THE ASYMPTOTIC NORMALITY OF LINEAR COMBINATIONS OF ORDER STATISTICS

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Two theorems are presented on the asymptotic normality of statistics of the form $T_n = n^{-1} \sum_{i=1}^{n} J(\frac{i}{n+1}) X_n^{(i)}$ where it is assumed that the underlying distribution function F has a finite variance. Theorem 1 is an extension of Theorem 2 of Stigler (1974) to depending on F possibly unbounded score functions. Theorem 2 of this paper is analogous to Theorem 1 of Shorack (1972) as his theorem applies to statistics of the above form, but with fewer conditions of F^{-1} and J. Under the conditions of Theorem 2, the variances of $\sqrt{n}\, T_n$ are also shown to converge to the variance of the limiting distribution; a question not considered in Shorack (1972). Finally an example is given for which Theorem 1 implies asymptotic normality, but for which Shorack's Theorem 1 does not apply. In the construction of the example a useful bound for the variance of a transformed uniform order statistic is obtained.

Keywords and phrases: Order statistics, unbounded score function, asymptotic normality.

0. Introduction, Summary and Preliminaries. We will assume throughout this paper that $X_1,\ldots,X_n$ are independent, identically distributed random variables with common distribution function F, such that $\mathrm{Var}X_1 = \sigma^2 < \infty$. $X_n^{(1)} \leq \cdots \leq X_n^{(n)}$ will denote their corresponding ordered values. J will be a real valued score function defined on (0,1). Let $T_n = \sum_{i=1}^{n} J(\frac{i}{n+1})X_n^{(i)}/n$, $\mu_n = ET_n$ and

$$\sigma^2(J,F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(F(x))J(F(y))[F(x \wedge y) - F(x)F(y)]dxdy.$$

F^{-1} will be the right continuous inverse of F, that is $F^{-1}(u) = \sup\{x: F(x) \leq u\}$ for $u \in [0,1]$.

Theorem 2 Stigler (1974) says that if J is bounded and continuous a.e. with respect to F^{-1} then $\sqrt{n}(T_n - \mu_n) \xrightarrow[n\to\infty]{d} N(0,\sigma^2(J,F))$. In Section 1, Stigler's theorem is extended to a more general class of score functions, which depending on F can include unbounded score functions. This extension is Theorem 1 Section 1. In Section 2, Theorem 1 is applied to obtain a theorem on the asymptotic normality of linear combinations of order statistics with unbounded score functions, which is as strong as Shorack's Theorem 1 (1972), as it applies to statistics of this type. Fewer conditions on F^{-1} and J are required and in addition it is shown that $\mathrm{Var}(\sqrt{n}T_n) \xrightarrow[n\to\infty]{} \sigma^2(J,F) < \infty$, a problem not considered in Shorack (1972). Finally in Section 3, an example of an unbounded score function and a distribution function is given which satisfy the conditions of Theorem 1 to imply $\sqrt{n}(T_n-\mu_n) \xrightarrow[n\to\infty]{d} N(0,\sigma^2(J,F))$, but which do not satisfy the conditions of Theorem 1 Shorack (1972). Included is a bound for the variance of a transformed uniform order statistic, which might be of separate interest.

Now for some notation and observations.

Let $g_{ij}^{(n)} = \text{cov}(X_n^{(i)}, X_n^{(j)})$ for $1 \leq i,j \leq n$. Let $F_i^{(n)}(x) = P(X_n^{(i)} \leq x)$ and $F_{ij}^{(n)}(x,y) = P(X_n^{(i)} \leq x, X_n^{(j)} \leq y)$.

We can write $g_{ij}^{(n)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F_{ij}^{(n)}(x,y) - F_i^{(n)}(x)F_j^{(n)}(y)]dxdy \geq 0$. (See Stigler (1974)). Let $g_{0j}^{(n)} = g_{i0}^{(n)} = g_{in+1}^{(n)} = g_{n+1j}^{(n)} = 0$ for $0 \leq i,j \leq n+1$. Set $V_n(u,v) = \sum_{i=0}^{[(n+1)u]} \sum_{j=0}^{[(n+1)v]} g_{ij}^{(n)}/n$ for $(u,v) \in [0,1]x[0,1]$.

($[a]$ = greatest integer $\leq$ a)

V_n defines a measure on $[0,1] \times [0,1]$ which gives nonnegative mass $g_{ij}^{(n)}/n$ to each point $(\frac{i}{n+1}, \frac{j}{n+1})$ for $0 \leq i,j \leq n+1$. In particular $V_n([0,1] \times [0,1]) \equiv V_n(1,1) = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} g_{ij}^{(n)}/n = \sigma^2$. Set $V(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_{[0,u]}(F(x)) I_{[0,v]}(F(y)) [F(x,y) - F(x)F(y)]dxdy$. V also defines a postive measure on $[0,1] \times [0,1]$, which gives measure σ^2 to $[0,1] \times [0,1]$. $V(u,v)$ can also be written: $V(u,v) = \int_0^u \int_0^v (s \wedge t - st)dF^{-1}(s)dF^{-1}(t)$. See Proposition 4.6 (Appendix). Proposition 4.2 (Appendix) shows that $V_n(u,v) \underset{n\to\infty}{\to} V(u,v)$ at every continuity point (u,v) of V.

Observe that if f and g are a.e. continuous with respect to F^{-1}, then $f(u)g(v)$ is a.e. continuous with respect to V.

Finally we record the following representations for $\text{Var}(\sqrt{n}T_n)$.

$$\text{Var}(\sqrt{n}T_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} J(\tfrac{i}{n+1})J(\tfrac{j}{n+1})\text{cov}(X_n^{(i)}, X_n^{(j)})/n$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} J(\tfrac{i}{n+1})J(\tfrac{j}{n+1})g_{ij}^{(n)}/n = \int_0^1 \int_0^1 J(u)J(v)dV_n(u,v).$$

The last representation of the $\text{Var}(\sqrt{n}T_n)$ will be exploited extensively in what follows.

1. The Main Theorem.

Theorem 1.

Let J be a real valued function on (0,1), bounded in absolute value on $(\frac{1}{a}, 1-\frac{1}{a})$ for all $a > 2$, continuous a.e. with respect to F^{-1} such that

$$\int_0^1 \int_0^1 |J(u)||J(v)|dV(u,v) < \infty.$$

If 1.i. $\lim_{n\to\infty} \int_0^1 \int_0^1 |J(u)||J(v)|dV_n(u,v) = \int_0^1 \int_0^1 |J(u)||J(v)|dV(u,v)$

then

1.ii. $\quad \mathrm{Var}(\sqrt{n}(T_n-\mu_n)) \underset{n\to\infty}{\to} \sigma^2(J,F)$

and

1.iii. $\quad \sqrt{n}(T_n-\mu_n) \overset{d}{\underset{n\to\infty}{\to}} N(0,\sigma^2(J,F))$

Corollary 1.

With the same assumptions on J as in Theorem 1, but with 1.i. replaced by

1.i' $\quad \int_0^1 \int_0^1 |J(u)J(v)|^{1+\delta}dV_n(u,v)$ is uniformly bounded for some $\delta > 0$ and all $n \geq 1$, then 1.ii. and 1.iii.

Proof of Corollary 1.

Merely apply Lemma A.4.

Remark 1.1

Theorem 1 yields the standard central limit theorem. $\sqrt{n}(\bar{X} - \mu) \overset{d}{\to} N(0,\sigma^2)$, where $\mu = EX$.

Remark 1.2

If there exists an $M > 0$ such that for all $n \geq 1$ and $1 \leq i,j \leq n$, $g_{ij} \leq M/n$, and if in addition to the conditions of Theorem 1, J satisfies $\sum_{i=1}^{n} J^{1+\delta}(\frac{i}{n+1})/n \underset{n\to\infty}{\to} \int_0^1 J^{1+\delta}(u)du < \infty$, then we can conclude 1.iii..

<u>Proof</u>. Define $J^+ = J \vee 0$ and $J^- = -(J \wedge 0)$

By Lemma A.3 (Appendix) we have

1.1. $$\lim_{n\to\infty} \int_0^1 \int_0^1 J^+(u)J^+(v)dV_n(u,v) = \int_0^1 \int_0^1 J^+(u)J^+(v)dV(u,v)$$

and

1.2. $$\lim_{n\to\infty} \int_0^1 \int_0^1 J^-(u)J^-(v)dV_n(u,v) = \int_0^1 \int_0^1 J^-(u)J^-(v)dV(u,v) < \infty.$$

1.1. and 1.2. together with Lemma A.2 give

1.3. $$\lim_{n\to\infty} \int_0^1 \int_0^1 J^+(u)J^-(v)dV_n(u,v) = \int_0^1 \int_0^1 J^+(u)J^-(v)dV(u,V) < \infty.$$

Now 1.1., 1.2., and 1.3. imply 1.ii..

Define $T_n^+ = \sum_{i=1}^{n} J^+(\frac{i}{n+1})X_n^{(i)}/n$ and $\mu_n^+ = ET_n^+$. Similarly define T_n^- and μ_n^-. So that $T_n = T_n^+ - T_n^-$ and $\mu_n = \mu_n^+ - \mu_n^-$.

Let a_m and b_m be sequences of constants in (0,1) such that for all $m \geq 1$, $1 > b_{m+1} > b_m > a_m > a_{m+1} > 0$, $\lim_{m\to\infty} a_m = 0$, $\lim_{m\to\infty} b_m = 1$ and $F^{-1}(\{a_m\}) = F^{-1}(\{b_m\}) = 0$.

Let $$J_m^+ = \begin{cases} J^+(u) & \text{if } u \in (a_m, b_m), \\ 0 & \text{elsewhere} \end{cases},$$

$$T_{mn}^+ = \sum_{i=1}^{n} J_m^+(\frac{i}{n+1})X_n^{(i)}/n \text{ and } \mu_{mn}^+ = ET_{mn}^+.$$

Similarly define J_m^- , T_{mn}^- and μ_{mn}^-.

Set $T_{mn} = T_{mn}^+ - T_{mn}^-$ and $\mu_{mn} = \mu_{mn}^+ - \mu_{mn}^-$.

Using the inequality, $\mathrm{Var}(X+Y) \leq ((\mathrm{Var}X)^{1/2} + (\mathrm{Var}Y)^{1/2})^2$, we have

$$\mathrm{Var}(\sqrt{n}(T_n-\mu_n)-\sqrt{n}(T_{mn}-\mu_{mn})) \leq ((\mathrm{Var}(\sqrt{n}(T_n^+-T_{mn}^+)))^{1/2} + (\mathrm{Var}(\sqrt{n}(T_n^--T_{mn}^-)))^{1/2})^2 =$$

$$((\int_0^1 \int_0^1 (J^+(u) - J_m^+(u))(J^+(v)-J_m^+(v))dV_n(u,v))^{1/2} +$$

$$(\int_0^1 \int_0^1 (J^-(u)-J_m^-(u))(J^-(v)-J_m^-(v))dV_n(u,v))^{1/2})^2.$$

Lemma A.5. (Appendix), now gives

1.4. $\lim_{m\to\infty} \lim_{n\to\infty} \mathrm{Var}(\sqrt{n}(T_n-\mu_n)-\sqrt{n}(T_{mn}-\mu_{mn})) = 0.$

Now to complete the proof of Theorem 1. For each fixed m we have by Theorem 2 Stigler (1974),

$$\sqrt{n}(T_{mn}-\mu_{mn}) \xrightarrow[n\to\infty]{d} N(0,\sigma^2(J_m,F)).$$

By the Dominated Convergence Theorem

$\sigma^2(J_m,F) \underset{m\to\infty}{\to} \sigma^2(J,F)$. Hence by Theorem 4.2, page 25 Billingsley (1968) and 1.4., we have 1.iii. □

2. Unbounded Scores.

Theorem 2.

Let J be continuous a.e. with respect to F^{-1}. If there exist constants $0 < \varepsilon_1, \varepsilon_2 < 1$, $\delta > 0$ and $M_1 > 0$ such that

2.i. $|J(u)|^{1+\delta} \leq M_1 u^{-\varepsilon_1}(1-u)^{-\varepsilon_2}$ for all $u \in (0,1)$

and

2.ii. $\int_0^1 u^{1/2-\varepsilon_1}(1-u)^{1/2-\varepsilon_2}dF^{-1}(u) < \infty$

then

2.iii. $\sigma^2(J,F) < \infty$

2.iv. $\mathrm{Var}(\sqrt{n}(T_n-\mu_n)) \underset{n\to\infty}{\to} \sigma^2(J,F)$

and

2.v. $\sqrt{n}(T_n-\mu_n) \xrightarrow[n\to\infty]{d} N(0,\sigma^2(J,F))$

Remark 2.1.

Compare Theorem 2 with Theorem 1 Shorack (1972) and Corollary 4.1 Stigler (1969). In each, a bounding function for $|F^{-1}|$ is required of the form $M_2 u^{-1/2+\delta_1}(1-u)^{-1/2+\delta_2}$ where $\delta_1 > \varepsilon_1$ and $\delta_2 > \varepsilon_2$. Whether 2.iv. holds is not investigated in Shorack (1972). In Stigler (1969), F is also required to have a density function which satisfies some smoothness conditions near the tails. For other related smoothness conditions see Theorem 3 of Chernoff, Gastwirth and Johns (1967).

Proof

To prove 2.iii., set $M(u) = M_1 u^{-\varepsilon_1}(1-u)^{-\varepsilon_2}$. Now $\sigma^2(J,F) \leq \int_0^1 \int_0^1 |J(u)||J(v)|$ $(u\wedge v-uv)dF^{-1}(u)dF^{-1}(v) \leq (\int_0^1 M(u)(u(1-u))^{1/2}dF^{-1}(u))^2$. Which by 2.ii. is finite. To prove 2.iv. and 2.v., by Corollary 1 we need only show that $\int_0^1 \int_0^1 |J(u)J(v)|^{1+\delta}dV_n(u,v)$ is uniformly bounded for $n \geq 1$.

Note by 2.i. $\int_0^1 \int_0^1 |J(u)J(v)|^{1+\delta}dV_n(u,v)$ is $\leq$

(2.1) $\int_0^1 \int_0^1 M(u)M(v)dV_n(u,v)$. So it is sufficient to prove that (2.1) is uniformly bounded for $n \geq 1$.

It is easy to show that we can pick a $K > 0$ such that for all $u \in (0,1/2]$ (resp. $u \in (1/2,1)$) $|M(u)| \leq K u^{-\varepsilon_1}$ (resp. $|M(u)| \leq K(1-u)^{-\varepsilon_2}$). Let $M_a(u)$ (resp. $M_b(u)$) $= \begin{cases} M(u) & u \in (0,1/2] \text{ (resp. } u \in (1/2,1)) \\ 0 & \text{elsewhere} \end{cases}$

Also set $T_n^* = \sum_{i=1}^{n} M(\frac{i}{n+1})X_n^{(i)}/n$, $T_n^a = \sum_{i=1}^{n} M_a(\frac{i}{n+1})X_n^{(i)}/n$, and $T_n^b = \sum_{i=1}^{n} M_b(\frac{i}{n+i})X_n^{(i)}/n$

Note that (2.1) = $\mathrm{Var}(\sqrt{n}\ T_n^*) \le ((\mathrm{Var}(\sqrt{n}\ T_n^a))^{1/2} + (\mathrm{Var}(\sqrt{n}\ T_n^b))^{1/2})^2$.
So that we need only uniformly bound $\mathrm{Var}(\sqrt{n}\ T_n^a)$, and $\mathrm{Var}(\sqrt{n}\ T_n^b)$.

We will show that $\mathrm{Var}(\sqrt{n}\ T_n^a)$ is uniformly bounded; that $\mathrm{Var}(\sqrt{n}\ T_n^b)$ is uniformly bounded follows by an analogous argument.

Set $M_n^a(x,y) = \sum_{i=1}^{n} \sum_{j=1}^{n} M_a(\frac{i}{n+i})M_a(\frac{j}{n+1})(F_{ij}^{(n)}(x,y) - F_i^{(n)}(x)F_j^{(n)}(y))/n$

Note that $\mathrm{Var}(\sqrt{n}\ T_n^a) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M_n^a(x,y)dxdy$. It is easy to show that $M_n^a(x,y) \le (M_n^a(x,x))^{1/2}(M_n^a(y,y))^{1/2}$. So to verify that $\mathrm{Var}(\sqrt{n}\ T_n^a)$ is uniformly bounded it is sufficient to show that $\int_{-\infty}^{\infty} (M_n^a(x,x))^{1/2}dx$ is uniformly bounded for $n \ge 1$.

Let $A = \{x: \ 0 < F(x) \le 1/2\}$. We will first show that

(2.2) $\int_A (M_n^a(x,x))^{1/2}dx$ is uniformly bounded.

For $x \in A$, let $A_n(x) = \{\ell: \frac{\ell}{n+1} \le \frac{F(x)}{2}\}$. Now by Minkowski's inequality

$(M_n^a(x,x))^{1/2} \le$

(2.3) $\quad (\sum\sum_{i,j \in A_n^c(x)} M_a(\frac{i}{n+1})M_a(\frac{j}{n+1})\ (F_{ij}^{(n)}(x,x) - F_i^{(n)}(x)F_j^{(n)}(x))/n)^{1/2}$ +

(2.4) $\quad (\sum\sum_{i,j \in A_n(x)} M_a(\frac{i}{n+1})M_a(\frac{j}{n+1})(F_{ij}^{(n)}(x,x) - F_i^{(n)}(x)F_j^{(n)}(x))/n)^{1/2}$

Note that (2.3) is $\le (M_a^2(\frac{F(x)}{2}) \sum_{i=1}^{n} \sum_{j=1}^{n} (F_{ij}^{(n)}(x,x) - F_i^{(n)}(x)F_j^{(n)}(x))/n)^{1/2}$,

which by definition of M_a is $\le 2^{\varepsilon_1}K(F(x))^{1/2-\varepsilon_1}(1-F(x))^{1/2}$.

Now since

$0 \le F_{ij}^{(n)}(x,x) - F_i^{(n)}(x)F_j^{(n)}(x) \le (F_i^{(n)}(x)(1-F_i^{(n)}(x)))^{1/2}(F_j^{(n)}(x)(1-F_j^{(n)}(x)))^{1/2}$,

(2.4) is $\le \sum_{i \in A_n(x)} M_a(\frac{i}{n+1})(F_i^{(n)}(x)(1-F_i^{(n)}(x))^{1/2}/\sqrt{n}$ =

$$\sum_{i=1}^{[(n+1)F(x)/2]} M_a\left(\frac{i}{n+1}\right)\left(F_i^{(n)}(x)\left(1-F_i^{(n)}(x)\right)\right)^{1/2}/\sqrt{n} \tag{2.5}$$

(If $[(n+1)F(x)/2] = 0$, (2.5) is set equal to 0)

Claim 2.1.

There exists a $C > 0$ independent of n and $x \in A$ such that (2.5) $\le$ $C(F(x))^{1/2-\varepsilon_1}(1-F(x))^{1/2}$.

Proof

Let $S_n(x) = \sum_{\ell=1}^{n} I_{[-\infty,x]}(X_\ell)$ and $S_n^*(x) = \sum_{\ell=1}^{n} (1-I_{[-\infty,x]}(X_\ell))$.

Note that $F_i^{(n)}(x) = P(S_n(x) \ge i)$ and $1 - F_i^{(n)}(x) =$

$P(S_n(x) < i) = P(S_n^*(x) > n-i) = P(S_n^*(x) - n(1-F(x)) > -i+n(F(x))$,

which when $\frac{i}{n+1} \le F(x)/2$ is

$\le P(S_n^*(x) - n(1-F(x)) > nF(x)/3)$ (2.6) .

Applying Chebychev's inequality to (2.6), we have

$$(2.6) \qquad \le 9(F(x))^{-1}(1-F(x))/n.$$

Hence for all $i \in A_n(x)$,

$$\left(F_i^{(n)}(x)\left(1-F_i^{(n)}(x)\right)\right)^{1/2} \le 3(F(x))^{-1/2}(1-F(x))^{1/2}/\sqrt{n}.$$

It is easily checked that there exists a constant $K_0 > 0$ independent of $x \in A$ and n such that

$$K \sum_{i=1}^{[(n+1)F(x)/2]} \frac{\left(\frac{i}{n+1}\right)^{-\varepsilon_1}}{n} \le K_0 \int_0^{F(x)} u^{-\varepsilon_1} du = K_0(1-\varepsilon_1)^{-1}(F(x))^{1-\varepsilon_1}.$$

Now let $C = 3K_0(1-\varepsilon_1)^{-1}$.

Then (2.5) $\le C(F(x))^{1/2-\varepsilon_1}(1-F(x))^{1/2}$. □

Hence (2.2) $\leq (2^{\varepsilon_1}K+C)\int_{-\infty}^{\infty}(F(x))^{1/2-\varepsilon_1}(1-F(x))^{1/2}dx$, which by 2.ii. is finite.

Now to show that (2.7) $\int_{A^c}(M_n^a(x,x))^{1/2}dx$ is uniformly bounded.

Let $B_n(x) = \{\ell: \; |F(x) - \frac{\ell}{n+1}| \geq n^{-1/8}\}$. Now by Minkowski's inequality $(M_n^a(x,x))^{1/2} \leq$

(2.8) $$\Big(\sum_{i,j\in B_n^c(x)}\sum M_a(\tfrac{i}{n+1})M_a(\tfrac{j}{n+1})(F_{ij}^{(n)}(x,x) - F_i^{(n)}(x)F_j^{(n)}(x))/n\Big)^{1/2} +$$

(2.9) $$\Big(\sum_{i,j\in B_n(x)}\sum M_a(\tfrac{i}{n+1})M_a(\tfrac{j}{n+1})(F_{ij}^{(n)}(x,x) - F_i^{(n)}(x)F_j^{(n)}(x))/n\Big)^{1/2}$$

We will assume without loss of generality that n is sufficiently large so that $n^{-1/8} < 1/4$.

Hence if $i \in B_n^c(x)$ and $x \in A^c$, $\frac{i}{n+1} > 1/4$.

Therefore (2.8) $\leq M(1/4)\;(F(x)(1-F(x)))^{1/2}$. Now to bound (2.9).

<u>Claim 2.2</u>

If i or $j \in B_n(x)$, there exists a constant $C > 0$ independent of n and x such that $F_{ij}^{(n)}(x,x) - F_i^{(n)}(x)F_j^{(n)}(x) \leq CF(x)(1-F(x))/n^{3/2}$.

<u>Proof</u> Assume $i \in B_n(x)$.

Note that if $i \in B_n(x)$, either $i \geq (n+1)F(x) + n^{-1/8}(n+1)$ or $i \leq (n+1)F(x)-n^{-1/8}(n+1)$.

<u>Case I.</u> $i \leq (n+1)F(x)+n^{-1/8}(n+1)$

Then $F_i^{(n)}(x) = P(S_n(x) \geq i) \leq P(|S_n(x) - nF(x)| \geq n^{-1/8}(n+1))$.

<u>Case II.</u> $i \leq (n+1)F(x)-n^{-1/8}(n+1)$

Then $1-F_i^{(n)}(x) = P(S_n^*(x) > n-i) \leq P(S_n^*(x) > n-(n+1)F(x) + n^{-1/8}(n+1))$
$\leq P(S_n^*(x) > n-1-nF(x) + n^{-1/8}(n+1)) =$

$$P(S_n^*(x) \geq n - nF(x) + n^{-1/8}(n+1)) \leq P(|S_n^*(x) - n(1-F(x))| \geq n^{-1/8}(n+1)).$$

Now by taking 4th moments and applying Chebychev's inequality in both cases, we have $F_i^{(n)}(x)$ in Case I and $1 - F_i^{(n)}(x)$ in Case II $\leq C(1-F(x))F(x)/n^{3/2}$. Where C is a constant independent of n and x.

The fact that

$$F_{ij}^{(n)}(x,x) - F_i^{(n)}(x)F_j^{(n)}(x) \leq \min(F_i^{(n)}(x)(1-F_i^{(n)}(x)),\ F_j^{(n)}(x)(1-F_j^{(n)}(x)))$$

completes the proof. □

Hence $(2.9) \leq \sum_{i=1}^{n} \frac{M_a(\frac{i}{n+1})}{n} \frac{(F(x)(1-F(x)))^{1/2}}{n^{1/4}}$, which is $\leq C^*(F(x)(1-F(x)))^{1/2}$

for some constant $C^* > 0$. We easily see now that there exists a constant C independent of n and x such that

$$\int_{A^c} (M_n^a(x,x))^{1/2} dx \leq C \int_{-\infty}^{\infty} (F(x)(1-F(x)))^{1/2} dx < \infty .$$

Therefore $\mathrm{Var}(\sqrt{n}\, T_n^a)$ is uniformly bounded. That the $\mathrm{Var}(\sqrt{n}\, T_n^b)$ is uniformly bounded follows by an argument similar to the above with F(x) replaced by 1-F(x). □

3. An Example.

Proposition 3.1 (A Variance Bound).

Let $U_n^{(i)}$ $1 \leq i \leq n$ be the ith order statistic from a sample of n independent uniform (0,1) random variables and $-1/2 < \alpha < 0$.

Then there exists a $C_\beta > 0$ independent of $n \geq 1$ and $1 \leq i \leq n$ such that for all $-1/2 < \alpha \leq \beta$

(3.1.i) $$\mathrm{Var}((1-U_n^{(i)})^\alpha) \le \frac{C_\beta}{n}\left(1-\frac{i}{n+1}\right)^{2\alpha-1}$$

Proof.

First a lemma.

Lemma 3.1

Let $-1/2 < \beta < 0$. There exists a $K_\beta > 0$ independent of $n \ge 1$ and $1 \le i \le n$ such that for all $-1/2 < \alpha \le \beta$

$$\prod_{j=n-i+1}^{n} 1/(1+\frac{2\alpha}{j}) < K_\beta\left(\frac{n-i+1}{n+1}\right)^{2\alpha}.$$

Proof.

Since $\sum_{j=1}^{n} (-\ln(1+\frac{2\alpha}{j}) + \frac{2\alpha}{j})$ converges uniformly for $\alpha \in [-1/2,\beta]$, there exists a $\gamma > 0$ dependent only on β such that

$$\sum_{j=n-i+1}^{n} (-\ln(1+\frac{2\alpha}{j}) + \frac{2\alpha}{j}) < \gamma \text{ for all } n \ge 1,\ 1 \le i \le n \text{ and } -1/2 \le \alpha \le \beta.$$

Hence for all $-1/2 < \alpha \le \beta$

$$-\ln\left(\sum_{j=n-i+1}^{n} 1/(1+\frac{2\alpha}{j})\right) \le \sum_{j=n-i+1}^{n} -\frac{2\alpha}{n} + \gamma.$$

But, $\sum_{j=n-i+1}^{n} -\frac{2\alpha}{j} \le -2\alpha(\ln(n+1) - \ln(n-\ell+1))$.

Therefore $\prod_{j=n-i+1}^{n} 1/(1+\frac{2\alpha}{j}) < e^\gamma\left(\frac{n-i+1}{n+1}\right)^{2\alpha}$.

Let $K_\beta = e^\gamma$. □

Now $\mathrm{Var}(1-U_n^{(i)})^\alpha = E(1-U_n^{(i)})^{2\alpha} - (E(1-U_n^{(i)})^\alpha)^2$

$$= \sum_{j=n-i+1}^{n} 1/(1+\frac{2\alpha}{j}) - \sum_{j=n-i+1}^{n} 1/(1+\frac{\alpha}{j})^2.$$

Set $f(x) = \prod_{j=n-i+1}^{n} 1/(1 + \frac{2\alpha}{j} + \frac{x^2}{j^2})$.

We see that $\mathrm{Var}(1-U_n^{(i)})^{\alpha} = f(0) - f(\alpha)$.

Now $f'(x) = -\sum_{j=n-i+1}^{n} \frac{2x}{j^2}(1/(1 + \frac{2\alpha}{j} + \frac{x^2}{j^2}))f(x)$

Observe that for $x \in [\alpha,0]$,

$$|f'(x)| \le 2|\alpha| \sum_{j=n-i+1}^{n} 1/j^2(1/(1+2\alpha))f(0).$$

which by Lemma 3.1 is

$$\le 2|\alpha| \sum_{j=n-i+1}^{n} 1/j^2(1/(1+2\alpha))K_{\beta}(\frac{n+1-i}{n+1})^{2\alpha},$$

which in turn is

$$\le 2|\alpha| \quad (1/(1+2\beta))K_{\beta}(1/(n-i+1))(\frac{n+1-i}{n+1})^{2\alpha}$$

$$\le 2|\alpha| \quad (1/(1+2\beta))K_{\beta}(\frac{n+1-i}{n+1})^{2\alpha-1}/n.$$

Let $C_{\beta} = (2/(1+2\beta))K_{\beta}$. The Mean Value Theorem completes the proof. □

We are now able to give an example of a distribution function F and an unbounded function J which satisfy the conditions of Corollary 1 to give

$\sqrt{n}(T_n - \mu_n) \xrightarrow[n\to\infty]{d} N(0,\sigma^2(J,F))$, but for which Theorem 1, Shorack (1972) is not applicable.

<u>Example</u>

Let $q(u) = F^{-1}(u) = \sum_{\ell=3}^{\infty} 2\ell 2^{-\ell}[(1-u)^{-1/2+\ell^{-1}} - u^{-1/2+\ell^{-1}}]/(\ell-2)$,

and $(J(u))^{1+\delta} = (\sum_{\ell=3}^{\infty} 2^{-\ell/2}(u(1-u))^{\ell^{-1}})^{-1/2}$ for some $\delta > 0$. Observe that $q'(u) =$

$\sum_{\ell=3}^{\infty} 2^{-\ell}[(1-u)^{-3/2+\ell^{-1}} + u^{-3/2} + \ell^{-1}]$. It is easy to see that for all $M_1 > 0$ and $0 < \delta^* < 1/2$ there exists a $u \in (0,1)$ such that $|F^{-1}(u)| > M_1(u(1-u))^{-1/2+\delta^*}$. We also see that J is unbounded on (0,1), but for all $\varepsilon > 0$ there exists an $M_\varepsilon > 0$ such that $M_\varepsilon(u(1-u))^{-\varepsilon} > J(u)$ for all $u \in (0,1)$. With these remarks it is a simple matter to check that F^{-1} and J do not satisfy the boundedness conditions of Theorem 1, Shorack (1972), which would require that J be a bounded function.

We will apply Corollary 1. First we must verify that the variance of F is finite. It is sufficient to show that $\int_0^1 (F^{-1}(u))^2 du < \infty$.

Claim 3.1.

$$\int_0^1 (F^{-1}(u))^2 du < \infty$$

Proof.

$\int_0^1 (F^{-1}(u))^2 du \leq$ by Schwarz's inequality and the c_r-inequality

$$\left(\sum_{\ell=3}^{\infty} (2\ell/(\ell-2))^2 2^{-\ell}\right)\left(\sum_{k=3}^{\infty} 2^{-k+1}\left(\int_0^1 (1-u)^{-1+2k^{-1}} du + \int_0^1 u^{-1+2k^{-1}} du\right)\right)$$

$$\equiv K_0 \sum_{\ell=3}^{\infty} \ell 2^{-\ell} < \infty \text{ for some } K_0 > 0. \quad \square$$

To verify 1.i' of Corollary 1, we will get suitable bounds for the covariances $g_{ij}^{(n)}$, $1 \leq i,j \leq n$.

Claim 3.2.

There exists a $C > 0$ independent of $n \geq 1$ and $1 \leq i \leq n$ such that $Var(F^{-1}(U_n^{(i)})) \leq$

$$C\left(\left(\frac{i}{n+1}\right)\left(\frac{n+1-i}{n+1}\right)\right)^{-2}\left(\sum_{\ell=3}^{\infty} 2^{-\ell}\left(\left(\frac{i}{n+1}\right)\left(\frac{n+1-i}{n+1}\right)\right)^{2\ell^{-1}}\right)/n.$$

Proof.

Let $\beta_\ell = E((1-U_n^{(i)})^{-1/2+\ell^{-1}} - (U_n^{(i)})^{-1/2+\ell^{-1}})$ and

$$\overline{\beta} = \sum_{\ell=3}^{\infty} (2\ell/(\ell-2))2^{-\ell}\beta_\ell = EF^{-1}(U_n^{(i)}).$$

Now by continuity of F^{-1}, we can pick an $\alpha \in (0,1)$ such that $F^{-1}(\alpha) = \overline{\beta}$. Hence $\mathrm{Var}(F^{-1}(U_n^{(i)})) =$

$$E\left(\int_\alpha^{U_n^{(i)}} q'(u)du\right)^2 = E(F^{-1}(U_n^{(i)}) - \overline{\beta})^2 =$$

$$E\left(\sum_{\ell=3}^{\infty} (2\ell/(\ell-2))2^{-\ell}[(1-U_n^{(i)})^{-1/2+\ell^{-1}} - (U_n^{(i)})^{-1/2+\ell^{-1}} - \beta_\ell]\right)^2$$

which by Schwarz's inequality is

$$\leq \sum_{\ell=3}^{\infty} (2\ell/(\ell-2))^2 2^{-\ell} \sum_{k=3}^{\infty} 2^{-k} E((1-U_n^{(i)}))^{-1/2+k^{-1}} - (U_n^{(i)})^{-1/2+k^{-1}} - \beta_K)^2$$

Note $E((1-U_n^{(i)}))^{-1/2+k^{-1}} - (U_n^{(i)})^{-1/2+k^{-1}} - \beta_K)^2$ by definition of β_K and Minkowski's inequality is

$$\leq ((\mathrm{Var}(1-U_n^{(i)})^{-1/2+k^{-1}})^{1/2} + (\mathrm{Var}(U_n^{(i)})^{-1/2+k^{-1}})^{1/2})^2.$$

which by application of Proposition 3.1 is

$$\leq \frac{C_\beta}{n}\left(\left(1-\frac{i}{n+1}\right)^{-1+k^{-1}} + \left(\frac{i}{n+1}\right)^{-1+k^{-1}}\right)^2$$

where $\beta = -1/2 + 1/3$,

which in turn is

$$\leq \frac{C^*_\beta}{n}\left(\left(1-\frac{i}{n+1}\right)\left(\frac{i}{n+1}\right)\right)^{-2+2k^{-1}}$$ for some $C^*_\beta > 0$.

Hence $\mathrm{Var}(F^{-1}(U_n^{(i)})) \leq C \sum_{\ell=3}^{\infty} 2^{-\ell}((1-\frac{i}{n+1})(\frac{i}{n+1}))^{-2+2\ell^{-1}}/n$

for some constant $C > 0$. □

Therefore $g_{ij}^{(n)} \leq$

$$\frac{C}{n}((\frac{i}{n+1})(1-\frac{i}{n+1})(\frac{j}{n+1})(1-\frac{j}{n+1}))^{-1}(\sum_{\ell=3}^{\infty} 2^{-\ell}((\frac{i}{n+1})(1-\frac{i}{n+1}))^{2\ell^{-1}} \sum_{k=3}^{\infty} 2^{-k}((\frac{j}{n+1})\cdot(1-\frac{j}{n+1}))^{2k^{-1}})^{1/2}$$

To verify 1.i', it is sufficient to show that there exists a nonnegative Riemann integrable function H on (0,1) such that for all $n \geq 1$, $1 \leq i,j \leq n$

$$\frac{H(\frac{i}{n+1})H(\frac{j}{n+1})}{n^2} \geq (J(\frac{i}{n+1})J(\frac{j}{n+1}))^{1+\delta} g_{ij}^{(n)}/n.$$

Note that $(\sum_{\ell=3}^{\infty} 2^{-\ell}((\frac{i}{n+1})(1-\frac{i}{n+1}))^{2\ell^{-1}})^{1/2} \leq \sum_{\ell=3}^{\infty} 2^{-\ell/2}((\frac{i}{n+1})(1-\frac{i}{n+1}))^{\ell^{-1}}$.

Hence by the above bound on $g_{ij}^{(n)}$ and definition of J, there exists a $C > 0$ such th[illegible]

$$(J(\frac{i}{n+1})J(\frac{j}{n+1}))^{1+\delta} g_{ij}^{(n)}/n \leq$$

$$C((\frac{i}{n+1})(1-\frac{i}{n+1}))^{-1}((\frac{j}{n+1})(1-\frac{j}{n+1}))^{-1}.$$

$$(\sum_{\ell=3}^{\infty} 2^{-\ell/2}((\frac{i}{n+1})(1-\frac{i}{n+1}))^{\ell^{-1}} \sum_{k=3}^{\infty} 2^{-k/2}((\frac{j}{n+1})(1-\frac{j}{n+1}))^{k^{-1}})^{1/2}/n^2$$

which is $\leq H(\frac{i}{n+1})H(\frac{j}{n+1})/n^2$.

Where $H(u) = \sqrt{C} \sum_{\ell=3}^{\infty} 2^{-\ell/4}(u(1-u))^{-1+(2\ell)^{-1}}$

Now $\int_0^1 H(u)du = \sqrt{C} \sum_{\ell=3}^{\infty} 2^{-\ell/4} \int_0^1 (u(1-u))^{-1+(2\ell)^{-1}} du.$

$$\leq 2\sqrt{C} \sum_{\ell=3}^{\infty} 2^{-\ell/4} \int_0^1 u^{1+(2\ell)^{-1}} du = 2\sqrt{C} \sum_{\ell=3}^{\infty} 2\ell 2^{-\ell/4} < \infty .$$

Hence by the above remarks 1.i' is satisfied, which by Corollary 1 implies

$$\sqrt{n}(T_n - \mu_n) \underset{n\to\infty}{\overset{d}{\to}} N(0,\sigma^2(J,F)) .$$

4. Appendix.

In Section 0, we defined F^{-1} to be $F^{-1}(u) = \sup\{x: F(x) \leq u\}$ for $u \in (0,1)$, Some easily verifiable properties of F^{-1} are: For $u \in (0,1)$,

4.i. If F^{-1} is continuous at u then $F(x) < F(F^{-1}(u)) < F(y)$ whenever $x < F^{-1}(u) < y$.

4.ii. $F^{-1}(u+) = F^{-1}(u)$ (right continuity on (0,1))

4.iii. If F^{-1} is constant on [0,u], then $F(x) = 0$ whenever $x \leq F^{-1}(u)$.

4.iv. $F^{-1}(u)$ can also be written: $F^{-1}(u) = \inf\{x: F(x) > u\}$.

Proposition 4.1

V may be written $V(u,v) = \int_0^u \int_0^v (s \wedge t - st) dF^{-1}(s) dF^{-1}(t)$.

Proof.

Let μ be Lebesque measure on $\mathcal{B}(R \times R)$, $T_1(x) = F(x)$, and $T_2(x,y) = (F(x),F(y))$. Note that T_2 is a measurable function from RxR to [0,1]x[0,1]. For $B \in \mathcal{B}([0,1]\times[0,1])$. Set $\mu^*(B) = \mu(T_2^{-1}(B)) = \int_{T_2^{-1}(B)} dxdy$. In particular, we have for B = CxD, where C, $D \in \mathcal{B}([0,1])$,

$$\mu^*(B) = (T^{-1}(B)) = \int_{T_2^{-1}(C \times D)} dxdy = \int_{T_1^{-1}(C)} dx \int_{T_1^{-1}(D)} dy.$$

Observation 4.1

Let $C = (c_1,c_2]$, $0 < c_1 < c_2 < 1$ and $D = (d_1,d_2]$, $0 < d_1 < d_2 < 1$.

Then $\int_{T_1^{-1}(C)} dx = \sup\{x: F(x) \leq c_2\} - \inf\{x: F(x) > c_1\}$

$= F^{-1}(c_2) - F^{-1}(c_1)$,

and similarly $\int_{T_1^{-1}(D)} dy = F^{-1}(d_2) - F^{-1}(d_1)$.

Thus for C and D of the form in Observation 4.1,

$$\int\int_{T_2^{-1}(C \times D)} dxdy = \int\int_{C \times D} d\mu^*(s,t) =$$

$$(F^{-1}(c_2) - F^{-1}(c_1))\ (F^{-1}(d_2) - F^{-1}(d_1)) = \int\int_{C \times D} dF^{-1}(s)dF^{-1}(t).$$

Let $g(s,t) = I_{[0,u]}(s)I_{[0,u]}(t)(s \wedge t - st)$. By Lemma 2, page 38 Lehmann (1959), we have:

$$V(u,v) = \int_0^1\int_0^1 g(s,t)d\mu^*(s,t) = \int\int_{\{0\} \times [0,1]} g(s,t)d\mu^*(s,t) +$$

$$\int\int_{[0,1] \times \{0\}} g(s,t)d\mu^*(s,t) + \int\int_{(0,1] \times (0,1]} g(s,t)d\mu^*(s,t).$$ The first two integrals are zero and the third equals

$$\int_{(0,u] \times (0,v]} (s \wedge t - st)d\mu^*(s,t).$$

Now, by using the continuity of $s \wedge t - st$, taking partitions of $(0,u] \times (0,v]$ of the form $(c_1,c_2] \times (d_1,d_2]$, and applying Observation 4.1. standard arguments complete the proof. □

Proposition 4.2

$V_n \xrightarrow{d} V$

proof

We need only show that

$\int_0^1\int_0^1 f(u,v)dV_n(u,v) \underset{n\to\infty}{\longrightarrow} \int_0^1\int_0^1 f(u,v)dV(u,v)$ for all bounded continuous functions on $[0,1] \times [0,1]$.

Let f be any bounded continuous function on [0,1] x [0,1]. It is sufficient to show that

$$(4.2.1)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} f(\tfrac{i}{n+1},\tfrac{j}{n+1}) \left(\frac{F_{ij}^{(n)}(x,y)-F_i^{(n)}(x)F_j^{(n)}(y)}{n}\right)dxdy \underset{n\to\infty}{\to}$$

$$(4.2.2)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(F(x),F(y))(F(x\wedge y)-F(x)F(y))dxdy$$

Let $H_n(x,y)$ be the integrand in (4.2.1) and $H(x,y)$ be the integrand in (4.2.2).

<u>Claim 4.2</u>

For each (x,y), $H_n(x,y) \underset{n\to\infty}{\to} H(x,y)$.

<u>proof</u>

Let $B_n(x)$ be as in Claim 2.2.

It can be shown that there exists a $C_{xy} > 0$ independent of $n \geq 1$ such that if $i \in B_n(x)$ or $j \in B_n(y)$

$F_{ij}^{(n)}(x,y) - F_i^{(n)}(x)F_j^{(n)}(y) \leq C_{xy}/n^{3/2}$.

(modify proof of Claim 2.2).

Now

$$| H_n(x,y) - H(x,y)| \leq$$

$$(4.2.3)\qquad \sum_{(i,j)\in B_n^c(x)\times B_n^c(y)}\sum \left|f(\tfrac{i}{n+1}, \tfrac{j}{n+1}) - f(F(x),F(y))\right|\left(\frac{F_{ij}^{(n)}(x,y)-F_i^{(n)}(x)F_j^{(n)}(y)}{n}\right)$$

+

$$(4.2.4)\qquad \sum_{(i,j)\notin B_n^c(x)\times B_n^c(y)}\sum \left|f(\tfrac{i}{n+1},\tfrac{j}{n+1}) - f(F(x),F(y))\right|\frac{(F_{ij}^{(n)}(x,y)-F_i^{(n)}(x)F_j^{(n)}(y))}{n}$$

Now

$$(4.2.3)\quad \leq \sigma^2 \max_{(i,j)\in B_n^c(x)\times B_n^c(y)} \left|f(\tfrac{i}{n+1},\tfrac{j}{n+1})-f(F(x),F(y))\right| \qquad (4.2.5)$$

and

$$(4.2.4)\quad \leq 2MC_{xy}/n^{1/2} \qquad (4.2.6),$$

where $M = \max|f|$.

By continuity of f, (4.2.5) $\underset{n\to\infty}{\to} 0$ and obviously (4.2.6) $\underset{n\to\infty}{\to} 0$. □

Now by application of the Bounded Convergence Theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_n(x,y)\,dxdy \underset{n\to\infty}{\to} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y)\,dxdy$$ □

This proof is merely a modification of a proof given by Stigler (1974) page 682.

In the following $C_m = \{u: \; |f(u)| > m\}$.

<u>Lemma A.1.</u>

Let f be a nonnegative function on (0,1), bounded on $(\frac{1}{a}, 1-\frac{1}{a})$ for all a > 2, continuous a.e. with respect to F^{-1} such that

$$\int_0^1\int_0^1 f(u)f(v)\,dV(u,v) < \infty, \text{ then}$$

A.1.i. $$\lim_{n\to\infty} \int_0^1\int_0^1 f(u)f(v)\,dV_n(u,v) = \int_0^1\int_0^1 f(u)f(v)\,dV(u,v)$$

iff

A.1.ii. $$\lim_{m\to\infty} \int_0^1\int_0^1 1_{C_m}(u)f(u)1_{C_m}(v)f(v)\,dV_n(u,v) = 0 \text{ uniformly in } n \geq 1.$$

<u>Lemma A.2.</u>

Let f and g be nonnegative functions on (0,1), bounded on $(\frac{1}{a}, 1-\frac{1}{a})$ for all a > 2, continuous a.e. with respect to F^{-1} such that

A.2.i. $$\lim_{n\to\infty} \int_0^1\int_0^1 f(u)f(v)\,dV_n(u,v) = \int_0^1\int_0^1 f(u)f(v)\,dV(u,v) < \infty$$

and

A.2.ii. $$\lim_{n\to\infty} \int_0^1\int_0^1 g(u)g(v)\,dV_n(u,v) = \int_0^1\int_0^1 g(u)g(v)\,dV(u,v) < \infty$$

then

A.2.iii. $\lim_{n\to\infty} \int_0^1 \int_0^1 f(u)\,g(v)\,dV_n(u,v) = \int_0^1 \int_0^1 f(u)\,g(v)\,dV(u,v) < \infty$

Lemma A.3.

Let f be a real valued function on (0,1), bounded in absolute value on $(\frac{1}{a}, 1-\frac{1}{a})$ for all $a > 2$, continuous a.e. with respect to F^{-1} such that

$$\int_0^1 \int_0^1 |f(u)|\ |f(v)|\,dV(u,v) < \infty,$$ then

A.3.i. $\lim_{n\to\infty} \int_0^1 \int_0^1 |f(u)||f(v)|\,dV_n(u,v) = \int_0^1 \int_0^1 |f(u)|\,|f(v)|\,dV(u,v)$

iff

A.3.ii. $\lim_{n\to\infty} \int_0^1 \int_0^1 f^+(u)f^+(v)\,dV_n(u,v) = \int_0^1 \int_0^1 f^+(u)f^+(v)\,dV(u,v)$

and

A.3.iii. $\lim_{n\to\infty} \int_0^1 \int_0^1 f^-(u)f^-(v)\,dV_n(u,v) = \int_0^1 \int_0^1 f^-(u)f^-(v)\,dV(u,v)$

Lemma A.4.

Let f be as in Lemma A.1, if

A.4.i $\int_0^1 \int_0^1 |f(u)f(v)|^{1+\delta}\,dV_n(u,v)$ is uniformly bounded for some $\delta > 0$ and $n \geq 1$.

then

A.4.ii. $\lim_{n\to\infty} \int_0^1 \int_0^1 f(u)f(v)\,dV_n(u,v) = \int_0^1 \int_0^1 f(u)f(v)\,dV(u,v)$

Lemma A.5.

Let f and g be nonnegative functions on (0,1), bounded on $(\frac{1}{a}, 1-\frac{1}{a})$ for all $a > 2$, continuous a.e. with respect to F^{-1} such that

$$\lim_{n\to\infty} \int_0^1 \int_0^1 f(u)g(v)dV_n(u,v) = \int_0^1 \int_0^1 f(u)g(v)dV(u,v) < \infty$$

Let a_m and b_m be sequences of constants in (0,1) such that for all $m \geq 1$, $1 > b_{m+1} > b_m > a_m > a_{m+1} > 0$, $\lim_{m\to\infty} a_m = 0$, $\lim_{m\to\infty} b_m = 1$ and $F^{-1}(\{a_m\}) = F^{-1}(\{b_m\}) = 0$.

Let $f_m(\text{resp. } g_m) = \begin{cases} f(u) & (\text{resp. } g(u)) \text{ if } u \in (a_m, b_m) \\ 0 & \text{elsewhere.} \end{cases}$

then

A.5.1. $\lim_{m\to\infty} \lim_{n\to\infty} \int_0^1 \int_0^1 (f(u)-f_m(u))(g(v)-g_m(v))dV_n(u,v) = 0$

Proof of Lemma A.1

That A.1.i. implies A.1.ii. is trivial.

Now assume A.2.ii.. Observe that $\int_0^1 \int_0^1 f(u)f(v)dV_n(u,v) =$

(A.1.1) $\int_0^1 \int_0^1 f(u)1_{C_m^c}(u)f(v)1_{C_m^c}(v)dV_n(u,v) +$

(A.1.2) $\int_0^1 \int_0^1 f(u)1_{C_m}(u)f(v)1_{C_m}(v)dV_n(u,v) +$

(A.1.3) $2\int_0^1 \int_0^1 f(u)1_{C_m^c}(u)f(v)1_{C_m}(v)dV_n(u,v).$

Since f is a.e. continuous with respect to F^{-1}, for every $m > 0$ there exists an $m' > m$ such that $1_{C_{m'}^c}(u)f(u)$ is bounded and continuous a.e. with respect to F^{-1}. Applying the Helly-Bray Theorem Loeve (1963) page 187, we get

$$\int_0^1 \int_0^1 1_{C^c_{m'}}(u)f(u)1_{C^c_{m'}}(v)f(v)dV_n(u,v) \underset{n\to\infty}{\to}$$

$$\int_0^1 \int_0^1 1_{C^c_{m'}}(u)f(u)1_{C^c_{m'}}(v)f(v)dV(u,v).$$

Now by A.1.ii., $\lim_{m\to\infty}$ (A.1.2) = 0 uniformly in $n \geq 1$.

By Schwarz's inequality (A.1.3) $\leq$

$$2\left(\int_0^1 \int_0^1 f(u)1_{C^c_m}(u)f(v)1_{C^c_m}(v)dV_n(u,v)\right)^{1/2}\left(\int_0^1 \int_0^1 f(u)1_{C_m}(u)f(v)1_{C_m}(v)dV_n(u,v)\right)^{1/2}$$

So that $\lim_{m\to\infty}$ (A.1.2) = 0 implies that $\lim_{m\to\infty}$ (A.1.3) = 0. Now it is a simple matter to show

$$\lim_{m'\to\infty} \lim_{n\to\infty} \int_0^1 \int_0^1 f(u)f(v)dV_n(u,v) - \int_0^1 \int_0^1 f(u)f(v)dV(u,v) = 0. \quad \square$$

Proof of Lemma A.5

$$\int_0^1 \int_0^1 (f(u)-f_m(u))(g(v)-g_m(v))dV_n(u,v) =$$

(A.5.1) $$\int_0^1 \int_0^1 f(u)g(v)dV_n(u,v) -$$

(A.5.2) $$\int_0^1 \int_0^1 f_m(u)g(v)dV_n(u,v) -$$

(A.5.3) $\int_0^1 \int_0^1 f(u)g_m(v)dV_n(u,v)$ +

(A.5.4) $\int_0^1 \int_0^1 f_m(u)g_m(v)dV_n(u,v)$.

So that we need only show that

$$\lim_{m\to\infty} \lim_{n\to\infty} \text{(A.5.1)} = \lim_{m\to\infty} \lim_{n\to\infty} \text{(A.5.2)} = \lim_{m\to\infty} \lim_{n\to\infty} \text{(A.5.3)} =$$

$$\lim_{m\to\infty} \lim_{n\to\infty} \text{(A.5.4)} = \int_0^1 \int_0^1 f(u)g(v)dV(u,v).$$

Note that (A.5.1) $\geq$ (A.5.2) V (A.5.3) $\geq$ (A.5.2) Λ (A.5.3) $\geq$ (A.5.4)

Thus it is sufficient to show:

A.5.ii. $\lim_{m\to\infty} \lim_{n\to\infty} \text{(A.5.4)} = \int_0^1 \int_0^1 f(u)f(v)dV(u,v)$.

Observe that for each fixed $m > 2$, $f_m(u)g_m(v)$ is bounded and continuous a.e. with respect to V. Thus by the Helly-Bray Theorem $\lim_{n\to\infty}$ (A.5.4) = $\int_0^1 \int_0^1 f_m(u)g_m(v)dV(u,v)$. Note $f_m(u)g_m(v) \uparrow f(u)g(v)$ as $m \to \infty$ a.e., with respect to V. Hence by the Monotone Convergence Theorem, A.5.ii. holds. This completes the proof. □

The proof of Lemma A.2 follows from steps analogous to Lemma A.1. The proofs of Lemmas A.3 and A.4 are merely applications of Lemma A.1.

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