L^p Estimates for stopping times of Bessel Processes

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Remark: If $f \in C_0^2(R_+)$ and satisfies the boundary condition $f'(0) = 0$ then obviously $f \in D(G)$.

Definition 1.3. The Bessel process of order $\gamma+1$ is the Markov process $y(t)$ with state space R_+ whose Kolmogorov backward differential equation is given by

$$(1.3) \quad \begin{cases} u_t(t,x) = Gu(t,x), \text{ G as in definition 1.2,} \\ \lim\limits_{t\to 0+} u(t,x) = u(0,x) = f \in D(G). \end{cases}$$

Remark: The choice $\gamma = (n-1)/2$ corresponds to the radial component of n-dimensional Brownian motion. For additional details concerning existence and uniqueness of solutions to the parabolic partial differential equation (1.3) the reader may insult Brezis, Rosenkrantz and Singer [1]. We can now state our extension of Theorem 1.1.

Theorem 1.4. Let τ be a stopping time for the Bessel process $y(t)$. Then for each $p \geq 1$ there exist constants $a(p,\gamma)$, $A(p,\gamma)$, independent of τ, such that

$$(1.4) \quad a(p,\gamma)E_0(\tau^p) \leq E_0\{y(\tau)^{2p}\} \leq A(p,\gamma)E_0\{\tau^p\}.$$

In addition $a(p,\gamma) \geq ((2\gamma+1)/p)^p$ for $p \geq 2$ and $A(p,\gamma) \leq (2p\gamma + p(2p-1))^p$, $p \geq 1$ and $a(1,\gamma) = A(1,\gamma) = 2\gamma + 1$.

Although there are some technical details that are not completely trivial the basic idea of the proof is extremely simple. All one needs to do is check that the function $(t,x) = t^p - Cx^2t^{p-1}, p > 1, C \geq p/2\gamma + 1)$, satisfies the differential inequality

$$(1.5) \quad v_t(t,x) + Gv(t,x) \leq 0.$$

In part II we show that this implies $v(t,y(t))$ is a supermartingale and hence for bounded stopping times τ, using Doob's optional stopping theorem [4], we deduce the inequality

(1.6) $E_0\{v(\tau,y(\tau))\} = E_0\{\tau^p - Cy(\tau)^2\tau^{p-1}\} \leq 0$. So

$$E_0\{\tau^p\} \leq CE_0\{y(\tau)^2\tau^{p-1}\}$$
$$\leq CE_0\{y(\tau)^{2p}\}^{1/p}\, E_0\{\tau^p\}^{(p-1)/p} \text{ or}$$

(1.7) $E_0(\tau^p)^{1/p} \leq CE_0\{y(\tau)^{2p}\}^{1/p}$, $p > 1$, $C \geq (p/(2\gamma+1))$.

The other inequality is deduced in the same way by considering the function $v(t,x) = x^{2p} - Ctx^{2p-2}$ where $p \geq 2$. The case $1 < p < 2$ is more delicate and requires a more careful construction of the supermartingale generating function $v(t,x)$. Finally the case p=1 follows from the fact that $v(t,x) = x^2-(2\gamma+1)t$ satisfies the partial differential equation

(1.8) $v_t(t,x) + Gv(t,x) = 0$

and then, as shown in part II, $v(t,y(t))$ is a martingale. Thus for every bounded stopping τ we have

$$E_0\{v(\tau,y(\tau))\} = E_0\{y(\tau)^2 - (2\gamma+1)\tau\} = 0, \text{ i.e.,}$$

(1.9) $E_0\{y(\tau)^2\} = (2\gamma+1)E_0(\tau)$. But this is the case p=1 of theorem 1.4. This result has a consequence of independent interest. Consider the process $Z(t) = y(t)/(2\gamma+1)^{\frac{1}{2}}$. It not only has continuous sample paths but in addition has the property that

(1.10) $E_0\{Z(\tau)^2\} = E_0\{\tau\}$ for every bounded stopping time τ. Professors D. Burkholder and B. Davis have pointed out to the authors, that this implies that Burkholder's distribution inequalities for Brownian motion (see Burkholder [2] chapter II) extend immediately to the class of Bessel processes considered here.

II. Some Martingales associated with the Bessel process.

The principal difficulty in establishing the supermartingale property for the process $v(t,y(t))$ is that ITO's formula cannot be applied for the simple reason that the drift term γ/x is unbounded in a neighborhood of the origin. In fact it cannot be applied even when $\gamma=0$. In this case $y(t) =$

$|w(t)|$, the reflected Brownian motion process. If one applies Ito's formula to $v(t,x)=x$ one obtains $v_t(t,x)+Gv(t,x) = 0$ and hence one might be tempted to conclude $v(t,y(t)) = y(t) = |w(t)|$ is a martingale, which is obviously false. The reason for this is $v(t,x) \notin D(G)$ because $v(t,x)$ does not satisfy the boundary condition $v_x(t,0) = 0$. For this reason a semi group version of Ito's formula due to Rosenkrantz [6] is not without interest.

Theorem 2.1. Suppose $v(t,x)$ and $v_t(t,x)$ are both jointly continuous in $R_+ \times R_+$ and that as functions of x we have $v(t,x) \in D(G)$, $v_t(t,x) \in D(G)$ all $t \geq 0$. Then the stochastic process

(2.1) $$v(t,y(t)) - \int_0^t \{v_s(s,y(s)) + Gv(s,y(s))\}ds \text{ is a martingale.}$$

Corollary 2.2. (a) If, in addition to the hypotheses of theorem 2.1, v satisfies the differential inequality

(2.2) $v_t(t,x) + Gv(t,x) \leq 0$ then $v(t,y(t))$ is a supermargingale.

If v satisfies the partial differential equation

(2.3) $v_t(t,x) + Gv(t,x) = 0$ then $v(t,y(t))$ is a martingale.

It often happens that $v(t,x)$ satisfies the smoothness conditions and boundary conditions of Theorem 2.1 but as a function of x, $v(t,x) \notin C_0(R^+)$. This difficulty is circumvented by means of the following device. Let $\varphi_n(x) \in C_0^\infty(R_+)$ and have the additional property that $\varphi_n(x) \equiv 1$, $0 \leq x \leq n$. Let τ_n devote the first passage time of the $y(t)$ process to the point $n > 0$. Clearly $v_n(t,x) = v(t,x)\varphi_n(x)$ now satisfies the hypotheses of Theorem 2.1. We are thus led to the following result which is more useful in the applications to come.

Theorem 2.3. Suppose $v(t,x) \in C^{0,2}(R_+)$, $v_t(t,x) \in C^{0,2}(R_+)$; $v_x(t,0) = 0$, $v_{tx}(t,0) = 0$. Then $v(t\wedge\tau_n, y(t\wedge\tau_n)) - \int_0^{t\wedge\tau_n} \{v_s(s,y(s)) + Gv(s,y(s))\}\, ds$ is a martingale.

Corollary 2.4. (a) If in addition to the hypotheses of theorem 2.3 v satisfies the differential inequality (2.2) then $v(t\wedge\tau_n, y(t\wedge\tau_n)$ is a supermartingale. (b). If v satisfies the differential equation (2.3) then $v(t\wedge\tau_n, y(t\wedge\tau_n))$ is a martingale.

A question which naturally arises in the present context is under what conditions is $v(t,y'(t))$ itself a supermartingale (or martingale)? More precisely can we let $n\to\infty$ in corollary 2.4? To answer this question we derive the following estimate:

(2.4) $$\lim_{n\to\infty} n^p P_x(\tau_n \le t) = 0 \text{ for every fixed } x,t.$$

To derive this estimate bring in the function $g(x,\lambda)$ which is monotonic increasing in x, satisfies the boundary condition $g'(0,\lambda) = 0$ and the differential equation $Gg(x,\lambda) = \lambda g(x,\lambda)$. This is just a second order linear differential equation with the origin a regular singular point. A routine calculation which we omit (see Rosenkrantz [6] p. 277-278) yields

(2.5) $$g(x,\lambda) = \sum_0^\infty g_k \lambda^k x^{2k} \text{ where}$$

$$g_k = (2^k k!\Gamma(\gamma + k + (\tfrac{1}{2})))^{-1}.$$

Now $v(t,x) = \exp(-\lambda t)g(x,\lambda)$ satisfies the differential equation $v_t + Gv = 0$ as well as the other conditions of corollary (2.4), so we may conclude $\exp(-\lambda(t\wedge\tau_n))g(y(t\wedge\tau_n),)$ is a martingale - in fact uniformly bounded by $g(n,\lambda)$. Thus $g(x,\lambda) = \lim_{t\to\infty} E_x\{\exp(-\lambda(t\wedge\tau_n))g(y(t\wedge\tau_n),\lambda)\}$

$$= E_x\{\exp(-\lambda\tau_n)g(n,\lambda)\}. \text{ So}$$

(2.6) $$E_x\{\exp(-\lambda\tau_n)\} = g(x,\lambda)/g(n,\lambda).$$

In particular $E_x\{\exp(-\lambda\tau_n);\ \tau_n \le t\} \le g(x,\lambda)/g(n,\lambda)$, and $E_x\{\exp(-\lambda\tau_n);\ \tau_n \le t\} \ge \exp(-\lambda t\ )P_x(\tau_n \le t)$.

(2.7) $\quad P_x(\tau_n \leq t) \leq \exp(\lambda t)g(x,\lambda)/g(n,\lambda)$.

From (2.5) we easily derive the result $\lim_{n\to\infty} n^p/g(n,\lambda) = 0$ and this suffices to to establish (2.4). We now proceed to the proof of Theorem 1.4.

Lemma 2.5.

(a) If $v(t,x) = x^2-(2\gamma+1)t$ then $v(t,y(t))$ is a martingale

(b) If $v(t,x) = t^p- C(p)x^2t^{p-1}$, $C(p) = p/(2\gamma+1)$, then $v(t,y(t))$ is a supermartingale

(c) If $v(t,x) = x^{2p}-c(p)tx^{2p-2}$, where $c(p) = 2p\gamma + p(2p-1)$ then $v(t,y(t))$ is a supermartingale for $p \geq 2$.

Remark: The case $1 < p < 2$ requires a separate, more delicate, argument and is therefore postponed.

Proof of Lemma 2.5. (a): It is clear that $v_x(t,0) = 0$, $v_{tx}(t,0) = 0$ so $v(t,x) = x^2-(2\gamma+1)t$ satisfies the conditions of corollary 2.4(6). Thus $v(t\wedge\tau_n,y(t\wedge\tau_n))$ is a martingale. Now

$$E_x\{y(t\wedge\tau_n)^2-(2\gamma+1)(t\wedge\tau_n)\} = E_x\{n^2-(2\gamma+1)\tau_n;\tau_n \leq t\}$$
$$+ E_x\{y(t)^2-(2\gamma+1)t;\ \tau_n > t\} = x^2.$$

Now $\lim_{n\to\infty} E_x\{n^2-(2\gamma+1)\tau_n;\tau_n \leq t\} = 0$. The first summand is just $n^2\ P(\tau_n \leq t)$ and tends to zero by (2.4). The second summand in absolute value is less than $(2\gamma+1)tP(\tau_n \leq t)$ which also goes to zero as $n\to\infty$. This proves $E_xv(t,y(t)) = v(0,x)$ which implies that $v(t,y(t))$ is a martingale.

Proof of lemma 2.5. (b): A straight forward computation which we omit shows that $v^\epsilon(t,x) = v(t+\epsilon,x)$ satisfies the hypotheses of corollary 2.4 (a) and hence $v^\epsilon(t\wedge\tau_n,y(t\wedge\tau_n))$ is a supermartingale. Now $E_x\{v^\epsilon(t\wedge\tau_n,y(t\wedge\tau_n)) =$
$E_x\{(\tau_n+\epsilon)^p-C(p)n^2(\tau_n+\epsilon)^{p-1};\ \tau_n \leq t\} + E_x\{(t+\epsilon)^p-C(p)y(t)^2(t+\epsilon)^{p-1};\ \tau_n > t\}$.

Exactly the same reasoning used in the proof of (a) shows that the first summand goes to zero as $n\to\infty$. Thus $E_x\{v^\epsilon(t,y(t))\} \le v^\epsilon(0,x)$ which proves that $v^\epsilon(t,y(t))$ is a supermartingale. Now we let ϵ decrease to 0 and conclude

$$E_x\{v(t,y(t))\} \le v(0,x).$$

Proof of lemma 2.5. (c): Once again it is easy to check that $v(t,x)$ satisfies the differential inequality (2.2) as well as the other conclusions of corollary 2.4 (a). Thus $v(t\wedge\tau_n, y(t\wedge\tau_n))$ is a supermartingale. Just as in the proofs of parts (a) and (b) we may let $n\to\infty$, using estimate (2.4), and conclude that $v(t,y(t))$ itself is a supermartingale.

As we observed in part I (just after the statement of Theorem 1.4) the inequality $a(p,\gamma)E_0\{\tau^p\} \le E_0\{y(\tau)^{2p}\}$, $p \ge 1$, follows at and from the fact that $x^2-(2\gamma+1)t$, and $t^p-C(p)x^2t^{p-1}$ generate martingales and supermartingales respectively. To get the inequality $E_0\{y(\tau)^{2p}\} \le A(p,\gamma)E_0(\tau^p)$, $p \ge 2$ we use the supermartingale generating function $v(t,x) = x^{2p}-c(p)tx^{2p-2}$ of lemma 2.5 (c). For τ a bounded stopping time Doob's optimal stopping theorem yields

$$E_0\{y(\tau)^{2p}-C(p)y(\tau)^{2p-2}\tau\} \le 0 \text{ - equivalently}$$

$$E_0\{y(\tau)^{2p}\} \le C(p)E_0\{y(\tau)^{2p-2}\tau\}$$

$$\le C(p)E_0\{y(\tau)^{2p}\}^{(p-1)/p}\, E_0\{\tau^p\}^{1/p}$$

where Holder's inequality has been used at the last step. Dividing both sides through by $E_0\{y(\tau)^{2p}\}^{(p-1)/p}$ completes the proof, at least in the case $p \ge 2$.

Lemma 2.6. For every bounded stopping time τ

$$E_0\{y(\tau)^{2p}\} = c(p)E_0\{\int_0^\tau y(s)^{2p-2}ds\}.$$

Proof: Apply Theorem 2.1 to $v(t,x) = x^{2p}$, $p \geq 1$ and taking note of the fact that $v_t = 0$, $Gv(t,x) = c(p)x^{2p-2}$ we deduce $y(t)^{2p} - \int_0^t c(p)y(s)^{sp-2}ds$ is a martingale. An application of Doob's optional stopping theorem completes the proof. We now assume $1 < p < 2$ and in particular that $(n+1)^{-1} \leq p-1 < n^{-1}$. For such p set $v(t,x) = t^{1-n(p-1)}x^{2(n+1)(p-1)}$ and $v^{\epsilon}(t,x) = v(t+\epsilon,(x^2+\epsilon)^{\frac{1}{2}})$. An elementary but tedious computation yields the inequality $v_t^{\epsilon}(t,x) + Gv^{\epsilon}(t,x) \geq ax^{2p-2}$ where a is a constant depending only on p,γ,n. By theorem 2.1 we have

$$E_0\{v^{\epsilon}(\tau,y(\tau))\} = \epsilon^p + E_0\ \{\int_0^{\tau}[v_s^{\epsilon}(s,y(s)) + Gv^{\epsilon}(s,y(s))]ds$$

$$\geq \epsilon^p + E_0\{\int_0^{\tau} ay(s)^{2p-2}ds\}$$

$$= \epsilon^p + (a/c(p))E_0\{y(\tau)^{2p}\}, \text{ by lemma 2.6.}$$

Let ϵ decrease to 0 and include

$$E_0\{\tau^{1-n(p-1)}y(\tau)^{2(n+1)(p-1)}\} \geq (a/c(p))E_0\{y(\tau)^{2p}\}.$$

The proof is now completed by applying Holder's inequality to the left hand side with exponents α,β so chosen that $(n+1)(p-1)\alpha = p$, $\beta = \alpha/(\alpha-1) = p/(1-n(p-1))$. The details are left to the reader. Incidentally the idea of this proof is due to L. Gordon [5].

III. Concluding remarks.

The methods used in part II are easily extended to a more general class of diffusion processes x(t) whose infinitesimal generater we denote by A. Suppose for example $A\varphi(x) \equiv 1$ and $\varphi(x) > 0$. Then $v(t,x) = t^p - p\varphi(x)t^{p-1}$ satisfies the differential inequality $v_t + Av \leq 0$. Thus $v(t,x(t))$ is a supermartingale and proceeding along a by now familiar route we get

$$E(\tau^p) \leq p^pE\{\varphi(x(\tau))^p\}.$$

Finally we observe that in the case $\gamma = (n-1)/2$ the supermartingales of the form $v(t,y(t))$ constructed in part II are actually supermartingales with respect to the larger sigma fields generated by the n-dimensional Brownian motion process itself. Hence theorem 2.1 remains valid for all τ which are stopping times relative to the n-dimensional Brownian motion process.

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