L_p Estimates for the exit times of a Bessel process

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Mimeograph Series #482

December 1976

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I. Introduction

Let $y(t)$ denote the Bessel process of order $\gamma + 1$ where $\gamma > -(1/2)$. Associated with $y(t)$ is the semigroup $T(t)f(x) = E\{f(y(t))|y(0) = x\} = E_x f(y(t))$ whose infinitesimal generator G is the singular second order linear differential operator

(1.1) $Gf(x) = (1/2)f''(x) + (\gamma/x)f'(x)$, $x \geq 0$ acting on the domain

(1.2) $$D(G) = \{f: f \in C_0[0,\infty), Gf \in C_0[0,\infty), f'(0) = 0\}.$$

$C_0[0,\infty)$ is the Banach space of bounded continuous functions with domain the right half line $[0,\infty)$, vanishing at infinity, and equipped with the sup norm $|f| = \sup_{x \geq 0}|f(x)|$. For further details on the semigroup $T(t) = \exp(tG)$ the reader should consult Brezis, Rosenkrantz and Singer (1971).

Theorem 1.1. Let τ be a stopping time for the Bessel process $y(t)$, and let $k = 1,2,\ldots,n$, denote a positive integer. Then there exist constants $a(\gamma,k)$ and $A(\gamma,k)$, <u>independent of τ</u>, such that

(1.3) $$a(\gamma,k)E_0(\tau^k) \leq E_0\{y(\tau)^{2k}\} \leq A(\gamma,k)E_0(\tau^k).$$

Remarks: When $\gamma = (n-1)/2$ then $y(t)$ is the radial component of n dimensional Brownian motion and for this choice of γ inequality (1.3) is a special case of recent results of Burkholder (1976) who stated them for the maximal function $y^*(\tau) = \sup_{0 \leq t < \infty} y(\tau \wedge t)$. More precisely he showed that

(1.4) $$c(p,n)E_0([n\tau]^{p/2}) \leq E_0(y^*(\tau)^p) \leq C(p,n)E_0([n\tau]^{p/2}),$$

for all p in the range $p > 0$. In addition, he obtains the result

$\lim_{n\to\infty} c(p,n) = \lim_{n\to\infty} C(p,n) = 1$. Our constants are less sharp. In fact, our estimates yield

$$(1.5)\quad \begin{cases} A(\gamma,k) \le [4k(\gamma+k-(1/2)]^k \\ a(\gamma,k) \ge [(\gamma+(1/2))/k]^k \end{cases}$$

Putting $\gamma = (n-1)/2$ in (1.5) yields

$$(1.6)\quad \begin{cases} n^{-k}A((n-1)/2,\ k) \le \{4k[(1/2)+(k-1)/n]\}^k \\ n^{-k}a((n-1)/2,\ k) \ge (2k)^{-k}, \end{cases}$$

and this is clearly less sharp than Burkholder's estimates. On the other hand, our estimates are valid for a much larger class of processes and we believe this extension as well as the method of proof are of independent interest. Incidentally, the method of proof used here appeared earlier in an unpublished paper of Rosenkrantz and Sawyer (1972).

II. A martingale generating function

Our proof of Theorem 1.1 is based upon a martingale generating function for the Bessel process $y(t)$ that we have constructed elsewhere - cf. Theorem 2 p. 278 of Rosenkrantz (1975).

Theorem 2.1. The function

$$u(t,x,\lambda) = \exp(-\lambda t)g(x,\lambda), \ \lambda > 0, \text{ where}$$

$$g(x,\lambda) = x^{(1/2)-\gamma} I_{\gamma-(1/2)} ([2\lambda]^{1/2} x),$$

$$I_\nu(y) = (y/2)^\nu \sum_{k=0}^{\infty} (y/2)^{2k}/[k!\Gamma(\nu+k+1))$$

is the modified Bessel function of order ν, is a martingale generating function i.e. $u(t,y(t),\lambda)$ is a martingale for every value of the parameter $\lambda > 0$.

If we set $\theta = (2\lambda)^{1/2}$ (so $\lambda = \theta^2/2$), perform some routine manipulations and cancel out the nonessential factor $(\theta/2)^{\gamma-1/2}$ we get the martingale generating function

$$(2.1) \quad v(t,x,\theta) = \exp(-\theta^2 t/2) \sum_{k=0}^{\infty} (\theta x)^{2k}/(4^k k!\Gamma(a+k)$$

for every value of the parameter θ, $a = \gamma+(1/2) > 0$, and Γ denotes the gamma function. Since $v(t,x,\theta)$ is analytic in θ we can expand it in a power series in θ like so:

$$(2.2) \qquad v(t,x,\theta) = \sum_{k=0}^{\infty} \theta^{2k} u_k(t,x).$$

Lemma 2.1. $u_k(t,y(t))$ is a martingale for $k = 0,1,2,\ldots$

Proof: Immediate.

Lemma 2.2. $u_k(t,x)$ is a polynomial of the $2k^{\underline{th}}$ degree in x and $k^{\underline{th}}$ degree in t of the form

$$(2.3) \qquad u_k(t,x) = \sum_{j=0}^{k} \alpha_j \beta_{k-j} x^{2k-2j} t^j \qquad \text{where}$$

$$(2.4) \qquad \alpha_j = (-1)^j/(2^j\, j!),\ \beta_j = 1/(4^j j!\Gamma(a+j)),\ j=0,1,\ldots$$

Proof: Just expand $\exp(-\theta^2 t/2)$ in a power series, then multiply the two power series occurring in (2.1) and collect the coefficients of θ^{2k}.

We now have at our disposal all the tools we shall need for proving Theorem 1.1. In what follows, we shall assume that τ is a bounded stopping time replacing τ by $\tau \wedge n$ if necessary. One can then let $n \to \infty$ and pass to the limit in the usual way.

Step 1. There exists a constant $a(\gamma,k)$, independent of the stopping τ, for which the estimate

$$(2.5) \qquad a(\gamma,k)E_0(\tau^k) \le E_0(y(\tau)^{2k}) \quad \text{holds.}$$

Proof: The optional stopping theorem of Doob (1953) applied to the martingale $u_k(t,y(t))$ yields

$$(2.6) \qquad E_0 u(\tau,y(\tau)) = \sum_{j=0}^{k} \alpha_j \beta_{k-j} E_0\{y(\tau)^{2k-2j}\tau^j\} = 0.$$

Thus

$$(2.7) \qquad |\alpha_k \beta_0|\ E_0(\tau^k) \le \sum_{j=0}^{k-1} |\alpha_j \beta_{k-j}|\ E_0\{y(\tau)^{2k-2j}\tau^j\}.$$

To each term on the right we apply Holder's inequality with $q = k/j$, $p = k/k-j$ and get $E_0\{y(\tau)^{2k-2j}\tau^j\} \le E_0\{y(\tau)^{2k}\}^{1-(j/k)} E_0(\tau^k)^{j/k}$.

Thus from (2.7) we deduce the inequality

$$(2.8) \qquad |\alpha_k\beta_0|\ E_0(\tau^k) \leq \sum_{j=0}^{k-1} |\alpha_j\beta_{k-j}| E_0\{y(\tau)^{2k}\}^{1-(j/k)} E_0\{\tau^k\}^{j/k}.$$

set $Z = [E_0\{\tau^k\}/E_0\{y(\tau)^{2k}\}]^{1/k}$ and divide both sides of (2.8) by $E_0\{y(\tau)^{2k}\}$ and $|\alpha_k\beta_0|$.

Inequality (2.8), rewritten in terms of Z, becomes

$$(2.9) \qquad Z^k - \sum_{j=0}^{k-1} (|\alpha_j\beta_{k-j}|/|\alpha_k\beta_0|)Z^j \leq 0.$$

The right hand side (2.9) is a polynomial $Q_k(Z)$ which has a root of largest modulus ρ_k. Clearly $Z \leq \rho_k$. But this is equivalent to the statement

$$E_0\{\tau^k\}/E_0\{y(\tau)^{2k}\} \leq \rho_k^{\ k}.$$

So $a(\gamma,k) \geq \rho_k^{-k}$.

The root of largest modulus ρ_k of the polynomial $Q_k(Z)$ may be estimated by means of an inequality to be found in Mitrinovic (1970):

Lemma 2.3. Let $r_1,\dots,r_k$ denote the roots of the polynomial

$$Q(Z) = Z^k + \sum_{1=i}^{k} u_j Z^{k-i}, \quad |u_i| \neq 0,\ i=1,\dots,k.$$

Let $r = \max_{1 \leq j \leq k} |r_j|$. Then

$$(2.10) \qquad r \leq \text{Max}\ (2|u_1|,\ 2|u_2/u_1|,\dots,2|u_{k-1}/u_{k-2}|,\ |u_k/u_{k-1}|).$$

We now apply this estimate to the polynomial $Q_k(Z)$ with $u_\ell = |\alpha_{k-\ell}\beta_\ell/\alpha_k\beta_0|$ and obtain the estimate $Z \leq k/a$. Hence

$$(2.11) \qquad a(\gamma,k) \geq (a/k)^k = [(\gamma+1/2)/k]^k.$$

Step 2: There exists a constant $A(\gamma,k)$, independent of the stopping time τ, for which the estimate $E_0(y(\tau)^{2k}) \leq A(\gamma,k)E_0(\tau^k)$ holds.

Proof: As in step 1 we apply Doob's optional stopping theorem to the martingale $u_k(t,y(t))$. This leads to the equation

$$(2.12)\quad E_0\{u_k(\tau,y(\tau))\} = 0 = \beta_k E_0\{y(\tau)^{2k}\} + \sum_{j=1}^{k-1}\alpha_j\beta_{k-j}\, E_0\{y(\tau)^{2k-2j}\tau^j\}$$

Applying Holder's inequality as before leads to the inequality

$$(2.13)\quad \beta_k E_0\{y(\tau)^{2k}\} - \sum_{j=1}^{k-1} |\alpha_j\beta_{k-j}| E_0\{y(\tau)^{2k}\}^{1-j/k} E_0\{\tau^k\}^{j/k} + \alpha_k\beta_0 E_0(\tau^k) \leq 0.$$

Dividing through by $E_0(\tau^k)$ and β_k and setting $s^k = E_0\{y(\tau)^{2k}\}/E_0\{\tau^k\}$ transforms (2.13) into

$$(2.14)\qquad P_k(s) = s^k - \sum_{j=1}^{k-1} |\alpha_j\beta_{k-j}/\beta_k| s^{k-j} + (\alpha_k\beta_0)/\beta_k \leq 0.$$

Thus s is smaller than ρ_k' where $\rho_k' = \max_{1\leq j\leq k} |s_j|$, $s_1,\ldots,s_k$ are the roots of the polynomial $P_k(s)$. We now apply estimate (2.10) to the polynomial $P_k(s)$ and after a routine calculation obtain the estimate

$$(2.15)\qquad A(\gamma,k) \leq (\rho_k')^k \leq [4k(\gamma+k-(1/2))]^k \quad \text{q.e.d.}$$

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