The effect of dead time on the physical generation of random digits

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Introduction. It has frequently been proposed to use physically generated random digits for simulation. One of the methods is to use particle counts from radioactive decay or similar processes. This has the advantage over other methods in that it is already digital. If counting takes place modulo B, the results at the sampling time are used as random digits to base B. If the arrival process is Poisson with rate 1, the distribution of the digits approaches the uniform at the rate $\exp(-(1-\cos \frac{2\pi}{B})t)$. It is easily seen from this that 3 is the best base, and 2 and 4 are next.

However, counters have a "dead time". We are then led to the question as to the effect of this on the generation. The effect can be looked at in two ways: the dead time can be adjusted given the arrival rate, or the arrival rate can be adjusted given the dead time. We conjecture that for $B \geq 3$ the optimal dead time for a given arrival rate is 0, but that in general the arrival rate for given dead time should be 2-2.5 dead times. However for $B = 2$ the optimal dead time for a fixed arrival rate is always positive for sufficiently regular distributions of dead time and frequently the asymptotically optimal dead time for fixed arrival rate and the asymptotically optimal arrival rate for fixed dead time are reciprocals. This can even be approximately true for relatively short times. This

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phenomenon makes the convenient base 2 optimal in the presence of dead time.

However, the asymptotic nature is not all that important. Since, from the previous discussion, the counter should be operated at a high rate, it will frequently be interrogated while in transition. It is unlikely that, even with the counter not being directly interrogated but driving a flip-flop circuit, that the reading mechanism can be unbiased to much better than one part in 10^5. In addition, since anything that can go wrong is likely to happen, the output should be tested. As the capacity of a tape is $<10^9$ bits and it is unlikely that more than a few tens of millions of bits can be handled for a single test, the reliability of the output unit cannot be tested for much better than 10^{-3}. This is much too low an accuracy for random numbers, and we propose that several tapes should be added (mod 2) to obtain the final random information for computational use. But this makes the performance of the counter system for relatively short periods important. For the case $B = 2$, we have computed the actual performance for Type I and Type II counters, and charts are appended give the time required from the start of a dead time to achieve a given level of $|P(0) - P(1)|$ for levels .1, .05, .02, .01,...,.00001. The discontinuites arise after the approach to 0 becomes oscillatory.

Methods and results. The times at which dead times begin form a renewal process. Let μ, s be a complex numbers. Then for $|\mu| \leq 1$ and $R(s) < 0$, it is well known that the generating function of the number of renewals starting at a renewal has the Laplace transform [1]

$$(1) \qquad g(s,\mu) = \int_0^\infty e^{st}\, E(\mu^{N(t)})\,dt$$

$$= \frac{1}{s}\,\frac{\varphi(s) - 1}{1-\mu\varphi(s)},$$

where $\varphi(s) = E(e^{st})$, T the renewal time. (If we do not start at a renewal, the φ in the numerator only is changed.) Now since the renewal period is the refractory period plus the wait for an arrival after its termination, $\varphi(s) = \frac{f(s)}{1-\lambda s}$. If we use arrival rate units, $\lambda = 1$, and 1 becomes

$$(2) \qquad g(s,\mu) = \frac{f(s)+s-1}{s(1-s-f(s))}.$$

If f is analytic, this formula is valid for all s whose real part is less than both the abscissa of analyticity of f and the smallest real part of a nontrivial zero of the denominator. Now we are only interested in (2) for μ a non-trivial B-th root of unity. We can get some insight by letting $f(s) = h(\alpha s)$, $\alpha > 0$. Then for small α, the real part of the critical zero increases with α if $R(\mu) < -.5$, and decreases otherwise. Thus if $B > 2$, the rate determined by the root $\mu = \exp \pm \frac{2\pi i}{B}$ gets worse with increasing α. If B=2, $\mu=-1$ and the improvement occurs, the convergence rate can be asymptotically more than twice as good. As a simple example, let the dead time distribution be exponential with mean 1/6. Then

$$(3) \qquad P_t(0) - P_t(1) = \begin{matrix} 4e^{-3t}-3e^{-4t} \\ 2e^{-3t}-e^{-4t} \end{matrix}$$

according as the counter is or is not dead at t=0.

The graphs were obtained by finding the last t with the desired accuracy, by power series until the solution could be replaced by the asymptotic solution using either the two real roots or the two complex roots, and by the asymptotic expression afterwards. Since $\delta(t) = P_t(0) - P_t(1)$ satisfies the equation, with dead time as the unit,

$$\delta'(t) = -\alpha(\delta(t) + \delta(t-1)) \tag{4}$$

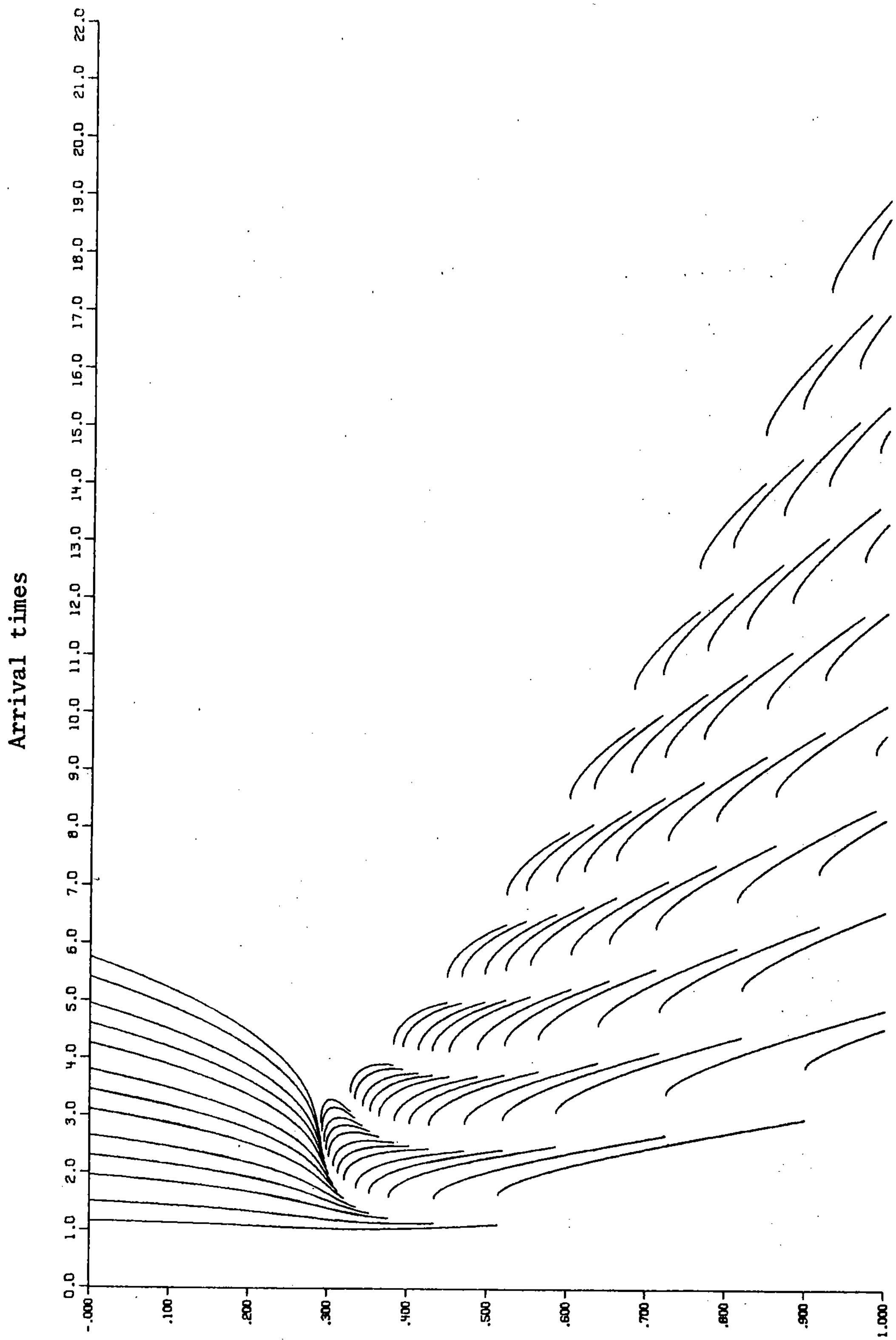
for a type I counter and

$$\delta'(t) = -2\alpha e^{-\alpha}\delta(t-1) \tag{5}$$

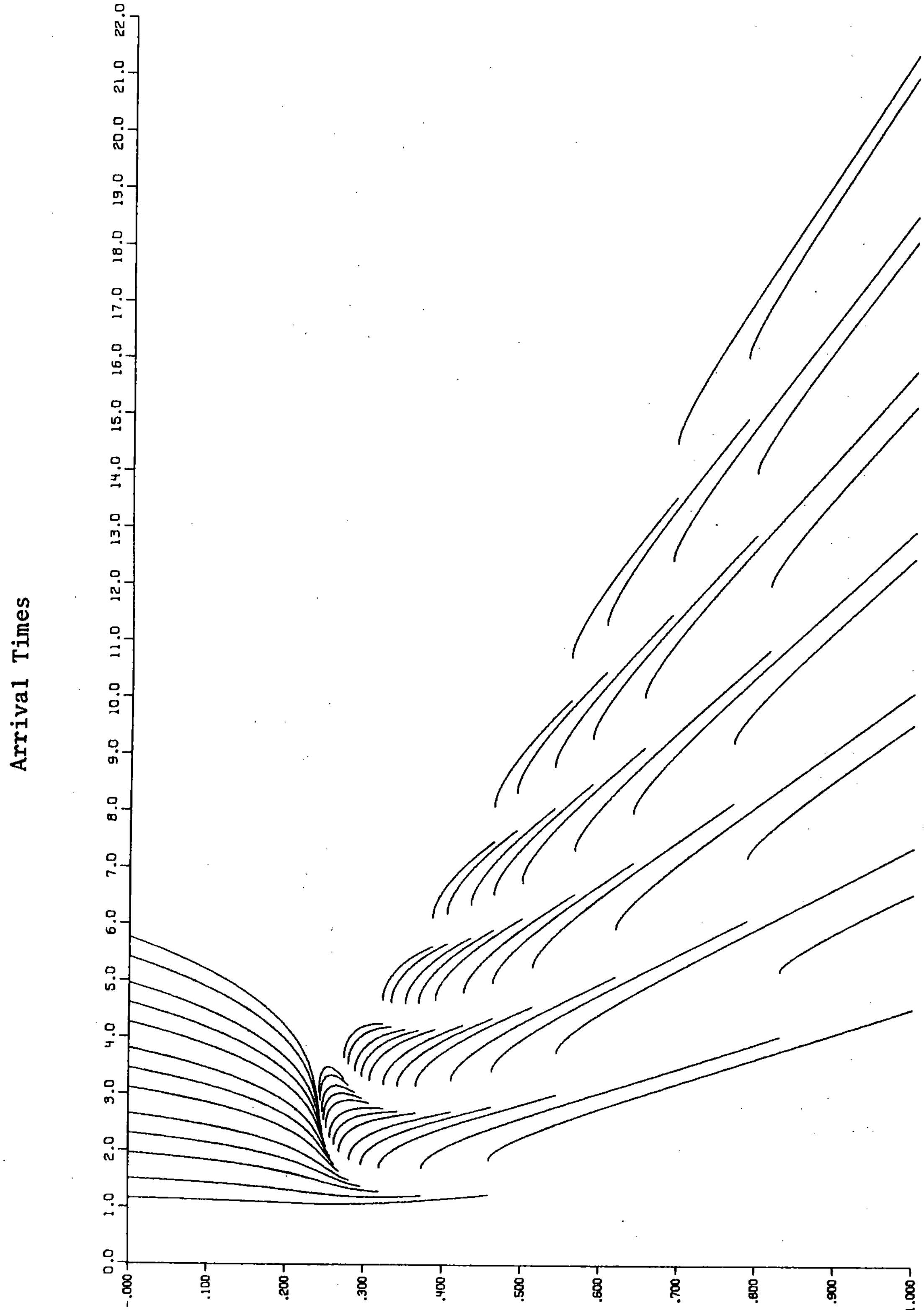
for a type II counter, this was rather easily done.

REFERENCES

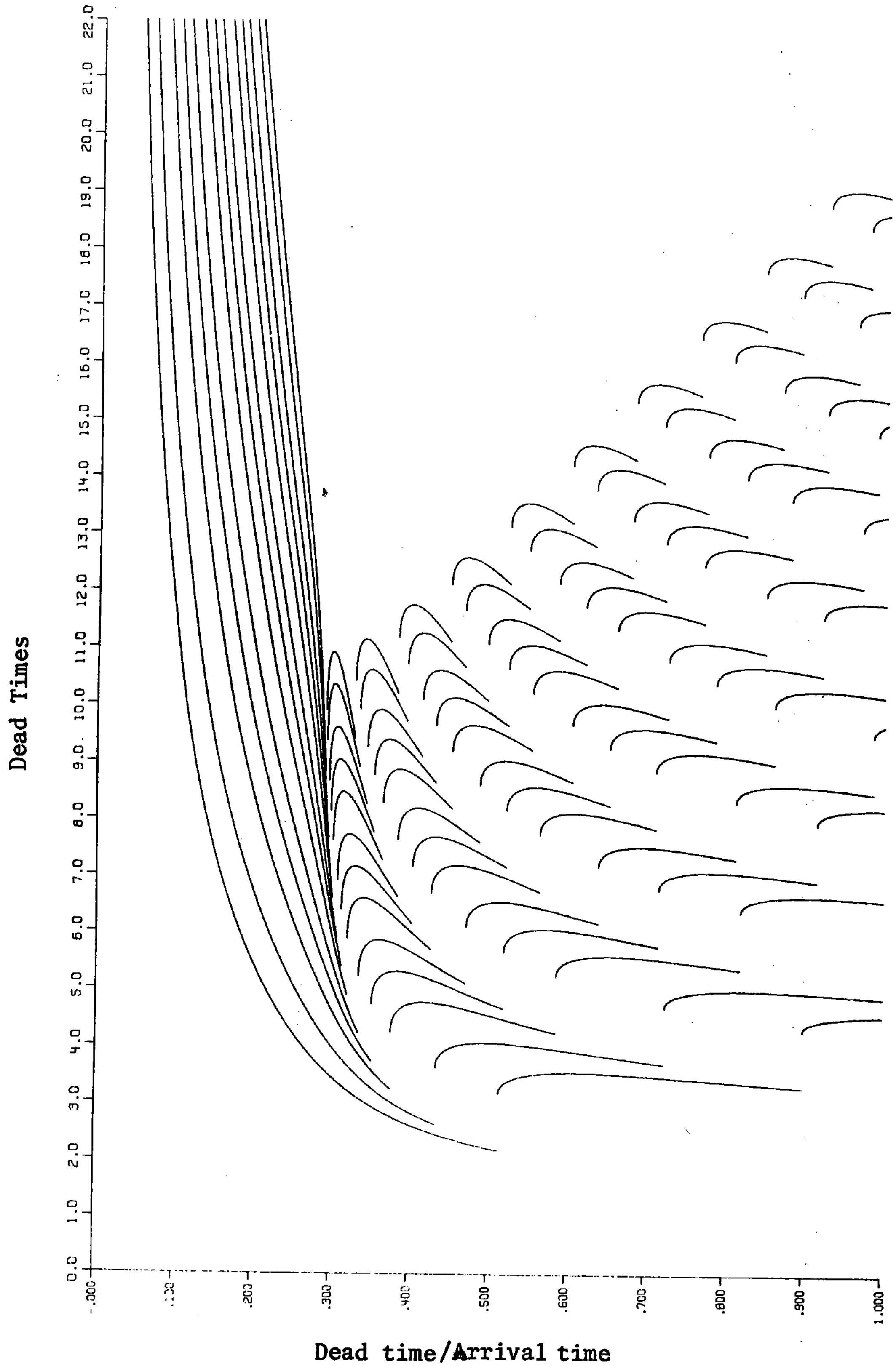
[1] Cox, D.R. (1962). Renewal Theory. Mathow. London.



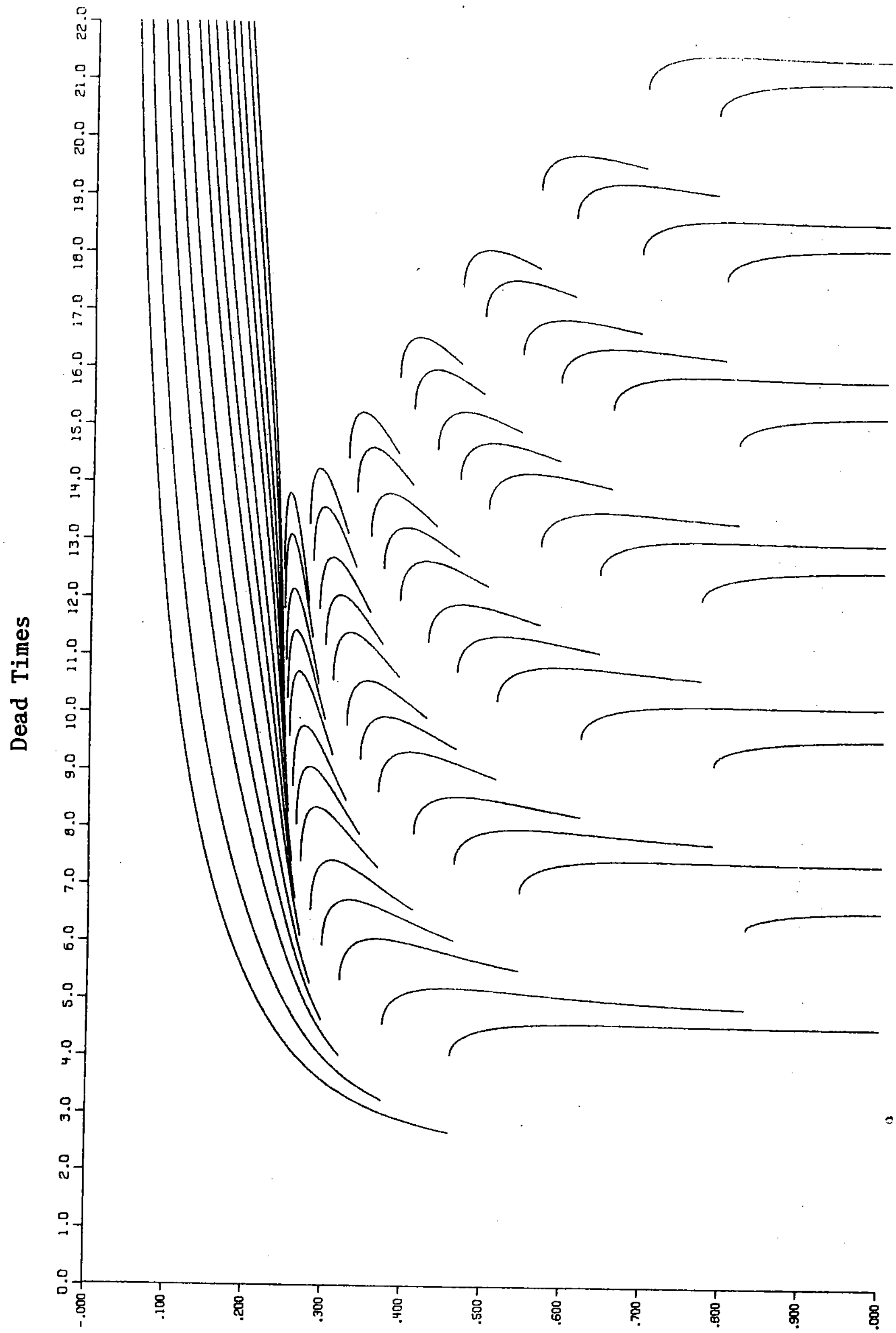
Dead time/Arrival time
Type I



Dead time/Arrival time
Type II


Dead Times
0.0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
11.0
12.0
13.0
14.0
15.0
16.0
17.0
18.0
19.0
20.0
21.0
22.0
.200
.300
.400
.500
.600
.700
.800
.900
1.000
Dead time/Arrival time

Type I



Dead time/Arrival time
Type II