| 1 | 3 | .474 | .473 | .466 | .471 | .454 | .484 | .487 | .480 | .485 |
| 2.5 | 1.5 | .478 | .486 | .487 | .487 | .456 | .490 | .501 | .503 | .503 |
| 2 | 2 | .483 | .493 | .496 | .495 | .460 | .495 | .508 | .512 | .512 |
| 1.00001 | 0.9 | .051410 | .051478 | .051407 | .051455 | .050925 | .049846 | .049467 | .049150 | .049284 |
| 1.00001 | 0.8 | .053888 | .054221 | .053848 | .054110 | .051623 | .050829 | .050254 | .049371 | .049818 |
| 1.01 | 0.99 | .050011 | .050008 | .050003 | .050006 | .050013 | .050011 | .050008 | .050002 | .050005 |
| 1.1 | 0.9 | .051077 | .050760 | .050285 | .050558 | .051303 | .051130 | .050751 | .050216 | .050523 |
| 1.1 | 0.8 | .052946 | .052472 | .051469 | .052057 | .053034 | .051518 | .050474 | .049105 | .049849 |
| 1.05 | 0.95 | .050270 | .050190 | .050071 | .050139 | .050324 | .050283 | .050188 | .050054 | .050131 |
| 1.2 | 0.99 | .0548 | .0549 | .0544 | .0547 | .0533 | .0578 | .0586 | .0585 | .0587 |
| 1.2 | 0.8 | .0540 | .0527 | .0508 | .0519 | .0550 | .0542 | .0527 | .0505 | .0518 |
| 2 | 0.9 | .176 | .173 | .164 | .170 | .162 | .187 | .186 | .178 | .184 |
| 2 | 0.7 | .150 | .140 | .123 | .133 | .144 | .160 | .151 | .134 | .144 |
| 3 | 0.9 | .45 | .44 | .43 | .44 | .43 | .46 | .46 | .45 | .45 |
| 0.99999 | 0.9 | .051411 | .051478 | .051407 | .051456 | .050925 | .049846 | .049467 | .049150 | .049284 |
| 0.9 | 0.999 | .051428 | .051500 | .051432 | .051479 | .050930 | .049848 | .049468 | .049153 | .049286 |
| 0.9 | 0.9 | .054587 | .055133 | .055279 | .055230 | .052141 | .051384 | .051072 | .050759 | .050864 |
| 0.9 | 0.8 | .0518 | .0535 | .0536 | .0537 | .0518 | .0479 | .0478 | .0473 | .0476 |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .0563 | .0572 | .0574 | .0573 | .0522 | .0523 | .0521 | .0518 | .0519 |
| 0.85 | 0.8 | | | | | | | | | |
| 0.81 | 0.9 | .0540 | .0554 | .0556 | .0556 | .0518 | .0498 | .0498 | .0494 | .0496 |

TABLE 8 (continued)

$n_1 = 3, \quad n_2 = 32$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050023 | .050028 | .050029 | .050028 |
| 1 | 1.1 | .051551 | .051565 | .051498 | .051546 | .051008 | .053856 | .054327 | .054327 | .054353 |
| 1.05 | 1.05 | .051228 | .051371 | .051383 | .051388 | .050662 | .053508 | .054133 | .054222 | .054200 |
| 1 | 1.5 | .0887 | .0883 | .0866 | .0877 | .0789 | .1000 | .1016 | .1001 | .1011 |
| 1.25 | 1.25 | .0795 | .0829 | .0831 | .0831 | .0691 | .0906 | .0964 | .0971 | .0969 |
| 1 | 2 | .2053 | .2039 | .1995 | .2022 | .1842 | .2229 | .2243 | .2204 | .2229 |
| 1.333 | 1.5 | .1388 | .1458 | .1462 | .1462 | .1181 | .1554 | .1662 | .1672 | .1670 |
| 1 | 3 | .487 | .485 | .480 | .483 | .464 | .504 | .504 | .500 | .503 |
| 2.5 | 1.5 | .491 | .500 | .500 | .500 | .465 | .508 | .521 | .521 | .521 |
| 2 | 2 | .496 | .507 | .508 | .508 | .469 | .514 | .529 | .530 | .530 |
| 1.00001 | 0.9 | .051633 | .051683 | .051630 | .051655 | .051038 | .049350 | .048910 | .048770 | .048827 |
| 1.00001 | 0.8 | .054988 | .055234 | .054969 | .055114 | .052227 | .050557 | .049802 | .049347 | .049567 |
| 1.01 | 0.99 | .050013 | .050008 | .050006 | .050007 | .050014 | .050014 | .050008 | .050005 | .050007 |
| 1.1 | 0.9 | .051290 | .050846 | .050554 | .050713 | .051449 | .051389 | .050843 | .050509 | .050691 |
| 1.1 | 0.8 | .053638 | .052912 | .052292 | .052628 | .053475 | .051596 | .050176 | .049385 | .049798 |
| 1.05 | 0.95 | .050323 | .050211 | .050138 | .050178 | .050361 | .050348 | .050211 | .050127 | .050172 |
| 1.2 | 0.99 | .0557 | .0556 | .0553 | .0555 | .0539 | .0601 | .0609 | .0606 | .0608 |
| 1.2 | 0.8 | .0548 | .0531 | .0519 | .0525 | .0556 | .0553 | .0531 | .0517 | .0525 |
| 2 | 0.9 | .188 | .182 | .176 | .180 | .170 | .204 | .201 | .195 | .199 |
| 2 | 0.7 | .162 | .147 | .136 | .142 | .152 | .177 | .162 | .151 | .157 |
| 3 | 0.9 | .46 | .45 | .45 | .45 | .44 | .48 | .47 | .47 | .47 |
| 0.99999 | 0.9 | .051634 | .051684 | .051631 | .051655 | .051038 | .049350 | .048910 | .048770 | .048827 |
| 0.9 | 0.999 | .051654 | .051709 | .051658 | .051681 | .051044 | .049346 | .048908 | .048769 | .048825 |
| 0.9 | 0.9 | .055304 | .055883 | .055933 | .055901 | .052458 | .050644 | .050314 | .050213 | .050234 |
| 0.9 | 0.8 | .0544 | .0560 | .0560 | .0560 | .0519 | .0489 | .0483 | .0481 | .0482 |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .0575 | .0585 | .0585 | .0585 | .0527 | .0517 | .0515 | .0514 | .0514 |
| 0.85 | 0.8 | | | | | | | | | |
| 0.81 | 0.9 | .0562 | .0576 | .0576 | .0576 | .0519 | .0501 | .0500 | .0498 | .0498 |

TABLE 8 (continued)

$n_1 = 3, \quad n_2 = 62$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050028 | .050032 | .050033 | .050033 |
| 1 | 1.1 | .051703 | .051695 | .051653 | .051685 | .051096 | .054492 | .054931 | .054907 | .054929 |
| 1.05 | 1.05 | .051342 | .051487 | .051491 | .051499 | .050723 | .054097 | .054722 | .054749 | .054744 |
| 1 | 1.5 | .0921 | .0912 | .0901 | .0907 | .0812 | .1058 | .1067 | .1057 | .1063 |
| 1.25 | 1.25 | .0822 | .0856 | .0857 | .0857 | .0707 | .0956 | .1016 | .1017 | .1017 |
| 1 | 2 | .2132 | .2104 | .2079 | .2093 | .1899 | .2346 | .2345 | .2320 | .2335 |
| 1.333 | 1.5 | .1446 | .1517 | .1518 | .1519 | .1220 | .1649 | .1758 | .1760 | .1760 |
| 1 | 3 | .497 | .493 | .490 | .492 | .471 | .516 | .515 | .512 | .514 |
| 2.5 | 1.5 | .500 | .509 | .508 | .509 | .471 | .521 | .533 | .533 | .533 |
| 2 | 2 | .505 | .516 | .517 | .517 | .475 | .526 | .541 | .542 | .542 |
| 1.00001 | 0.9 | .051776 | .051808 | .051776 | .051784 | .051110 | .049040 | .048583 | .048522 | .048546 |
| 1.00001 | 0.8 | .055687 | .055850 | .055690 | .055758 | .052612 | .050401 | .049554 | .049327 | .049432 |
| 1.01 | 0.99 | .050014 | .050009 | .050007 | .050008 | .050015 | .050016 | .050009 | .050007 | .050008 |
| 1.1 | 0.9 | .051429 | .050895 | .050732 | .050819 | .051542 | .051564 | .050898 | .050710 | .050809 |
| 1.1 | 0.8 | .054080 | .053173 | .052825 | .053002 | .053751 | .051664 | .050002 | .049575 | .049794 |
| 1.05 | 0.95 | .050358 | .050224 | .050183 | .050205 | .050384 | .050392 | .050224 | .050177 | .050202 |
| 1.2 | 0.99 | .0563 | .0561 | .0559 | .0560 | .0542 | .0617 | .0623 | .0621 | .0622 |
| 1.2 | 0.8 | .0554 | .0533 | .0526 | .0530 | .0560 | .0560 | .0533 | .0526 | .0530 |
| 2 | 0.9 | .196 | .188 | .184 | .186 | .175 | .216 | .210 | .207 | .209 |
| 2 | 0.7 | .170 | .152 | .145 | .149 | .158 | .188 | .169 | .163 | .167 |
| 3 | 0.9 | .47 | .46 | .46 | .46 | .45 | .49 | .48 | .48 | .48 |
| 0.99999 | 0.9 | .051777 | .051809 | .051776 | .051784 | .051110 | .049040 | .048583 | .048521 | .048546 |
| 0.9 | 0.999 | .051798 | .051836 | .051804 | .051812 | .051116 | .049034 | .048578 | .048517 | .048541 |
| 0.9 | 0.9 | .055760 | .056341 | .056356 | .056328 | .052658 | .050190 | .049884 | .049855 | .049852 |
| 0.9 | 0.8 | .0560 | .0576 | .0575 | .0576 | .0522 | .0488 | .0487 | .0486 | .0486 |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .0583 | .0592 | .0592 | .0592 | .0531 | .0514 | .0512 | .0512 | .0512 |
| 0.85 | 0.8 | | | | | | | | | |
| 0.81 | 0.9 | .0575 | .0590 | .0589 | .0589 | .0527 | .0502 | .0501 | .0500 | .0500 |

TABLE 8 (continued)

$n_1 = 4, \quad n_2 = 7$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050001 | .050000 | .050000 | .050000 | .049995 | .049997 | .050007 | .050003 |
| 1 | 1.1 | .050977 | .051105 | .051010 | .051079 | .050813 | .050451 | .050754 | .051719 | .051376 |
| 1.05 | 1.05 | .050812 | .050980 | .051121 | .051056 | .050524 | .050289 | .050627 | .051844 | .051356 |
| 1 | 1.5 | .0856 | .0871 | .0847 | .0867 | .0824 | .0833 | .0856 | .0876 | .0880 |
| 1.25 | 1.25 | .0763 | .0791 | .0826 | .0813 | .0714 | .0739 | .0775 | .0859 | .0827 |
| 1 | 2 | .2443 | .2468 | .2415 | .2458 | .2382 | .2412 | .2448 | .2453 | .2475 |
| 1.333 | 1.5 | .1595 | .1646 | .1708 | .1687 | .1511 | .1564 | .1624 | .1753 | .1706 |
| 1 | 3 | .605 | .607 | .603 | .606 | .601 | .603 | .606 | .605 | .607 |
| 2.5 | 1.5 | .633 | .637 | .642 | .641 | .627 | .631 | .636 | .644 | .642 |
| 2 | 2 | .647 | .651 | .657 | .655 | .640 | .645 | .650 | .659 | .656 |
| 1.00001 | 0.9 | .051057 | .051142 | .051172 | .051210 | .050913 | .051609 | .051515 | .050404 | .050894 |
| 1.00001 | 0.8 | | | | | | | | | |
| 1.01 | 0.99 | .050007 | .050006 | .050000 | .050002 | .050013 | .050007 | .050006 | .049996 | .050002 |
| 1.1 | 0.9 | .050676 | .050559 | .050018 | .050184 | .051316 | .060665 | .050564 | .049648 | .050172 |
| 1.1 | 0.8 | | | | | | .050128 | .049955 | .046989 | .048558 |
| 1.05 | 0.95 | .050169 | .050140 | .050006 | .050046 | .050327 | .050167 | .050141 | .049911 | .050043 |
| 1.2 | 0.99 | .0536 | .0540 | .0534 | .0538 | .0531 | .0526 | .0533 | .0547 | .0544 |
| 1.2 | 0.8 | | | | | | | | | |
| 2 | 0.9 | .214 | .215 | .205 | .212 | .211 | .211 | .213 | .208 | .213 |
| 2 | 0.7 | .167 | .164 | .139 | .153 | .171 | .164 | .163 | .141 | .155 |
| 3 | 0.9 | .57 | .57 | .56 | .57 | .56 | .57 | .57 | .56 | .57 |
| 0.99999 | 0.9 | .051057 | .051142 | .051173 | .051210 | .050913 | .051610 | .051516 | .050404 | .050895 |
| 0.9 | 0.999 | .051071 | .051159 | .051198 | .051232 | .050916 | .051630 | .051537 | .050422 | .050913 |
| 0.9 | 0.9 | .052702 | .053245 | .054101 | .053812 | .051149 | .053845 | .054009 | .052571 | .053174 |

TABLE 8 (continued)

$n_1 = 4, \quad n_2 = 17$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050011 | .050017 | .050020 | .050019 |
| 1 | 1.1 | .051612 | .051691 | .051567 | .051656 | .051222 | .052722 | .053332 | .053518 | .053513 |
| 1.05 | 1.05 | .051300 | .051512 | .051573 | .051558 | .050801 | .052398 | .053153 | .053543 | .053424 |
| 1 | 1.5 | .0980 | .0987 | .0954 | .0977 | .0910 | .1032 | .1062 | .1041 | .1061 |
| 1.25 | 1.25 | .0863 | .0908 | .0921 | .0918 | .0778 | .0914 | .0986 | .1015 | .1007 |
| 1 | 2 | .2680 | .2688 | .2619 | .2665 | .2553 | .2750 | .2788 | .2735 | .2778 |
| 1.333 | 1.5 | .1786 | .1869 | .1890 | .1887 | .1638 | .1856 | .1977 | .2018 | .2009 |
| 1 | 3 | .625 | .625 | .619 | .623 | .615 | .629 | .632 | .627 | .631 |
| 2.5 | 1.5 | .650 | .656 | .657 | .658 | .639 | .655 | .663 | .666 | .665 |
| 2 | 2 | .663 | .670 | .672 | .672 | .651 | .667 | .677 | .680 | .679 |
| 1.00001 | 0.9 | .051696 | .051828 | .051744 | .051805 | .051301 | .050576 | .050149 | .049717 | .049894 |
| 1.00001 | 0.8 | | | | | | | | | |
| 1.01 | 0.99 | .050012 | .050008 | .050001 | .050005 | .050018 | .050013 | .050008 | .050000 | .050005 |
| 1.1 | 0.9 | .051225 | .050787 | .050126 | .050493 | .051825 | .051276 | .050779 | .050044 | .050454 |
| 1.1 | 0.8 | .051750 | .051118 | .050114 | .050486 | .052882 | .050798 | .049504 | .047530 | .048585 |
| 1.05 | 0.95 | .050307 | .050197 | .050031 | .050123 | .050453 | .050320 | .050195 | .050011 | .050114 |
| 1.2 | 0.99 | .0560 | .0561 | .0555 | .0559 | .0547 | .0581 | .0592 | .0591 | .0594 |
| 1.2 | 0.8 | .0540 | .0522 | .0509 | .0510 | .0566 | .0542 | .0522 | .0492 | .0509 |
| 2 | 0.9 | .239 | .235 | .224 | .231 | .228 | .246 | .245 | .235 | .242 |
| 2 | 0.7 | .193 | .180 | .158 | .170 | .193 | .200 | .188 | .166 | .179 |
| 3 | 0.9 | .59 | .59 | .58 | .58 | .58 | .59 | .59 | .59 | .59 |
| 0.99999 | 0.9 | .051696 | .051828 | .051745 | .051805 | .051301 | .050576 | .050149 | .049717 | .049894 |
| 0.9 | 0.999 | .051718 | .051856 | .051777 | .051836 | .051307 | .050586 | .050161 | .049730 | .049906 |
| 0.9 | 0.9 | .054924 | .055860 | .056117 | .056039 | .052264 | .052627 | .052480 | .052081 | .052213 |
| 0.9 | 0.8 | | | | | | | | | |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .0514 | .0532 | .0536 | .0535 | .0516 | .0496 | .0492 | .0488 | .0489 |

TABLE 8 (continued)

$n_1 = 4, \quad n_2 = 32$

| $\lambda_1$ | $\lambda_2$ | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050020 | .050026 | .050027 | .050027 |
| 1 | 1.1 | .051948 | .051993 | .051894 | .051962 | .051435 | .053914 | .054554 | .054557 | .054595 |
| 1.05 | 1.05 | .051553 | .051790 | .051812 | .051814 | .050946 | .053491 | .054351 | .054489 | .054454 |
| 1 | 1.5 | .1049 | .1047 | .1022 | .1037 | .0959 | .1142 | .1165 | .1144 | .1158 |
| 1.25 | 1.25 | .0917 | .0969 | .0974 | .0973 | .0814 | .1009 | .1092 | .1102 | .1100 |
| 1 | 2 | .2813 | .2802 | .2750 | .2781 | .2651 | .2940 | .2962 | .2914 | .2945 |
| 1.333 | 1.5 | .1890 | .1985 | .1992 | .1992 | .1710 | .2016 | .2155 | .2169 | .2167 |
| 1 | 3 | .635 | .634 | .630 | .633 | .623 | .644 | .645 | .641 | .644 |
| 2.5 | 1.5 | .659 | .666 | .666 | .667 | .645 | .668 | .677 | .678 | .678 |
| 2 | 2 | .671 | .680 | .681 | .681 | .657 | .679 | .691 | .692 | .692 |
| 1.00001 | 0.9 | .052018 | .052129 | .052058 | .052092 | .051486 | .050073 | .049548 | .049349 | .049428 |
| 1.00001 | 0.8 | .050073 | .050528 | .050131 | .050276 | .050165 | .046600 | .045821 | .044958 | .045402 |
| 1.01 | 0.99 | .050015 | .050009 | .050005 | .050007 | .050021 | .050016 | .050009 | .050004 | .050006 |
| 1.1 | 0.9 | .051521 | .050881 | .050450 | .050681 | .052069 | .051635 | .050877 | .050389 | .050651 |
| 1.1 | 0.8 | .052853 | .051799 | .050815 | .051347 | .053732 | .051248 | .049336 | .048113 | .048751 |
| 1.05 | 0.95 | .050381 | .050220 | .050112 | .050170 | .050514 | .050410 | .050219 | .050097 | .050163 |
| 1.2 | 0.99 | .0573 | .0572 | .0567 | .0570 | .0556 | .0611 | .0621 | .0617 | .0620 |
| 1.2 | 0.8 | .0553 | .0527 | .0509 | .0518 | .0577 | .0558 | .0527 | .0507 | .0518 |
| 2 | 0.9 | .252 | .246 | .238 | .242 | .239 | .265 | .261 | .253 | .258 |
| 2 | 0.7 | .208 | .188 | .173 | .181 | .205 | .220 | .201 | .185 | .194 |
| 3 | 0.9 | .60 | .60 | .59 | .59 | .59 | .61 | .61 | .60 | .61 |
| 0.99999 | 0.9 | .052018 | .052129 | .052059 | .052093 | .051486 | .050073 | .049548 | .049349 | .049428 |
| 0.9 | 0.999 | .052044 | .052163 | .052094 | .052127 | .051494 | .050079 | .049556 | .049358 | .049436 |
| 0.9 | 0.9 | .056029 | .057046 | .057138 | .057089 | .052806 | .052057 | .051883 | .051750 | .051774 |
| 0.9 | 0.8 | | | | | | | | | |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .0536 | .0556 | .0557 | .0556 | .0518 | .0499 | .0494 | .0493 | .0493 |

Table 8 (continued)

$n_1 = 4,\ n_2 = 62$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050025 | .050032 | .050032 | .050032 |
| 1 | 1.1 | .052178 | .052193 | .052123 | .052172 | .051589 | .054719 | .055319 | .055282 | .055317 |
| 1.05 | 1.05 | .051725 | .051975 | .051976 | .051987 | .051054 | .054225 | .055100 | .055142 | .055135 |
| 1 | 1.5 | .1097 | .1086 | .1070 | .1079 | .0994 | .1219 | .1232 | .1216 | .1225 |
| 1.25 | 1.25 | .0955 | .1009 | .1010 | .1010 | .0839 | .1074 | .1160 | .1163 | .1162 |
| 1 | 2 | .2906 | .2877 | .2844 | .2863 | .2721 | .3072 | .3073 | .3042 | .3060 |
| 1.333 | 1.5 | .1962 | .2061 | .2062 | .2063 | .1761 | .2127 | .2270 | .2273 | .2273 |
| 1 | 3 | .643 | .640 | .638 | .639 | .629 | .654 | .654 | .651 | .653 |
| 2.5 | 1.5 | .666 | .673 | .673 | .673 | .650 | .676 | .686 | .686 | .686 |
| 2 | 2 | .677 | .686 | .687 | .687 | .662 | .688 | .700 | .700 | .700 |
| 1.00001 | 0.9 | .052231 | .052311 | .052271 | .052280 | .051597 | .049749 | .049189 | .049099 | .049131 |
| 1.00001 | 0.8 | .051492 | .051743 | .051380 | .051554 | .051098 | .047082 | .046072 | .045603 | .045828 |
| 1.01 | 0.99 | .050017 | .050009 | .050007 | .050008 | .050022 | .050019 | .050009 | .050006 | .050008 |
| 1.1 | 0.9 | .051725 | .050934 | .050685 | .050817 | .052230 | .051889 | .050935 | .050650 | .050801 |
| 1.1 | 0.8 | .053590 | .052212 | .051645 | .051938 | .054280 | .051576 | .049242 | .048557 | .048909 |
| 1.05 | 0.95 | .050432 | .050234 | .050171 | .050204 | .050554 | .050474 | .050234 | .050163 | .050200 |
| 1.2 | 0.99 | .0582 | .0579 | .0576 | .0578 | .0562 | .0631 | .0639 | .0636 | .0638 |
| 1.2 | 0.8 | .0562 | .0529 | .0519 | .0524 | .0584 | .0568 | .0530 | .0518 | .0524 |
| 2 | 0.9 | .262 | .252 | .248 | .250 | .246 | .278 | .271 | .266 | .269 |
| 2 | 0.7 | .219 | .193 | .184 | .189 | .213 | .234 | .209 | .200 | .204 |
| 3 | 0.9 | .61 | .60 | .60 | .60 | .60 | .62 | .62 | .61 | .61 |
| 0.99999 | 0.9 | .052232 | .052312 | .052271 | .052281 | .051598 | .049749 | .049189 | .049099 | .049131 |
| 0.9 | 0.999 | .052260 | .052348 | .052309 | .052318 | .051606 | .049751 | .049194 | .049106 | .049137 |
| 0.9 | 0.9 | .056758 | .057778 | .057819 | .057769 | .053146 | .051699 | .051555 | .051516 | .051501 |
| 0.9 | 0.8 | | | | | | | | | |
| 0.9 | 0.76 | | | | | | | | | |
| 0.85 | 0.9 | .0551 | .0570 | .0571 | .0570 | .0526 | .0498 | .0497 | .0496 | .0496 |

TABLE 8 (continued)

$n_1 = 7,\ n_2 = 7$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .049985 | .049988 | .050000 | .049995 |
| 1 | 1.1 | .051263 | .052055 | .051485 | .051512 | .051293 | .049766 | .050421 | .051485 | .051034 |
| 1.05 | 1.05 | .051062 | .052013 | .051633 | .051519 | .050847 | .049579 | .050161 | .051633 | .051036 |
| 1 | 1.5 | .1636 | .1703 | .1649 | .1659 | .1631 | .1582 | .1641 | .1649 | .1642 |
| 1.25 | 1.25 | .1467 | .1533 | .1553 | .1535 | .1440 | .1412 | .1465 | .1553 | .1517 |
| 1 | 2 | .5394 | .5453 | .5401 | .5413 | .5387 | .5353 | .5407 | .5401 | .5401 |
| 1.333 | 1.5 | .4163 | .4231 | .4255 | .4237 | .4135 | .4115 | .4170 | .4255 | .4221 |
| 1 | 3 | .905 | .906 | .905 | .905 | .905 | .904 | .905 | .905 | .905 |
| 2.5 | 1.5 | .937 | .938 | .939 | .938 | .937 | .937 | .938 | .939 | .938 |
| 2 | 2 | .947 | .948 | .948 | .948 | .946 | .946 | .947 | .948 | .948 |
| 1.00001 | 0.9 | .050797 | .050448 | .051204 | .051172 | .050970 | .052377 | .052468 | .051204 | .051688 |
| 1.00001 | 0.8 | | | | | | | | | |
| 1.01 | 0.99 | .050007 | .050009 | .050001 | .050000 | .050020 | .050006 | .050010 | .049996 | .050000 |
| 1.1 | 0.9 | .050679 | .050882 | .050008 | .050012 | .052020 | .050623 | .050955 | .049567 | .050037 |
| 1.1 | 0.8 | | | | | | | | | |
| 1.05 | 0.95 | .050174 | .050221 | .050003 | .050006 | .050502 | .050160 | .050239 | .049893 | .050012 |
| 1.2 | 0.99 | .0561 | .0584 | .0564 | .0566 | .0562 | .0533 | .0551 | .0564 | .0558 |
| 1.2 | 0.8 | | | | | | | | | |
| 2 | 0.9 | .467 | .473 | .463 | .466 | .467 | .462 | .469 | .463 | .464 |
| 2 | 0.7 | .334 | .344 | .312 | .321 | .344 | .328 | .340 | .312 | .320 |
| 3 | 0.9 | .87 | .88 | .87 | .87 | .87 | .87 | .88 | .87 | .87 |
| 0.99999 | 0.9 | .050797 | .050448 | .051204 | .051172 | .050970 | .052378 | .052468 | .051204 | .051689 |
| 0.9 | 0.999 | .050795 | .050444 | .051220 | .051183 | .050955 | .052392 | .052485 | .051220 | .051705 |

TABLE 8 (continued)

$n_1 = 7, \quad n_2 = 17$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050000 | .050000 | .050000 | .050000 | .050000 | .050001 | .050008 | .050014 | .050012 |
| 1 | 1.1 | .052319 | .052308 | .052377 | .052497 | .052095 | .052435 | .053346 | .053709 | .053682 |
| 1.05 | 1.05 | .051888 | .052102 | .052479 | .052441 | .051382 | .052002 | .053141 | .053833 | .053635 |
| 1 | 1.5 | .1783 | .1792 | .1766 | .1790 | .1746 | .1788 | .1831 | .1814 | .1834 |
| 1.25 | 1.25 | .1590 | .1649 | .1681 | .1676 | .1528 | .1594 | .1690 | .1735 | .1723 |
| 1 | 2 | .5536 | .5544 | .5514 | .5538 | .5499 | .5540 | .5573 | .5550 | .5572 |
| 1.333 | 1.5 | .4300 | .4367 | .4399 | .4394 | .4235 | .4304 | .4404 | .4448 | .4437 |
| 1 | 3 | .909 | .909 | .908 | .909 | .908 | .909 | .910 | .909 | .909 |
| 2.5 | 1.5 | .940 | .940 | .941 | .941 | .939 | .940 | .941 | .941 | .941 |
| 2 | 2 | .949 | .949 | .950 | .950 | .948 | .949 | .950 | .950 | .950 |
| 1.00001 | 0.9 | .051898 | .052495 | .052188 | .052255 | .051824 | .051781 | .051423 | .050780 | .051021 |
| 1.00001 | 0.8 | | | | | | | | | |
| 1.01 | 0.99 | .050015 | .050008 | .050000 | .050003 | .050031 | .050015 | .050008 | .049997 | .050003 |
| 1.1 | 0.9 | .051497 | .050808 | .050084 | .050301 | .053111 | .051506 | .050804 | .049699 | .050264 |
| 1.1 | 0.8 | | | | | | | | | |
| 1.05 | 0.95 | .050378 | .050206 | .050018 | .050078 | .050772 | .050380 | .050204 | .049927 | .050[illegible]69 |
| 1.2 | 0.99 | .0599 | .599 | .0594 | .0600 | .0592 | .0601 | .0619 | .0618 | .062[illegible] |
| 1.2 | 0.8 | | | | | | | | | |
| 2 | 0.9 | .484 | .481 | .474 | .479 | .482 | .484 | .484 | .478 | .482 |
| 2 | 0.7 | .359 | .344 | .322 | .334 | .368 | .360 | .347 | .325 | .337 |
| 3 | 0.9 | .88 | .88 | .88 | .88 | .88 | .88 | .88 | .88 | .88 |
| 0.99999 | 0.9 | .051898 | .052495 | .052189 | .052256 | .051824 | .051781 | .051423 | .050781 | .051021 |
| 0.9 | 0.999 | .051912 | .052524 | .052222 | .052287 | .051815 | .051794 | .051441 | .050801 | .051040 |

TABLE 8 (continued)

$n_1 = 7, \quad n_2 = 32$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050012 | .050021 | .050001 | .050001 | .050000 | .050012 | .050021 | .050023 | .050022 |
| 1 | 1.1 | .054133 | .055168 | .053042 | .053132 | .052512 | .054133 | .055168 | .055187 | .055248 |
| 1.05 | 1.05 | .053514 | .054936 | .053044 | .053013 | .051638 | .053514 | .054936 | .055209 | .055139 |
| 1 | 1.5 | .1926 | .1964 | .1855 | .1875 | .1818 | .1926 | .1964 | .1935 | .1955 |
| 1.25 | 1.25 | .1712 | .1833 | .1764 | .1761 | .1580 | .1712 | .1833 | .1851 | .1847 |
| 1 | 2 | .5667 | .5692 | .5600 | .5620 | .5571 | .5667 | .5692 | .5661 | .5682 |
| 1.333 | 1.5 | .4429 | .4554 | .4491 | .4489 | .4297 | .4429 | .4554 | .4571 | .4568 |
| 1 | 3 | .912 | .913 | .910 | .911 | .910 | .912 | .913 | .912 | .912 |
| 2.5 | 1.5 | .941 | .943 | .942 | .942 | .940 | .941 | .943 | .943 | .943 |
| 2 | 2 | .950 | .952 | .951 | .951 | .949 | .950 | .952 | .952 | .952 |
| 1.00001 | 0.9 | .051430 | .050873 | .052749 | .052830 | .052352 | .051430 | .050873 | .050541 | .050671 |
| 1.00001 | 0.8 | | | | | | | | | |
| 1.01 | 0.99 | .050022 | .050009 | .050002 | .050005 | .050037 | .050022 | .050009 | .050001 | .050005 |
| 1.1 | 0.9 | .052140 | .050913 | .050145 | .050540 | .053714 | .052140 | .050913 | .050062 | .050500 |
| 1.1 | 0.8 | | | | | | | | | |
| 1.05 | 0.95 | .050539 | .050231 | .050039 | .050138 | .050921 | .050539 | .050231 | .050018 | .050128 |
| 1.2 | 0.99 | .0646 | .0663 | .0617 | .0622 | .0610 | .0646 | .0663 | .0657 | .0662 |
| 1.2 | 0.8 | | | | | | | | | |
| 2 | 0.9 | .500 | .497 | .484 | .488 | .491 | .500 | .497 | .490 | .494 |
| 2 | 0.7 | .383 | .359 | .334 | .344 | .384 | .383 | .359 | .340 | .350 |
| 3 | 0.9 | .88 | .88 | .88 | .88 | .88 | .88 | .88 | .88 | .88 |
| 0.99999 | 0.9 | .051431 | .050874 | .052750 | .052830 | .052352 | .051431 | .050874 | .050541 | .050672 |
| 0.9 | 0.999 | .051442 | .050893 | .052792 | .052872 | .052347 | .051442 | .050893 | .050563 | .050692 |

TABLE 8 (continued)

$n_1 = 7,\ n_2 = 62$

| | | With local unbiasedness property | | | | | With equal tail areas | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_1$ | $\lambda_2$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ | $LS^{(2)}$ | $L_2^{(2)}$ | $U^{(2)}$ | $V^{(2)}$ | $W^{(2)}$ |
| 1 | 1.001 | .050019 | .050029 | .050029 | .050029 | .050000 | .050019 | .050028 | .050029 | .050029 |
| 1 | 1.1 | .055413 | .056409 | .056335 | .056404 | .053743 | .055413 | .056409 | .056335 | .056404 |
| 1.05 | 1.05 | .054644 | .056160 | .056250 | .056235 | .052766 | .054644 | .056160 | .056249 | .056235 |
| 1 | 1.5 | .2034 | .2057 | .2033 | .2047 | .1825 | .2034 | .2057 | .2033 | .2047 |
| 1.25 | 1.25 | .1802 | .1934 | .1940 | .1939 | .1571 | .1802 | .1934 | .1940 | .1939 |
| 1 | 2 | .5766 | .5777 | .5752 | .5766 | .5588 | .5766 | .5777 | .5752 | .5766 |
| 1.333 | 1.5 | .4524 | .4660 | .4666 | .4665 | .4298 | .4524 | .4660 | .4665 | .4665 |
| 1 | 3 | .915 | .915 | .914 | .914 | .910 | .915 | .915 | .914 | .914 |
| 2.5 | 1.5 | .943 | .945 | .945 | .945 | .940 | .943 | .945 | .945 | .945 |
| 2 | 2 | .951 | .953 | .953 | .953 | .949 | .951 | .953 | .953 | .953 |
| 1.00001 | 0.9 | .051184 | .050534 | .050372 | .050425 | .053683 | .051184 | .050534 | .050372 | .050425 |
| 1.00001 | 0.8 | | | | | | | | | |
| 1.01 | 0.99 | .050027 | .050010 | .050004 | .050007 | .050041 | .050027 | .050010 | .050004 | .050007 |
| 1.1 | 0.9 | .052647 | .050977 | .050433 | .050717 | .054102 | .052646 | .050977 | .050433 | .050717 |
| 1.1 | 0.8 | | | | | | | | | |
| 1.05 | 0.95 | .050666 | .050247 | .050111 | .050182 | .051017 | .050666 | .050247 | .050111 | .050182 |
| 1.2 | 0.99 | .0680 | .0694 | .0688 | .0691 | .0602 | .0680 | .0694 | .0688 | .0691 |
| 1.2 | 0.8 | | | | | | | | | |
| 2 | 0.9 | .512 | .506 | .501 | .504 | .493 | .512 | .506 | .501 | .504 |
| 2 | 0.7 | .401 | .367 | .355 | .361 | .391 | .401 | .367 | .355 | .361 |
| 3 | 0.9 | .89 | .89 | .88 | .89 | .88 | .89 | .89 | .88 | .89 |
| 0.99999 | 0.9 | .051184 | .050535 | .050372 | .050426 | .053683 | .051184 | .050534 | .050372 | .050426 |
| 0.9 | 0.999 | .051194 | .050555 | .050394 | .050447 | .053691 | .051194 | .050555 | .050394 | .050447 |