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ON SUBSET SELECTION PROCEDURES FOR POISSON PROCESSES AND SOME APPLICATIONS TO THE BINOMIAL AND MULTINOMIAL PROBLEMS*

by

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Mimeograph Series #457

July 1976

*This research was supported by the Office of Naval Research under Contract N00014-75-C-0455 at Purdue University. Reproduction in whole or part is permitted for any purpose of the United States Government.

Let $\pi_1,\ldots,\pi_k$ be k Poisson processes with mean rates $\lambda_1^{-1},\ldots,\lambda_k^{-1}$, respectively. Let $\lambda_{[1]} \leq \cdots \leq \lambda_{[k]}$ denote the ordered set of the values $\lambda_1,\ldots,\lambda_k$. The process associated with $\lambda_{[1]}$ ($\lambda_{[k]}$) is defined to be the best process. All through this paper we assume that λ's are unknown and that there is no a priori information available about the correct pairing of the ordered $\lambda_{[i]}$ values and the k given Poisson processes. Our problem is to define a subset selection procedure which selects a small, non-empty subset of the k processes and guarantees that the selected subset includes the best process with probability at least P^*, $k^{-1} < P^* < 1$. If CS stands for a correct selection then our goal is to define a selection rule R such that

$$\inf_{\Omega} P(CS|R) \geq P^* \tag{1.1}$$

where Ω is the set of all k-tuples $(\lambda_1,\ldots,\lambda_k)$, $\lambda_i > 0$, $i = 1,\ldots,k$.

In Section 2, some subset selection rules for selecting a subset containing the process with the smallest value $\lambda_{[1]}$ are proposed. The probability of a correct selection is evaluated. Some properties of the proposed selection rules are discussed. Section 3 deals with the analogous problem of selecting the process for which the associated value λ is the largest. In Section 4, applications to binomial and multinomial selection problems are considered.

2. Selection Procedures for the Process Associated with $\lambda_{[1]}$

In this section, four different selection rules are proposed.

(A) Procedure R_1 and Its Properties

Let $X_1(t),\ldots,X_k(t)$ denote the number of arrivals from processes $\pi_1,\ldots,\pi_k$ during time t, respectively. Let $X_{(i)}(t)$ and $\pi_{(i)}$ be associated with $\lambda_{[i]}$, $i = 1,\ldots,k$. Let N be a fixed positive integer. We propose a subset selection rule as follows:

R_1: Observe the processes until $\max_{1\le i\le k} X_i(t) = N$. Select process π_i if and only if

(2.1) $$X_i(t) \ge N-c_1$$

where $c_1 = c_1(k,P^*,N)$ is the smallest non-negative integer for which the condition (1.1) is satisfied.

Before we derive some properties of the selection rule, we introduce some definitions. Let $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k) \in \Omega$. Define

(2.2) $$p_{\underline{\lambda}}(i|R) = P_{\underline{\lambda}}(\pi_{(i)} \text{ is selected}|R).$$

Definition 2.1. A rule R is said to be (reverse) strongly monotone in $\pi_{(i)}$ if

$$p_{\underline{\lambda}}(i|R) \text{ is } \begin{cases} (\downarrow)\uparrow \text{ in } \lambda_{[i]} \text{ when all other components of } \underline{\lambda} \text{ are fixed.} \\ (\uparrow)\downarrow \text{ in } \lambda_{[j]}\ (j \neq i) \text{ when all other components of } \underline{\lambda} \text{ are fixed.} \end{cases}$$

Gupta [8] has proved that the subset selection rules which he studied possess the properties of monotonicity and unbiasedness. We recall these definitions (see Santner [17]).

Definition 2.2. The rule R is (reverse) monotone means for all $1 \le i < j \le k$, and $\underline{\lambda} \in \Omega$,

(2.3) $$p_{\underline{\lambda}}(i|R)\ (\ge) \le p_{\underline{\lambda}}(j|R).$$

Definition 2.3. The rule R is unbiased means for all $1 \le i \le k$, and $\underline{\lambda} \in \Omega$,

$p_{\underline{\lambda}}$(R does not select $\pi_{(i)}$) $\geq p_{\underline{\lambda}}$(R does not select the best process).

Remark 2.1. (1) If a rule R is (reverse) strongly monotone in $\pi_{(i)}$ for all $i = 1,\ldots,k$, then R is (reverse) monotone and

(2.4) $$\inf_{\Omega} P(CS|R) = \inf_{\Omega_0} P(CS|R)$$

where $\Omega_0 = \{\underline{\lambda}: \underline{\lambda} = (\lambda,\ldots,\lambda),\ \lambda > 0\}$.

(2) If R is (reverse) monotone, then it is unbiased.

Let $T_i(N)$ denote the waiting time for N arrivals for the process π_i, $i = 1,\ldots,k$. $T_i(N)$ is distributed according to gamma distribution with density given by

(2.5) $$f_{i,N}(t) = \frac{1}{\Gamma(N)\lambda_i^N} t^{N-1} e^{-\frac{t}{\lambda_i}},\ t > 0$$

It is easy to see that rule R_1 can be rewritten as follows: Select process π_i if and only if

(2.6) $$T_i(N-c_1) \leq \min_{1\leq j\leq k} T_j(N).$$

Let $T_{(i)}(N)$ denote the unknown waiting time for N arrivals for the process $\pi_{(i)}$, $i = 1,\ldots,k$. Then for any $\underline{\lambda} \in \Omega$,

$$p_{\underline{\lambda}}(i|R_1) = P_{\underline{\lambda}}(T_{(i)}(N-c_1) \leq \min_{1\leq j\leq k} T_{(j)}(N))$$

(2.7) $$= \int_0^\infty \prod_{\substack{j=1\\ j\neq i}}^{k} \{1-G_N(\frac{\lambda_{[i]}}{\lambda_{[j]}} t)\} dG_{N-c_1}(t),$$

where

(2.8) $$G_r(x) = \int_0^x \frac{1}{\Gamma(r)} t^{r-1} e^{-t}\, dt.$$

It follows from (2.7) that procedure R_1 is reverse strongly monotone in $\pi_{(i)}$ for all $i = 1,\ldots,k$. Furthermore

(2.9) $$\inf_{\Omega} P(CS|R_1) = \inf_{\Omega_0} P(CS|R_1) = \int_0^{\infty}\{1-G_N(t)\}^{k-1}\, dG_{N-c_1}(t)$$

which is independent of the common unknown parameter. Hence we have proved the following theorem.

Theorem 2.1. The procedure R_1 is reverse strongly monotone in $\pi_{(i)}$ for all $i = 1,\ldots,k$, and the infimum of the probability of a correct selection occurs when all the processes are identical and the infimum does not depend on the common unknown parameter.

Remark 2.2. In order to find the selection constant c_1 so as to satisfy the condition (1.1), we solve for the smallest integer ($0 \leq c_1 \leq N$) which satisfies

(2.10) $$\int_0^{\infty}\{1-G_N(t)\}^{k-1}\, dG_{N-c_1}(t) \geq P^*.$$

For given k, N and P*,values of c_1 have been computed along with the actual values of the probabilities.

Consistent with the basic probability requirement (1.1), we would like the size of the selected subset to be small. Now, S, the size of the selected subset is a random variable which takes values 1,2,...,k. Hence one criterion of the efficiency of the procedure R_1 is the expected value of the size of the selected subset. The expected value of S is given by

(2.11) $$\begin{aligned} E_{\underline{\lambda}}(S|R_1) &= \sum_{i=1}^{k} P_{\underline{\lambda}}(\pi_{(i)} \text{ is selected}|R_1) \\ &= \sum_{i=1}^{k} \int_0^{\infty} \prod_{\substack{j=1 \\ j\neq i}}^{k} \{1-G_N(\frac{\lambda_{[i]}}{\lambda_{[j]}}\, t)\}dG_{N-c_1}(t). \end{aligned}$$

It will now be shown that the maximum of $E_{\underset{\sim}{\lambda}}(S|R_1)$ takes place when all the parameters λ_i are equal. If we set the m largest parameter $\lambda_{[i]}$ $(1 \leq m < k)$ equal to a common value λ (say), we obtain from (2.11) that

$$(2.12) \qquad E_{\underset{\sim}{\lambda}}(S|R_1) = m \int_0^\infty \{1-G_N(t)\}^{m-1} \cdot \prod_{j=1}^{k-m} \{1-G_N(\frac{\lambda}{\lambda_{[j]}} t)\} dG_{N-c_1}(t)$$

$$+ \sum_{i=1}^{m-k} \int_0^\infty \{1-G_N(\frac{\lambda_{[i]}}{\lambda} t)\}^m \prod_{\substack{j=1 \\ j \neq i}}^{m-k} \{1-G_N(\frac{\lambda_{[i]}}{\lambda_{[j]}} t)\} dG_{N-c_1}(t).$$

We now show that the right hand member of (2.12) is a decreasing function of λ for $k^{-1} < P^* < 1$. Since this holds for integer $m < k$, this proves that the maximum value of $E_{\underset{\sim}{\lambda}}(S|R_1)$ occurs when $\lambda = \lambda_{[1]}$, and the desired result will follow. To show that $E_{\underset{\sim}{\lambda}}(S|R_1)$ is monotone, we differentiate $E_{\underset{\sim}{\lambda}}(S|R_1)$ with respect to λ and show that the derivative is negative for $k^{-1} < P^* < 1$. Differentiation gives

$$(2.13) \qquad \frac{\partial}{\partial \lambda} E_{\underset{\sim}{\lambda}}(S|R_1) = -m \sum_{i=1}^{k-m} \int_0^\infty \{1-G_N(t)\}^{m-1} \prod_{\substack{j=1 \\ j \neq i}}^{k-m} \{1-G_N(\frac{\lambda}{\lambda_{[j]}} t)\} \cdot$$

$$\frac{1}{\Gamma(N)} (\frac{\lambda}{\lambda_{[j]}} t)^{N-1} e^{-\frac{\lambda_{[j]}}{\lambda} t} \frac{t}{\lambda_{[j]}} dG_{N-c_1}(t)$$

$$+ m \sum_{i=1}^{k-m} \int_0^\infty \{1-G_N(\frac{\lambda_{[i]}}{\lambda} t)\}^{m-1} \prod_{\substack{j \neq 1 \\ j=i}}^{k-m} \{1-G_N(\frac{\lambda_{[i]}}{\lambda_{[j]}} t)\} \cdot$$

$$\frac{1}{\Gamma(N)} (\frac{\lambda_{[i]}}{\lambda} t)^{N-1} e^{-\frac{\lambda}{\lambda_{[i]}} t} \frac{\lambda_{[i]}}{\lambda^2} t \, dG_{N-c_1}(t).$$

If we let $\lambda t = \lambda_{[i]} t'$ in the first integral and drop primes then (2.13) becomes

(2.13) $$\frac{\partial}{\partial\lambda} E_{\underline{\lambda}}(S|R_1) = m\sum_{i=1}^{k-m}\int_0^\infty \{1-G_N(\frac{\lambda_{[i]}}{\lambda} t)\}^{m-1} \prod_{\substack{j=1\\ j\neq i}}^{k-m} \{1-G_N(\frac{\lambda_{[i]}}{\lambda_{[j]}} t)\} \frac{\lambda_{[i]}}{\lambda^2} \cdot$$
$$[(\frac{\lambda_{[i]}}{\lambda} t)^N e^{-\frac{\lambda}{\lambda_{[i]}} t} - (\frac{\lambda_{[i]}}{\lambda_{[j]}} t)^N e^{-\frac{\lambda_{[i]}\lambda_{[j]}}{\lambda^2} t}]$$
$$\leq 0.$$

Hence we have proved the following theorem.

Theorem 2.2.

(2.14) $$\sup_{\Omega} E_{\underline{\lambda}}(S|R_1) = k\int_0^\infty \{1-G_N(t)\}^{k-1}\, dG_{N-c_1}(t).$$

Invariance and Minimax Properties

Let $X_1,\ldots,X_k$ be a set of observations from k populations (processes) $\pi_1,\ldots,\pi_k$, respectively and R be a procedure which selects π_i with probability $\varphi_i(X_1,\ldots,X_k)$. Then the procedure R is said to be invariant if

$$\varphi_i(X_1,\ldots,X_i,\ldots,X_j,\ldots,X_k) = \varphi_j(X_1,\ldots,X_j,\ldots,X_i,\ldots,X_k)$$

for all i and j.

It had been shown by Gupta and Studden [13] that for any invariant rule R^*,

$$E_{\underline{\lambda}_0}(S|R^*) = k\, P_{\underline{\lambda}_0}(CS|R^*)$$

where $\underline{\lambda}_0 = (\lambda_1,\ldots,\lambda) \in \Omega_0$. It follows from Theorem 2.1 and Theorem 2.2 that the rule R_1 is minimax in the sense that it minimizes $\sup_{\Omega} E_{\underline{\lambda}}(S|R)$ over the class of all invariant rule satisfying the basic P^* condition.

(B) Procedure R_2 and Its properties

Suppose that the Poisson processes are observed at successive intervals of time, t = 1,2,... . Observe the processes until time t_0, the smallest value of t, say, when the number of arrivals from one of the processes is equal to or greater than N. Let I denote the set of values i for which $X_i(t_0) \geq N$ and J the set of values j for which $X_j(t_0) \geq N-c_1$ where c_1 is the constant associated with R_1 defined in (2.1). Clearly $I \subseteq J$. For each $j \in J$, let t_{j0} be the time such that $X_j(t_{j0}) \geq N-c_1$ and $X_j(t_{j0}-1) < N-c_1$, and let $m_j = N-c_1-X_j(t_{j0}-1)$, $n_j = X_j(t_{j0})-X_j(t_{j0}-1)$. Similarly for each $i \in I$, let $m_i' = N-X_i(t_0-1)$ and $n_i' = X_i(t_0)-X_i(t_0-1)$. Let U(m,n) denote the mth smallest observation in a sample of size n from a uniform distribution on the unit interval (0,1). Now we compute

(2.15)
$$U_j = t_{j0}-1 + U(m_j,n_j) \qquad \text{for } j \in J$$
$$U_i' = t_0-1 + U(m_i',n_i') \qquad \text{for } i \in I$$

and propose the following selection rule:

R_2: Select process π_j $(j \in J)$ if and only if

(2.16)
$$U_j \leq \min_{i \in I} U_i'$$

Note that U_i' and U_j are simply the waiting times for N and $N-c_1$ arrivals from the processes π_i and π_j, respectively. To see this, observe that if n is a random variable distributed according to the Poisson distribution with mean λ, then for any given value of m,

(2.17)
$$\Pr(U(m,n) \leq t) = \sum_{n=m}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \frac{n!}{(m-1)!(n-m)!} \int_0^t x^{m-1}(1-x)^{n-m}dx$$
$$= G_m(\lambda t) \qquad 0 < t < 1.$$

Thus, the arrivals times for a Poisson process can be generated from the observed number of arrivals during the successive unit time intervals and random observations from a uniform distribution. It follows that for any $\underline{\lambda} \in \Omega$,

$$P_{\underline{\lambda}}(\pi_{(i)} \text{ is selected} | R_2) = P_{\underline{\lambda}}(\pi_{(i)} \text{ is selected} | R_1).$$

Hence the rule R_2 is reverse strongly monotone in $\pi_{(i)}$ for all $i = 1,\ldots,k$. Moreover R_2 is minimax among the class of invariant rules and

$$\text{(2.18)} \qquad \inf_{\Omega} P(CS|R_2) = \int_0^\infty \{1-G_N(t)\}^{k-1}\, dG_{N-c_1}(t)$$

$$= \frac{1}{k} \sup_{\Omega} E(S|R_2)$$

(C) Procedures R_3 and R_4 and Their Properties

Let t_0 be a fixed positive number. Observe the number of arrivals $X_1(t_0),\ldots,X_k(t_0)$ from processes $\pi_1,\ldots,\pi_k$ during time t_0, respectively. We propose a rule R_3 as follows:

R_3: Select process π_i if and only if

$$\frac{X_i(t_0)}{t_0} + 1 \geq c_3 \max_{1 \leq j \leq k} \frac{X_j(t_0)}{t_0}$$

where $c_3 = c_3(k,P^*,t_0)$ is the largest nonnegative number satisfied the condition (1.1).

It is easy to see that for any $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k) \in \Omega$,

$$p_{\underline{\lambda}}(i|R_3) = \sum_{x=0}^{\infty} e^{-\frac{t_0}{\lambda_{[i]}}} \frac{1}{x!} \left(\frac{t_0}{\lambda_{[i]}}\right)^x \prod_{j \neq i} \sum_{y=0}^{\left[\frac{x+t_0}{c_3}\right]} e^{-\frac{t_0}{\lambda_{[j]}}} \frac{1}{y!} \left(\frac{t_0}{\lambda_{[j]}}\right)^y$$

$$\text{(2.20)} \qquad = \sum_{x=0}^{\infty} e^{-\frac{t_0}{\lambda_{[i]}}} \frac{1}{x!} \left(\frac{t_0}{\lambda_{[i]}}\right)^x \prod_{j \neq i} \int_{\frac{t_0}{\lambda_{[j]}}}^{\infty} \frac{1}{\left[\frac{x+t_0}{c_3}\right]!} y^{\left[\frac{x+t_0}{c_3}\right]} e^{-y} dy,$$

where $[x]$ denotes the largest integer less than or equal to x

It follows from (2.20) that R_3 is reverse strongly monotone in $\pi_{(i)}$ for $i = 1,\ldots,k$. In particular

$$\inf_{\Omega} P(CS|R_3) = \inf_{\Omega_0} P(CS|R_3)$$

$$(2.21) \qquad = \inf_{\lambda>0} \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} \{ \sum_{y=0}^{[\frac{x+t_0}{1+c_3}]} e^{-\lambda} \frac{\lambda^y}{y!} \}^{k-1}.$$

By using a similar method of proof as given in [9], one can easily prove the following theorem.

Theorem 2.3. For given P^* and any nonnegative integer r, let $P_1^* = (P^*)^{\frac{1}{k-1}}$ and let $c_3(r)$ be the largest value such that

$$\sum_{i=0}^{[\frac{r+t_0}{1+c_3(r)}]} \binom{r}{i} \frac{1}{2^r} \geq P_1^*,$$

of $c_3 = \inf\{c_3(r): r \geq 0\}$, then

$$\inf_{\Omega} P(CS|R_3) \geq P^*.$$

Let $\Omega_1 = \{\underline{\lambda} = (\delta\lambda,\lambda,\ldots,\lambda),\ 0 < \lambda \leq \lambda_0\}$, $0 < \delta < 1$. Then for any $\underline{\lambda} \in \Omega_1$,

$$E_{\underline{\lambda}}(S|R_3) = P_{\underline{\lambda}}(X_{(1)}(t_0)+t_0 \geq c_3 \max_{2\leq j\leq k} X_{(j)}(t_0)) + (k-1)P_{\underline{\lambda}}(X_{(2)}(t_0) + t_0 \geq c_3 \max_{j\neq 2} X_j(t_0))$$

$$\leq k-P_{\underline{\lambda}}(X_{(1)}(t_0)+t_0 < c_3X_{(2)}(t_0)) + (k-1)P_{\underline{\lambda}}(X_{(2)}(t_0)+t_0 < c_3X_{(1)}(t_0))$$

$$= k-\sum_{x=1}^{\infty}\{ \sum_{i=0}^{[\frac{c_3x-t_0}{1+c_3}]} \binom{x}{i}(\frac{1}{1+\delta})^i(\frac{\delta}{1+\delta})^{x-i}\} e^{-\frac{t_0}{\lambda}(1+\frac{1}{\delta})} \frac{(\frac{t_0(1+\delta)}{\lambda\delta})^x}{x!}$$

$$- (k-1)\sum_{x=1}^{\infty}\{ \sum_{i=0}^{[\frac{c_3x-t_0}{1+c_3}]} \binom{x}{i}(\frac{\delta}{1+\delta})^i(\frac{1}{1+\delta})^{x-i}\} e^{-\frac{t_0}{\lambda}(1+\frac{1}{\delta})} \frac{(\frac{t_0(1+\delta)}{\lambda\delta})^x}{x!}$$

$$(2.22)\qquad < k-\{\inf_{x>1} g(x)+(k-1)\inf_{x>1} h(x)\}\,\{1-e^{-\frac{t_0(1+\delta)}{\lambda_0\delta}}\},$$

where g(x) and h(x) are defined in terms of incomplete beta function as follows:

$$(2.23)\qquad \begin{aligned} g(x) &= 1-I_{\frac{1}{1+\delta}}\left(\left[\frac{c_3x-t_0}{1+c_3}\right]+1,\, x-\left[\frac{c_3x-t_0}{1+c_3}\right]\right) \\ h(x) &= 1-I_{\frac{\delta}{1+\delta}}\left(\left[\frac{c_3x-t_0}{1+c_3}\right]+1,\, x-\left[\frac{c_3x-t_0}{1+c_3}\right]\right). \end{aligned}$$

Using the same sampling rule as in R_3, we propose the following conditional procedure.

R_4: Select process π_i if and only if

$$(2.24)\qquad \frac{X_i(t_0)}{t_0}+1 \ge c_4 \max_{1\le j\le k} \frac{X_j(t_0)}{t_0}, \text{ given } \sum_{i=1}^{k} X_i(t_0) = r,$$

where $c_4 \ge 0$ is the maximum value for which the condition (1.1) is satisfied.

Let $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k) \in \Omega$, and let

$$(2.25)\qquad \begin{aligned} s_i &= \left(\frac{1}{\lambda_{[k]}}+\ldots+\frac{1}{\lambda_{[k-i+1]}}\right), \qquad i \le k, \\ p_{ij} &= \frac{1}{\lambda_{[j]}s_i}, \qquad j = 1,\ldots,k. \end{aligned}$$

Then

$$P_{\underset{\sim}{\lambda}}(CS|R_4) = P_{\underset{\sim}{\lambda}}(X_{(1)}(t_0)+t_0 \geq c_4 \max_{2\leq j\leq k} X_{(j)}(t_0) | \sum_{i=1}^{k} X_{(i)}(t_0) = r)$$

$$(2.26) \qquad = \sum_{x=0}^{r} \binom{r}{x} p_{k1}^{r}(1-p_{k1})^{r-x} \sum (r-x)! \prod_{j=2}^{k} (\frac{p_{k-1,j}^{x_j}}{xj!})$$

where the second summation of the right hand side of (2.26) is over all $(k-1)$-tuples $(x_2,\ldots,x_k)$ of nonnegative integers, such that $0 \leq x_i \leq \min\{\frac{x+t_0}{c_4}, r-x\}$, $i = 2,\ldots,k$, and $\sum_{i=2}^{k} x_i = r-x$.

Recall that the vector $\underset{\sim}{x} = (x_1,\ldots,x_n)$ majorizes the vector $\underset{\sim}{y} = (y_1,\ldots,y_n)$ if

$$(2.27) \qquad \sum_{i=1}^{m} x_{[n+1-i]} \geq \sum_{i=1}^{m} y_{[n+1-i]} \text{ for } m = 1,\ldots,n-1, \text{ and } \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

and is written $\underset{\sim}{x} > \underset{\sim}{y}$. A real-valued function $\varphi(\underset{\sim}{x})$ is called a Schur-convex (concave) function if $\varphi(\underset{\sim}{x}) \geq (\leq) \varphi(\underset{\sim}{y})$ whenever $\underset{\sim}{x} > \underset{\sim}{y}$. It is known that (see Rinott [16]) if $\varphi(x_1,\ldots,x_k)$ is asymmetric Schur-concave function and $(X_1,\ldots,X_k)$ is a multinomial random vector with parameter N and $\underset{\sim}{p}$, then $E\{\varphi(X_1,\ldots,X_k)\}$ is Schur-concave in $\underset{\sim}{p}$.

Now for a fixed x, the second summation of the right hand member of (2.26) can be expressed as

$$(2.28) \qquad \sum (r-x)! \prod_{j=2}^{k} (\frac{p_{k-1,j}^{x_j}}{x_j!}) = E_{\underset{\sim}{p}}\{\psi(Y_1,\ldots,Y_{k-1})\}$$

where

$$(2.29) \qquad \psi(y_1,\ldots,y_{k-1}) = \begin{cases} 1 & \text{if } c_4 \max_{1\leq j\leq k-1} y_j \leq x+t_0 \\ 0 & \text{if } c_4 \max_{1\leq j\leq k-1} y_j > x+t_0, \end{cases}$$

and $(Y_1,\ldots,Y_{k-1})$ is a multinomial random vector with parameters r-x and

$\underline{p} = (p_{k-1,1},\ldots,p_{k-1,k-1})$. Since ψ is a symmetric Schur-concave function, it follows that $E_{\underline{p}}\{\psi(Y_1,\ldots,Y_{k-1})\}$ is Schur-concave in $\underline{p}$. In other words, if we fix s_{k-1} and $\lambda_{[1]}$, then $P_{\underline{\lambda}}(CS|R_4)$ decreases when $\lambda_{[2]} \to \lambda_{[1]}$. Hence the least favorable configuration is of the form $(\lambda,\ldots,\lambda,\delta\lambda,\infty,\ldots,\infty)$ where $\lambda > 0$, $\delta \geq 1$. It should be pointed out that the probability of a correct selection under the configuration $(\lambda,\ldots,\lambda,\delta\lambda,\infty,\ldots,\infty)$ does not depend on the unknown parameter λ. Also when $k = 2$, the infimum of $P(CS|R_4)$ takes place when the two processes are identical. However, when $k \geq 3$, the infimum of $P(CS|R_4)$ does not necessarily take place at the configuration of the type $(\lambda,\ldots,\lambda,\infty,\ldots,\infty)$ as shown by the following example.

First of all, we need some algebraic concepts. Let

$$(2.30)\qquad p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

be a polynomial of degree n. The coefficients $a_0,a_1,\ldots,a_n$ are assumed to be real. The <u>discriminant</u> of the polynomial of $p(x)$ is defined to be

$$(2.31)\qquad D(p) = \left|\begin{array}{cccccccc} a_0 & a_1 \ldots & a_{n-1} & a_n & 0 & 0 & \ldots & 0 \\ 0 & a_0 \ldots & & a_{n-1} & a_n & 0 & \ldots & 0 \\ \vdots & & & & \vdots & & & \\ 0 & 0 \ldots & & a_0 & a_1 & & \ldots & a_n \\ na_0 & (n-1)q_1 \ldots & a_{n-1} & 0 & 0 & & \ldots & 0 \\ 0 & & & & & & & \\ & & & 0 & na & & \ldots & a_{n-1} \end{array}\right| \begin{array}{l} \\ \\ (n-1) \text{ rows} \\ \\ \\ n \text{ rows} \\ \\ \end{array}$$

In particular, when $p(x) = a_0x^2 + a_1x + a_2$, then $D(p) = a_1^2-4a_0a_2$. It is well-known that (see [14]) if the polynomial $p(x)$ with real coefficients, not having multiple roots, then $D(p) > 0$, if the number of pairs of complex conjugate roots of $p(x)$ is even, and $D(p) < 0$, if this number is odd.

Moreoever, $D(p) = 0$ if and only if $p(x)$ has multiple roots. Now we consider the case when $k = 3$, $c = \frac{2}{9}, t_0 = 1$ and $\sum_{i=1}^{3} X_{(i)}(t_0) = 6$ under the configuration

$$(2.31) \qquad (\lambda,\lambda,\delta\lambda), \quad \lambda > 0, \quad \delta \geq 1.$$

Let $x = (1+2\delta)^{-1}$. Under the configuration (2.31),

$$(2.32) \qquad P(CS|R_4) = P_r(X_{(1)}(t_0)+1 \geq \frac{2}{9} \max_{2\leq i\leq 3} X_{(i)}(t_0) | \sum_{i=1}^{3} X_{(i)}(t_0) = 6)$$

$$= 1-P_r(X_{(1)}(t_0) = 0, \max_{2\leq i\leq 3} X_{(i)}(t_0) \geq 5 | \sum_{i=1}^{3} X_{(i)}(t_0)=6)$$

$$= 1-x^5(3-2x) - (\frac{1-x}{2})^5(\frac{1+11x}{2})$$

$$= p(x), \text{ say.}$$

The derivative $p'(x)$ of $p(x)$ is a polynomial of degree 5. It follows that $p'(x)$ has at most two pairs of complex conjugate roots. Direct computation shows that the discriminant of $p'(x)$ is negative. This implies that $p'(x)$ has three real roots. Since $p'(\frac{1}{3})p'(1) < 0$ and $p'(x) < 0$ for all $0 < x < \frac{1}{11}$, there are at most two real roots lying in the interval. Moreover $p'(\frac{33}{352}) < 0 < p'(\frac{33}{352} + \frac{1}{11264})$, and $p'(\frac{1}{4}) > 0 > p'(\frac{1}{3})$. Now

$$p(\frac{33}{352}) = 0.980578 < p(1) = 0.9805795 < p(0) = 0.984375.$$

This implies that the infimum of $p(x)$ takes place for some x in the closed interval $[\frac{33}{352}, \frac{33}{352} + \frac{1}{11264}]$. This shows that the least favorable configuration is of the type $(\lambda,\lambda,\delta\lambda)$ where $\delta > 1$.

The next theorem shows that R_4 is reverse monotone.

Theorem 2.4. For $1 \leq i \leq j \leq k$ and $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k) \in \Omega$,

$$P_{\underline{\lambda}}\{\pi_{(i)} \text{ is selected} | R_4\} \geq P_{\underline{\lambda}}\{\pi_{(j)} \text{ is selected} | R_4\}.$$

<u>Proof</u>. For $1 \le i \le j \le k$ and $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k) \in \Omega$,

$$P_{\underline{\lambda}}(i|R_4) = P_{\underline{\lambda}}(X_{(i)}(t_0)+t_0 \ge c_4 \max_{1\le j\le k} X_{(j)}(t_0) | \sum_{i=1}^{k} X_i(t_0) = r) \tag{2.33}$$

$$= \sum\{\sum \binom{x_i+x_j}{x_i} \left(\frac{\lambda_{[j]}}{\lambda_{[i]}+\lambda_{[j]}}\right)^{x_i} \left(\frac{\lambda_{[i]}}{\lambda_{[i]}+\lambda_{[j]}}\right)^{x_j}\}.$$

$$\{\frac{r!}{\prod_{\substack{\ell\ne i\\ \ell\ne j}} x_\ell!(x_i+x_j)!} \prod_{\substack{\ell\ne i\\ \ell\ne j}} p_{\ell k}^{x_\ell} (p_{ik}+p_{jk})^{x_i+x_j}\}$$

where the first summation is over all k-tuples $(x_1,\ldots,x_k)$ of nonnegative integer such that (i) $\sum_{i=1}^{k} x_i = r$ and (ii) $x_i+t_0 \ge c_4 \max_{\ell\ne i} x_\ell$; the second summation is over all (x_i,x_j) such that (i) holds and $x_i + t_0 \ge c_4 x_j$. Since the term in the first parenthesis can be written as

$$I_{\frac{\lambda_{[j]}}{\lambda_{[j]}+\lambda_{[i]}}}\left(\left[\frac{c_4(x_i+x_j)-t_0}{1+c_4}\right], x_i+x_j-\left[\frac{c_4(x_i+x_j)-t_0}{1+c_4}\right]+1\right)$$

where $I_p(\cdot)$ represnets the incomplete beta function, and $[x]$, the integral part of x. Similarly,

$$p_{\underline{\lambda}}(j|R_4) = \sum\{\sum \binom{x_i+x_j}{x_i} \left(\frac{\lambda_{[i]}}{\lambda_{[i]}+\lambda_{[j]}}\right)^{x_i} \left(\frac{\lambda_{[j]}}{\lambda_{[i]}+\lambda_{[j]}}\right)^{x_j}\} . \tag{2.34}$$

$$\{\frac{r!}{\prod_{\substack{\ell\ne i\\ \ell\ne j}} x_\ell!(x_i+x_j)!} \prod_{\substack{\ell\ne i\\ \ell\ne j}} p_{\ell k}^{x_\ell}(p_{ik} + p_{jk})^{x_i+x_j}\}$$

where the summations are respectively over the same regions as that of $p_{\underline{\lambda}}(i|R_4)$. From the fact that

(2.35) $I_{p_1}(a,b)$ [illegible] $I_{p_2}(a,b)$ if p_1 [illegible] p_2, it follows that $p_{\underline{\lambda}}(i|R_4) \geq p_{\underline{\lambda}}(j|R_4)$ whenever $i \leq j$. This completes the proof.

The following result provides a method to obtain a conservative selection constant for the procedure R_4.

Theorem 2.5. For a given P^*, let $P_1^* = \frac{1-P^*}{k-1}$ and let c_4 be the largest positive number such that $\sum_{i=0}^{[\frac{c_4 r - t_0}{1+c_4}]} \binom{r}{i} \frac{1}{2^r} \leq P_1^*$, then $\inf_{\Omega} P(CS|R_4) \geq P^*$.

Proof. For any $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k) \in \Omega$,

$$
\begin{aligned}
P_{\underline{\lambda}}(CS|R_4) &= P_r(X_{(1)}(t_0)+t_0 \geq c_4 \max_{2\leq j\leq k} X_{(j)}(t_0) \mid \sum_{i=1}^{k} X_{(i)}(t_0)=r) \\
&= 1-P_r(X_{(1)}(t_0)+t_0 < c_4 \max_{2\leq j\leq k} X_{(j)}(t_0) \mid \sum_{i=1}^{k} X_{(i)}(t_0)=r) \\
&\geq 1-\sum_{j=2}^{k} P_r(X_{(1)}(t_0)+t_0 < c_4 X_{(j)}(t_0) \mid \sum_{i=1}^{k} X_{(i)}(t_0)=r) \\
&\geq 1-\sum_{j=2}^{k} \Big\{ \sum_{i=0}^{[\frac{c_4 r-t_0}{1+c_4}]} \binom{r}{i} \Big(\frac{\lambda_{[j]}}{\lambda_{[1]}+\lambda_{[j]}}\Big)^i \Big(\frac{\lambda_{[1]}}{\lambda_{[1]}+\lambda_{[j]}}\Big)^{r-i} \Big\} \\
&\geq 1-(k-1) \sum_{i=0}^{[\frac{c_4 r-t_0}{1+c_4}]} \binom{r}{i} \frac{1}{2^r} \\
&= P^*.
\end{aligned}
$$

3. Selection Procedures for the Process Associated with $\lambda_{[k]}$

For the analogous problem of selecting the process for which the mean rate is the smallest, we propose the following subset selection rules.

(A) Let N be a fixed positive integer. We observed the processes until, say t_0, that $\min_{1\leq i\leq k} X_i(t_0) = N$.

R_1': Select the process π_i if and only if

(3.1) $$X_i(t_0) \leq N + c_1',$$

where c_1' is the smallest non-negative integer such that the basic probability requirement (1.1) is satisfied.

By using similar arguments as given in Section 2, one can show that the procedure R_1' is strongly monotone in $\pi_{(i)}$ for all $i = 1,\ldots,k$. This implies that the infimum of $P(CS|R_1')$ takes place when all the process are identical. In fact, the infimum of $P(CS|R_1')$ is given by

(3.2) $$\inf_{\Omega} P(CS|R_1') = \int_0^\infty G_N^{k-1}(t)dG_{N+c_1'}(t).$$

Also, one can show that

(3.3) $$\sup_{\Omega} E(S|R_1') = k \inf_{\Omega} P(CS|R_1').$$

(B) If the processes are observed at successive intervals of time $t = 1,2,\ldots$ We observe the processes until the first time t_0, say, when $\min\limits_{1\leq i\leq k} X_i(t_0) \geq N$. Let t_i be the time such that $X_i(t_i) \geq N$ and $X_i(t_i-1) < N$, $i = 1,\ldots,k$. Let $m_i = N-X_i(t_i-1)$ and $n_i = X_i(t_i)-X_i(t_i-1)$. As in the previous section, we compute

(3.4) $$U_i = t_i-1 + U(m_i,n_i), \quad i = 1,\ldots,k,$$

and propose a selection procedure R_2' as follows:

R_2': Retain process π_i in the selected subset if and only if

(3.5) $$U_i \geq c_2' \max_{1\leq j\leq k} U_j,$$

where $0 < c_2' < 1$ is the largest value for which the condition (1.1) is satisfied. Since U_i is distributed as $T_i(N)$, hence this reduces to the problem of selecting a subset of k gamma populations which includes the one with the smallest value of scale parameter. It follows from [7] that

(i) Rule R_2' is strongly monotone in $\pi_{(i)}$ for all $i = 1,..,k$.

(ii) $\sup_{\Omega} E(S|R_2') = k \inf_{\Omega} P(CS|R_2')$.

It should be pointed out that a rule similar to R_2' has been studied by Goel [6].

4. Applications

(A) A sequential (inverse sampling) subset selection rule for the most probable multinomial event.

Let $\underline{X} = (X_1,\ldots,X_k)$ have the multinomial distribution

$$(4.1) \qquad P(\underline{X} = \underline{x}) = \binom{n}{x_1,\ldots,x_k} \prod_{i=1}^{k} p_i^{x_i}$$

where $\underline{x} = (x_1,\ldots,x_k)$. Let $p_{[1]} \le \ldots \le p_{[k]}$ denote the ordered values of $p_1,\ldots,p_k$. The subset selection problem for the multinomial distribution has been considered by Gupta and Nagel [11], Gupta and Huang [9] and Panchapakesan [15]. A related problem has also been discussed by Alam, Seo and Thompson [2], Bechhofer, Elmarghrabi and Morse [3]. In [9] and [11], the authors considered the fixed-sample subset selection rules. The procedure given in [15] is based on a completely sequential sampling scheme in which one observation is taken at a time from the given distribution until the highest cell count is equal to a fixed number N, say.

We consider below a variation of the sampling scheme in [2]. The sampling scheme is given as follows: Let a positive integer N be given, and let $n_1,n_2,\ldots$ denote a sequence of random observations taken from a Poisson distribution with mean λ. Having observed these numbers, take n_i observations from the given multinomial distribution for the ith experiment, $i = 1,2,\ldots$. Let π_i denote the cell corresponding to p_i, and let Y_{ij} denote the cell count in π_i out of n_j observations. Stop sampling as soon as the total count from any cell is equal to or greater than N. Let t_0 denote the stage at which the experiment terminates, and let $X_i(t) = \sum_{j=1}^{t} Y_{ij}$. Then $X_i(t_0-1) < N$ for $i = 1,\ldots,k$ and $X_i(t_0) \geq N$ for some i. As in Section 2, let I be the set of values of i for which $X_i(t_0) \geq N$ and J be the set of values of j for which $X_j(t_0) \geq N-c_2$ where c_2 is the selection constant associated with rule R_2. Take the similar random observations from the uniform distribution on the unit interval (0,1) and obtain the statistics U_i' and U_j as defined in (2.15). Based on the statistics U_i''s and U_j's, we select the cell according to the rule R_2. Then the problem reduces to that of selecting the Poisson process with maximum mean rate. To see this, suppose the parameter n in (4.1) is a random variable distributed according to a Poisson distribution with mean λ. It is easy to show that the cell frequencies $X_1,\ldots,X_k$ are independently distributed according to the Poisson distribution with mean $\lambda p_1,\ldots,\lambda p_k$, respectively. It follows (2.18) that the least favorable configuration is $(\frac{\lambda}{k},\ldots,\frac{\lambda}{k})$, and the infimum of P(CS) is independent of the parameter λ. Moreover, the supremum of the expected subset size is obtained when all the cells are identical and is equal to k $\inf_{\Omega}$ P(CS). It should be pointed out that when $\lambda \to 0$, the rule reduces to the one proposed by Panchapakesan [15].

(B) <u>A sequential (inverse sampling) rule for selection procedure for k binomial populations</u>

Let $\pi_1,\ldots,\pi_k$ be k independent binomial populations with parameters $p_1,\ldots,p_k$ respectively. To select a subset of the k populations which contains the population associated with the largest p_i, Gupta and Sobel [12] proposed a fixed-sample procedure which is based on the statistics $\max_{1\leq j\leq k} X_j - X_i$, where X_i represents the number of successes in n independent trials from population π_i. Recently, Gupta, Huang and Huang [10] proposed a conditional procedure for this problem and gave a lower bound for the infimum of the probability of a correct selection. It should be pointed out that a related problem has been considered by Sobel and Weiss [17].

Suppose the number of observations taken as each stage from the k binomial populations, is a random variable distributed according to a Poisson distribution with mean λ. Using the same sampling procedure and selection rule as mentioned in part (A) of this section, the problem then reduces to that of selecting the Poisson process with largest mean rate. It follows that the infimum of the probability of a correct selection and the supremum of the expected subset size take place when all the populations are identical. Also the inf P(CS) and the sup E(S) do not depend on the common unknown parameter p and the mean λ. Moreover, the selection rule is strongly monotone in $\pi_{(i)}$ for all $i = 1,\ldots,k$.

Acknowledgment

The authors wish to thank Miss Carol de Branges for programming assistance.

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TABLE I

For given k, N, and P*, this table gives the actual minimum $P(CSIR_1)$ and the smallest integer c_1 (in parenthesis) necessary to apply the procedure R_1.

P*=.75

N \ k	2	3	4	5	6	7
2	0.7500 (1)	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)
3	0.8750 (2)	0.8025 (2)	0.7521 (2)	1.0000 (3)	1.0000 (3)	1.0000 (3)
4	0.8125 (2)	0.8967 (3)	0.8662 (3)	0.8419 (3)	0.8217 (3)	0.8044 (3)
5	0.7734 (2)	0.8236 (3)	0.7757 (3)	0.9147 (4)	0.9024 (4)	0.8915 (4)
6	0.8555 (3)	0.7710 (3)	0.8637 (4)	0.8380 (4)	0.8166 (4)	0.7981 (4)
7	0.8281 (3)	0.8523 (4)	0.8097 (4)	0.7762 (4)	0.8884 (5)	0.8758 (5)
8	0.8062 (3)	0.8163 (4)	0.7657 (4)	0.8542 (5)	0.8340 (5)	0.8165 (5)
9	0.7880 (3)	0.7861 (4)	0.8409 (5)	0.8112 (5)	0.7864 (5)	0.7652 (5)
10	0.7728 (3)	0.7604 (4)	0.8081 (5)	0.7737 (5)	0.8552 (6)	0.8392 (6)

P*=.80

N \ k	2	3	4	5	6	7
2	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)
3	0.8750 (2)	0.8025 (2)	1.0000 (3)	1.0000 (3)	1.0000 (3)	1.0000 (3)
4	0.8125 (2)	0.8967 (3)	0.8662 (3)	0.8419 (3)	0.8217 (3)	0.8044 (3)
5	0.8906 (3)	0.8236 (3)	0.9291 (4)	0.9147 (4)	0.9024 (4)	0.8915 (4)
6	0.8555 (3)	0.8955 (4)	0.8637 (4)	0.8380 (4)	0.8166 (4)	0.9411 (5)
7	0.8281 (3)	0.8523 (4)	0.8097 (4)	0.7027 (5)	0.8884 (5)	0.8758 (5)
8	0.8062 (3)	0.8163 (4)	0.8781 (5)	0.8542 (5)	0.8340 (5)	0.8165 (5)
9	0.8666 (4)	0.8780 (5)	0.8409 (5)	0.8112 (5)	0.8937 (6)	0.8813 (6)
10	0.8491 (4)	0.8516 (5)	0.8081 (5)	0.8735 (6)	0.8552 (6)	0.8392 (6)

TABLE I (cont'd.)

For given k, N, and P*, this table gives the actual minimum $P(CSIR_1)$ and the smallest integer (in parenthesis) necessary to apply the procedure R_1.

P*=.90

N \ k	2	3	4	5	6	7
2	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)
3	1.0000 (3)	1.0000 (3)	1.0000 (3)	1.0000 (3)	1.0000 (3)	1.0000 (3)
4	0.9375 (3)	1.0000 (4)	1.0000 (4)	1.0000 (4)	1.0000 (4)	1.0000 (4)
5	0.9688 (4)	0.9466 (4)	0.9291 (4)	0.9147 (4)	0.9024 (4)	1.0000 (5)
6	0.9375 (4)	0.9726 (5)	0.9629 (5)	0.9547 (5)	0.9475 (5)	0.9411 (5)
7	0.9102 (4)	0.9395 (5)	0.9194 (5)	0.9027 (5)	0.9722 (6)	0.9685 (6)
8	0.9453 (5)	0.9074 (5)	0.9534 (6)	0.9430 (6)	0.9338 (6)	0.9257 (6)
9	0.9270 (5)	0.9432 (6)	0.9239 (6)	0.9077 (6)	0.9616 (7)	0.9565 (7)
10	0.9102 (5)	0.9209 (6)	0.9535 (7)	0.9429 (7)	0.9336 (7)	0.9252 (7)

P*=.95

N \ k	2	3	4	5	6	7
2	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)	1.0000 (2)
3	1.0000 (3)	1.0000 (3)	1.0000 (3)	1.0000 (3)	1.0000 (3)	1.0000 (3)
4	1.0000 (4)	1.0000 (4)	1.0000 (4)	1.0000 (4)	1.0000 (4)	1.0000 (4)
5	0.9688 (4)	1.0000 (5)	1.0000 (5)	1.0000 (5)	1.0000 (5)	1.0000 (5)
6	0.9844 (5)	0.9726 (5)	0.9629 (5)	0.9547 (5)	1.0000 (6)	1.0000 (6)
7	0.9648 (5)	0.9860 (6)	0.9808 (6)	0.9762 (6)	0.9722 (6)	0.9685 (6)
8	0.9805 (6)	0.9656 (6)	0.9534 (6)	0.9876 (7)	0.9854 (7)	0.9834 (7)
9	0.9673 (6)	0.9807 (7)	0.9735 (7)	0.9672 (7)	0.9616 (7)	0.9565 (7)
10	0.9539 (6)	0.9658 (7)	0.9535 (7)	0.9814 (8)	0.9781 (8)	0.9750 (8)

TABLE II

For given k, N, P* (or c_1) and δ, this table gives the actual probability of a correct selection (top), the probability of selecting a non-best population (middle) and the expected proportion (bottom) of population selected in the subset when the rule R_1 is used and the parameters are given by slippage configurations $\delta\lambda, \lambda, \ldots, \lambda$. For given k and N, each of the four blocks of three numbers correspond to P*=.75, .80, .90 and .95, respectively.
Note that for fixed k, N and P*(large) if c_1=N (from Table I), all three entries in each block are 1, as expected.

$\delta = 0.1$

N \ k	2	3	4	5	6	7
3	0.9993	0.9985	0.9978	1.0000	1.0000	1.0000
	0.2487	0.2481	0.2475	1.0000	1.0000	1.0000
	0.6240	0.4982	0.4350	1.0000	1.0000	1.0000
	0.9993	0.9985	1.0000	1.0000	1.0000	1.0000
	0.2487	0.2481	1.0000	1.0000	1.0000	1.0000
	0.6240	0.4982	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

N \ k	2	3	4	5	6	7
4	0.9997	0.9999	0.9998	0.9997	0.9997	0.9996
	0.0686	0.3168	0.3167	0.3165	0.3163	0.3162
	0.5342	0.5445	0.4874	0.4531	0.4302	0.4138
	0.9997	0.9999	0.9998	0.9997	0.9997	0.9996
	0.0686	0.3168	0.3167	0.3165	0.3165	0.3162
	0.5342	0.5445	0.4874	0.4531	0.4302	0.4138
	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	0.3170	1.0000	1.0000	1.0000	1.0000	1.0000
	0.6585	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

δ=0.1

N \ k	2	3	4	5	6	7
5	0.9999	0.9999	0.9999	1.0000	1.0000	1.0000
	0.0199	0.0968	0.0968	0.3790	0.3789	0.3789
	0.5099	0.3979	0.3226	0.5032	0.4824	0.4676
	1.0000	0.9999	1.0000	1.0000	1.0000	1.0000
	0.0199	0.0968	0.3790	0.3790	0.3789	0.3789
	0.5484	0.3979	0.5342	0.5032	0.4824	0.4676
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.3791	0.3790	0.3790	0.3790	0.3789	1.0000
	0.6895	0.5860	0.5342	0.5032	0.4824	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.3791	1.0000	1.0000	1.0000	1.0000	1.0000
	0.6895	1.0000	1.0000	1.0000	1.0000	1.0000

N \ k	2	3	4	5	6	7
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.0297	0.0297	0.1276	0.1276	0.1276	0.1276
	0.5148	0.3531	0.3457	0.3021	0.2730	0.2522
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.0297	0.1276	0.1276	0.1276	0.1276	0.4355
	0.5148	0.6237	0.3457	0.3021	0.2730	0.5161
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.1276	0.4355	0.4355	0.4355	0.4355	0.4355
	0.5638	0.6237	0.5766	0.5484	0.5296	0.5161
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.4355	0.4355	0.4355	0.4355	1.0000	1.0000
	0.7178	0.6237	0.5766	0.5484	1.0000	1.0000

N \ k	2	3	4	5	6	7
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.0092	0.0415	0.0415	0.0415	0.1603	0.1603
	0.5046	0.3610	0.2812	0.2332	0.3002	0.2802
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.0092	0.0415	0.0415	0.1603	0.1603	0.1603
	0.5046	0.3610	0.2812	0.3282	0.3002	0.2802
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.0415	0.1603	0.1603	0.1603	0.4868	0.4868
	0.5208	0.6579	0.3702	0.3282	0.5724	0.5601
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.1603	0.4868	0.4868	0.4863	0.4868	0.4868
	0.5801	0.6579	0.6151	0.5895	0.5724	0.5601

δ=0.1

N \ k	2	3	4	5	6	7
	1.0000 0.0028 0.5014	1.0000 0.0134 0.3422	1.0000 0.0134 0.2600	1.0000 0.0554 0.2443	1.0000 0.0554 0.2129	1.0000 0.0554 0.1904
	1.0000 0.0028 0.5014	1.0000 0.0134 0.3422	1.0000 0.0554 0.2916	1.0000 0.0554 0.2443	1.0000 0.0554 0.2129	1.0000 0.0554 0.1904
8	1.0000 0.0554 0.5277	1.0000 0.0554 0.3703	1.0000 0.1942 0.3957	1.0000 0.1942 0.3554	1.0000 0.1942 0.3285	1.0000 0.1942 0.3093
	1.0000 0.1942 0.5971	1.0000 0.1942 0.4628	1.0000 0.1942 0.3957	1.0000 0.5335 0.6268	1.0000 0.5335 0.6112	1.0000 0.5335 0.6001

N \ k	2	3	4	5	6	7
	1.0000 0.0009 0.5004	1.0000 0.0043 0.3362	1.0000 0.0186 0.2640	1.0000 0.0186 0.2149	1.0000 0.0186 0.1822	1.0000 0.0186 0.1588
	1.0000 0.0043 0.5021	1.0000 0.0186 0.3457	1.0000 0.0186 0.2640	1.0000 0.0186 0.2149	1.0000 0.0712 0.2260	1.0000 0.0712 0.2039
9	1.0000 0.0186 0.5093	1.0000 0.0712 0.3808	1.0000 0.0712 0.3034	1.0000 0.0712 0.2570	1.0000 0.2289 0.3574	1.0000 0.2289 0.3391
	1.0000 0.0712 0.5356	1.0000 0.2289 0.4860	1.0000 0.2289 0.4217	1.0000 0.2289 0.3831	1.0000 0.2289 0.3574	1.0000 0.2289 0.3391

N \ k	2	3	4	5	6	7
	1.0000 0.0003 0.5001	1.0000 0.0014 0.3343	1.0000 0.0062 0.2546	1.0000 0.0062 0.2049	1.0000 0.0250 0.1875	1.0000 0.0250 0.1643
	1.0000 0.0014 0.5006	1.0000 0.0062 0.3374	1.0000 0.0250 0.2546	1.0000 0.0250 0.2200	1.0000 0.0250 0.1875	1.0000 0.0250 0.1643
10	1.0000 0.0062 0.5031	1.0000 0.0250 0.3500	1.0000 0.0887 0.3165	1.0000 0.0887 0.2710	1.0000 0.0887 0.2406	1.0000 0.0887 0.2189
	1.0000 0.0250 0.5125	1.0000 0.0887 0.3925	1.0000 0.0887 0.3165	1.0000 0.2640 0.4112	1.0000 0.2640 0.3866	1.0000 0.2640 0.3691

TABLE II (cont'd)

For given k, N, P* (or c_1) and δ, this table gives the actual probability of a correct selection (top), the probability of selecting a non-best population (middle) and the expected proportion (bottom) of populations selected in the subset when the rule R_1 is used and the parameters are given by slippage configurations $\delta\lambda, \lambda, \ldots, \lambda$. For given k and N, each of the four blocks of three numbers correspond to P*=.75, .80, .90 and .95, respectively.
Note that for fixed k, N and P* (large) if c_1=N (from Table I), all three entries in each block are 1, as expected.

δ = 0.3

N \ k	2	3	4	5	6	7
3	0.9877	0.9766	0.9665	1.0000	1.0000	1.0000
	0.5448	0.5326	0.5218	1.0000	1.0000	1.0000
	0.7663	0.6806	0.6330	1.0000	1.0000	1.0000
	0.9877	0.9766	1.0000	1.0000	1.0000	1.0000
	0.5448	0.5326	1.0000	1.0000	1.0000	1.0000
	0.7663	0.6806	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

N \ k	2	3	4	5	6	7
4	0.9884	0.9945	0.9919	0.9895	0.9872	0.9849
	0.3267	0.6440	0.6386	0.6336	0.6288	0.6244
	0.6576	0.7608	0.7269	0.7047	0.6885	0.6759
	0.9884	0.9945	0.9919	0.9895	0.9872	0.9849
	0.3267	0.6440	0.6386	0.6336	0.6288	0.6244
	0.6576	0.7608	0.7269	0.7047	0.6885	0.6759
	0.9972	1.0000	1.0000	1.0000	1.0000	1.0000
	0.6499	1.0000	1.0000	1.0000	1.0000	1.0000
	0.8235	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

δ=0.3

N \ k	2	3	4	5	6	7
5	0.9910	0.9938	0.9909	0.9975	0.9969	0.9963
	0.2048	0.4150	0.4105	0.7228	0.7204	0.7181
	0.5979	0.6080	0.5556	0.7777	0.7665	0.7579
	0.9968	0.9938	0.9981	0.9975	0.9969	0.9963
	0.4199	0.4150	0.7253	0.7228	0.7204	0.7181
	0.7084	0.6080	0.7935	0.7777	0.7665	0.7579
	0.9994	0.9987	0.9981	0.9975	0.9969	1.0000
	0.7307	0.7279	0.7253	0.7228	0.7204	1.0000
	0.8650	0.8182	0.7935	0.7777	0.7665	1.0000
	0.9994	1.0000	1.0000	1.0000	1.0000	1.0000
	0.7307	1.0000	1.0000	1.0000	1.0000	1.0000
	0.8650	1.0000	1.0000	1.0000	1.0000	1.0000

N \ k	2	3	4	5	6	7
6	0.9973	0.9634	0.9975	0.9967	0.9960	0.9952
	0.2743	0.2711	0.5006	0.4981	0.4957	0.4935
	0.6358	0.6618	0.6249	0.5979	0.5791	0.5651
	0.9973	0.9983	0.9975	0.9967	0.9960	0.9991
	0.2743	0.5032	0.5006	0.4981	0.4957	0.7868
	0.6358	0.6249	0.5979	0.5791	0.8171	
	0.9992	0.9997	0.9996	0.9994	0.9993	0.9991
	0.5060	0.7915	0.7903	0.7891	0.7879	0.7868
	0.7526	0.8609	0.8426	0.8312	0.8231	0.8171
	0.9999	0.9997	0.9996	0.9994	1.0000	1.0000
	0.7928	0.7915	0.7903	0.7891	1.0000	1.0000
	0.8963	0.8609	0.8426	0.8312	1.0000	1.0000

N \ k	2	3	4	5	6	7
7	0.9979	0.9984	0.9977	0.9969	0.9989	0.9987
	0.1810	0.3436	0.3416	0.3398	0.5775	0.5762
	0.5895	0.5618	0.5056	0.4712	0.6478	0.6366
	0.9979	0.9984	0.9977	0.9991	0.9989	0.9987
	0.1810	0.3436	0.3416	0.5789	0.5775	0.5762
	0.5895	0.5618	0.5056	0.6629	0.6478	0.6366
	0.9992	0.9996	0.9993	0.9991	0.9998	0.9998
	0.3456	0.5817	0.5803	0.5789	0.8383	0.8378
	0.6724	0.7210	0.6850	0.6629	0.8652	0.8609
	0.9998	0.9999	0.9999	0.9999	0.9998	0.9998
	0.5832	0.8400	0.8395	0.8389	0.8383	0.8378
	0.7915	0.8933	0.8796	0.8711	0.8652	0.8609

$\delta=0.3$

N \ k	2	3	4	5	6	7
3	0.9984	0.9987	0.9980	0.9991	0.9989	0.9987
	0.1206	0.2340	0.2327	0.4126	0.4115	0.4104
	0.5595	0.4889	0.4240	0.5299	0.5094	0.4944
	0.9984	0.9987	0.9993	0.9991	0.9989	0.9987
	0.1206	0.2340	0.4137	0.4126	0.4115	0.4104
	0.5595	0.4889	0.5601	0.5299	0.5094	0.4944
	0.9998	0.9996	0.9998	0.9998	0.9997	0.9997
	0.4161	0.4149	0.6495	0.6488	0.6480	0.6473
	0.7079	0.6098	0.7371	0.7190	0.7067	0.6977
	0.9999	0.9999	0.9998	1.0000	1.0000	1.0000
	0.6511	0.6503	0.6495	0.8766	0.8763	0.8761
	0.8255	0.7668	0.7371	0.9013	0.8970	0.8938

N \ k	2	3	4	5	6	7
9	0.9988	0.9990	0.9994	0.9992	0.9990	0.9988
	0.0808	0.1595	0.2910	0.2902	0.2894	0.2886
	0.5398	0.4393	0.4681	0.4320	0.4077	0.3901
	0.9995	0.9996	0.9994	0.9992	0.9997	0.9996
	0.1603	0.2918	0.2910	0.2902	0.4811	0.4805
	0.5799	0.5277	0.4681	0.4320	0.5676	0.5547
	0.9998	0.9999	0.9998	0.9998	0.9999	0.9999
	0.2927	0.4832	0.4825	0.4818	0.7082	0.7079
	0.6462	0.6554	0.6118	0.5854	0.7569	0.7496
	0.9999	1.0000	0.9999	0.9999	0.9999	0.9999
	0.4839	0.7094	0.7090	0.7086	0.7082	0.7079
	0.7419	0.8063	0.7818	0.7669	0.7569	0.7496

N \ k	2	3	4	5	6	7
10	0.9992	0.9992	0.9998	0.9993	0.9997	0.9996
	0.0545	0.1089	0.2033	0.2028	0.3494	0.3489
	0.5268	0.4056	0.5128	0.3621	0.4578	0.4419
	0.9996	0.9997	0.9998	0.9998	0.9997	0.9996
	0.1094	0.2038	0.2033	0.3499	0.3494	0.3489
	0.5545	0.4691	0.5128	0.4799	0.4578	0.4419
	0.9998	0.9999	1.0000	0.9999	0.9999	0.9999
	0.2044	0.3510	0.5468	0.5464	0.5460	0.5457
	0.6021	0.5673	0.6601	0.6371	0.6217	0.6106
	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000
	0.3515	0.5472	0.5468	0.7594	0.7592	0.7590
	0.6757	0.6981	0.6601	0.8075	0.7994	0.7935

TABLE II (cont'd.)

For given k, N, P* (or c_1) and δ, this table gives the actual probability of a correct selection (top), the probability of selecting a non-best population (middle) and the expected proportion (bottom) of populations selected in the subset when the rule R_1 is used and the parameters are given by slippage configurations $\delta\lambda, \lambda, \ldots, \lambda$. For given k and N, each of the four blocks of three numbers correspond to P*=.75, .80, .90 and .95 respectively.
Note that for fixed k, N and P* (large) if c_1=N (from Table I), all three entries in each block are 1, as expected.

$\delta = 0.5$

N \ k	2	3	4	5	6	7
3	0.9630	0.9336	0.9092	1.0000	1.0000	1.0000
	0.7037	0.6719	0.6462	1.0000	1.0000	1.0000
	0.8333	0.7591	0.7120	1.0000	1.0000	1.0000
	0.9630	0.9336	1.0000	1.0000	1.0000	1.0000
	0.7037	0.6719	1.0000	1.0000	1.0000	1.0000
	0.8333	0.7591	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

N \ k	2	3	4	5	6	7
4	0.9547	0.9771	0.9677	0.9593	0.9517	0.9446
	0.5391	0.7852	0.7704	0.7575	0.7460	0.7357
	0.7469	0.8491	0.8197	0.7976	0.7803	0.7656
	0.9547	0.9771	0.9677	0.9593	0.9517	0.9446
	0.5391	0.7852	0.7704	0.7575	0.7460	0.7357
	0.7469	0.8491	0.8197	0.7976	0.7803	0.7656
	0.9877	1.0000	1.0000	1.0000	1.0000	1.0000
	0.8025	1.0000	1.0000	1.0000	1.0000	1.0000
	0.8951	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

$\delta=0.5$

N \ k	2	3	4	5	6	7
5	0.9547	0.9669	0.9534	0.9856	0.9827	0.9799
	0.4294	0.6274	0.6092	0.8437	0.8370	0.8309
	0.6920	0.7405	0.6952	0.8721	0.8619	0.8522
	0.9822	0.9669	0.9888	0.9856	0.9827	0.9799
	0.6488	0.6274	0.8510	0.8437	0.8370	0.8309
	0.8155	0.7405	0.8854	0.8721	0.8619	0.8522
	0.9959	0.9922	0.9888	0.9856	0.9827	1.0000
	0.8683	0.8591	0.8510	0.8437	0.8370	1.0000
	0.9321	0.9035	0.8854	0.8721	0.8619	1.0000
	0.9959	1.0000	1.0000	1.0000	1.0000	1.0000
	0.8683	1.0000	1.0000	1.0000	1.0000	1.0000
	0.9321	1.0000	1.0000	1.0000	1.0000	1.0000

N \ k	2	3	4	5	6	7
6	0.9803	0.9634	0.9813	0.9761	0.9713	0.9667
	0.5318	0.5110	0.7115	0.7009	0.6914	0.6827
	0.7561	0.6618	0.7789	0.7560	0.7380	0.7233
	0.9803	0.9870	0.9813	0.9761	0.9713	0.9929
	0.5318	0.7232	0.7115	0.7009	0.6914	0.8918
	0.7561	0.8111	0.7789	0.7560	0.7380	0.9063
	0.9931	0.9974	0.9962	0.9950	0.9939	0.9929
	0.7366	0.9074	0.9030	0.8990	0.8953	0.8918
	0.8649	0.9374	0.9263	0.9182	0.9117	0.9063
	0.9986	0.9924	0.9962	0.9950	1.0000	1.0000
	0.9122	0.9074	0.9030	0.8990	1.0000	1.0000
	0.9554	0.9374	0.9263	0.9182	1.0000	1.0000

N \ k	2	3	4	5	6	7
7	0.9803	0.9843	0.9775	0.9712	0.9886	0.9867
	0.4407	0.6085	0.5959	0.5847	0.7767	0.7710
	0.7105	0.7737	0.6913	0.6620	0.8120	0.8018
	0.9803	0.9843	0.9775	0.9906	0.9886	0.9867
	0.4407	0.6085	0.5959	0.7829	0.7767	0.7710
	0.7105	0.7337	0.6913	0.8244	0.8120	0.8018
	0.9917	0.9950	0.9928	0.9906	0.9979	0.9975
	0.6228	0.7968	0.7896	0.7829	0.9325	0.9306
	0.8073	0.8983	0.8404	0.8244	0.9434	0.9401
	0.9974	0.9991	0.9987	0.9983	0.9979	0.9975
	0.8049	0.9390	0.9367	0.9345	0.9325	0.9306
	0.9012	0.9590	0.9522	0.9473	0.9434	0.9401

$\delta=0.5$

N \ k	2	3	4	5	6	7
8	0.9812	0.9832	0.9759	0.9877	0.9850	0.9825
	0.3685	0.5137	0.5017	0.6750	0.6678	0.6611
	0.6749	0.6702	0.6203	0.7375	0.7207	0.7070
	0.9812	0.9832	0.9905	0.9877	0.9850	0.9825
	0.3685	0.5137	0.6828	0.6750	0.6678	0.6611
	0.6749	0.6702	0.7597	0.7375	0.7207	0.7070
	0.9966	0.9934	0.9972	0.9964	0.9956	0.9948
	0.7009	0.6914	0.8478	0.8438	0.8400	0.8364
	0.8487	0.7920	0.8852	0.8743	0.8659	0.8590
	0.9990	0.9981	0.9972	0.9994	0.9993	0.9992
	0.8569	0.8522	0.8478	0.9574	0.9563	0.9552
	0.9280	0.9008	0.8852	0.9658	0.9634	0.9615

N \ k	2	3	4	5	6	7
9	0.9826	0.9832	0.9892	0.9860	0.9830	0.9802
	0.3102	0.4356	0.5884	0.5805	0.5731	0.5663
	0.6464	0.6181	0.6886	0.6616	0.6414	0.6255
	0.9912	0.9926	0.9892	0.9860	0.9938	0.9927
	0.4480	0.5972	0.5884	0.5805	0.7441	0.7395
	0.7196	0.7290	0.6886	0.6616	0.7857	0.7757
	0.9961	0.9973	0.9961	0.9949	0.9983	0.9980
	0.6069	0.7598	0.7542	0.7490	0.8860	0.8838
	0.8015	0.8390	0.8147	0.7982	0.9047	0.9001
	0.9986	0.9993	0.9990	0.9987	0.9983	0.9980
	0.7659	0.8932	0.8907	0.8883	0.8860	0.8838
	0.8823	0.9286	0.9178	0.9103	0.9047	0.9001

N \ k	2	3	4	5	6	7
10	0.9841	0.9838	0.9887	0.9853	0.9925	0.9912
	0.2626	0.3708	0.5069	0.4993	0.6538	0.6488
	0.6233	0.5751	0.6273	0.5962	0.7103	0.6977
	0.9915	0.9922	0.9887	0.9939	0.9925	0.9912
	0.3816	0.5152	0.5069	0.6591	0.6538	0.6488
	0.6866	0.6742	0.6273	0.7261	0.7103	0.6977
	0.9960	0.9968	0.9984	0.9979	0.9975	0.9970
	0.5245	0.6709	0.8115	0.8081	0.8050	0.8020
	0.7602	0.7795	0.8582	0.8461	0.8371	0.8298
	0.9984	0.9989	0.9984	0.9995	0.9994	0.9993
	0.6776	0.8151	0.8115	0.9205	0.9191	0.9178
	0.8380	0.8764	0.8582	0.9363	0.9325	0.9295

TABLE II (cont'd.)

For given k, N, P* (or c_1) and δ, this table gives the actual probability of a correct selection (top), the probability of selecting a non-best population (middle) and the expected proportion (bottom) of populations selected in the subset when the rule R_1 is used and the parameters are given by slippage configurations $\delta\lambda, \lambda, \ldots, \lambda$. For given k and N, each of the four blocks of three numbers correspond to P*=.75, .80, .90 and .95, respectively.
Note that for fixed k, N and P* (large) if c_1=N (from Table I), all three entries in each block are 1, as expected.

$\delta = 0.7$

N \ k	2	3	4	5	6	7
3	0.9302 0.7965 0.8633	0.8816 0.7458 0.7911	0.8445 0.7078 0.7420	1.0000 (3) 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000
	0.9302 0.7965 0.8633	0.8811 0.7458 0.7911	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000
	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000
	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000

N \ k	2	3	4	5	6	7
4	0.9036 0.6831 0.7933	0.9492 0.8518 0.8842	0.9310 0.8290 0.8545	0.9156 0.8100 0.8312	0.9022 0.7938 0.8118	0.8903 0.7795 0.7953
	0.9036 0.6831 0.7933	0.9492 0.8518 0.8842	0.9310 0.8290 0.8545	0.9156 0.8100 0.8312	0.9022 0.7938 0.8118	0.8903 0.7795 0.7953
	0.9713 0.8803 0.9258	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000
	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000	1.0000 1.0000 1.0000

δ=0.7

N \ k	2	3	4	5	6	7
5	0.8919	0.9183	0.8901	0.9627	0.9561	0.9501
	0.6055	0.7426	0.7100	0.8899	0.8800	0.8712
	0.7487	0.8012	0.7550	0.9044	0.8927	0.8825
	0.9534	0.9183	0.9701	0.9627	0.9561	0.9501
	0.7846	0.7426	0.9011	0.8899	0.8800	0.8712
	0.8690	0.8012	0.9183	0.9044	0.8927	0.8825
	0.9882	0.9784	0.9701	0.9627	0.9561	1.0000
	0.9296	0.9140	0.9011	0.8899	0.8800	1.0000
	0.9589	0.9355	0.9183	0.9044	0.8927	1.0000
	0.9882	1.0000	1.0000	1.0000	1.0000	1.0000
	0.9296	1.0000	1.0000	1.0000	1.0000	1.0000
	0.9589	1.0000	1.0000	1.0000	1.0000	1.0000

N \ k	2	3	4	5	6	7
6	0.9425	0.8998	0.9451	0.9320	0.9204	0.9100
	0.7087	0.6610	0.8078	0.7893	0.7733	0.7592
	0.8434	0.7406	0.8421	0.8179	0.7978	0.7807
	0.9425	0.9601	0.9451	0.9320	0.9204	0.9779
	0.7087	0.8296	0.8078	0.7893	0.7733	0.9258
	0.8434	0.8731	0.8421	0.8179	0.7978	0.9333
	0.9779	0.9909	0.9872	0.9838	0.9807	0.9779
	0.8562	0.9503	0.9431	0.9367	0.9310	0.9258
	0.9171	0.9638	0.9541	0.9462	0.9393	0.9333
	0.9951	0.9909	0.9872	0.9838	1.0000	1.0000
	0.9586	0.9503	0.9431	0.9367	1.0000	1.0000
	0.9769	0.9638	0.9541	0.9462	1.0000	1.0000

N \ k	2	3	4	5	6	7
7	0.9340	0.9467	0.9269	0.9098	0.9604	0.9547
	0.6468	0.7563	0.7294	0.7069	0.8525	0.8429
	0.7914	0.8198	0.7788	0.7476	0.8705	0.8589
	0.9340	0.9467	0.9269	0.9666	0.9604	0.9547
	0.6468	0.7563	0.7294	0.8632	0.8525	0.8429
	0.7914	0.8198	0.7788	0.8839	0.8705	0.8589
	0.9703	0.9810	0.9734	0.9666	0.9917	0.9904
	0.7897	0.8891	0.8753	0.8632	0.9606	0.9577
	0.8800	0.9197	0.8998	0.8839	0.9658	0.9623
	0.9897	0.9962	0.9946	0.9931	0.9917	0.9904
	0.9054	0.9712	0.9674	0.9638	0.9606	0.9577
	0.9476	0.9796	0.9742	0.9697	0.9658	0.9623

δ=0.7

N \ k	2	3	4	5	6	7
8	0.9322	0.9370	0.9139	0.9518	0.9431	0.9351
	0.5949	0.6937	0.6640	0.7938	0.7797	0.7669
	0.7635	0.7748	0.7265	0.8254	0.8068	0.7910
	0.9322	0.9370	0.9615	0.9518	0.9431	0.9351
	0.5949	0.6937	0.8099	0.7938	0.7797	0.7669
	0.7635	0.7748	0.8478	0.8254	0.8068	0.7910
	0.9850	0.9724	0.9874	0.9840	0.9808	0.9779
	0.8509	0.8286	0.9203	0.9128	0.9059	0.8997
	0.9180	0.8765	0.9371	0.9270	0.9184	0.9109
	0.9953	0.9911	0.9874	0.9971	0.9964	0.9959
	0.9384	0.9287	0.9203	0.9794	0.9776	0.9760
	0.9669	0.9495	0.9371	0.9829	0.9808	0.9788

N \ k	2	3	4	5	6	7
9	0.9301	0.9303	0.9520	0.9400	0.9293	0.9195
	0.5504	0.6399	0.7503	0.7315	0.7151	0.7005
	0.7403	0.7367	0.8007	0.7732	0.7508	0.7318
	0.9610	0.9655	0.9520	0.9400	0.9702	0.9657
	0.6789	0.7723	0.7503	0.7315	0.8482	0.8392
	0.8199	0.8367	0.8007	0.7732	0.8685	0.8573
	0.9811	0.9861	0.9803	0.9750	0.9909	0.9895
	0.7988	0.8815	0.8690	0.8580	0.9409	0.9370
	0.8900	0.9329	0.8969	0.8814	0.9492	0.9445
	0.9926	0.9959	0.9941	0.9925	0.9909	0.9895
	0.8960	0.9547	0.9496	0.9451	0.9409	0.9370
	0.9443	0.9684	0.9608	0.9545	0.9492	0.9445

N \ k	2	3	4	5	6	7
10	0.9293	0.9257	0.9445	0.9308	0.9608	0.9550
	0.5117	0.5931	0.6969	0.6765	0.7924	0.7314
	0.7205	0.7040	0.7588	0.7274	0.8205	0.8062
	0.9583	0.9600	0.9445	0.9671	0.9608	0.9550
	0.6327	0.7910	0.6969	0.8046	0.7924	0.7814
	0.7955	0.8007	0.7588	0.8371	0.8205	0.8062
	0.9781	0.9815	0.9901	0.9874	0.9848	0.9824
	0.7502	0.8339	0.9112	0.9040	0.8974	0.8913
	0.8642	0.8831	0.9309	0.9206	0.9119	0.9043
	0.9901	0.9931	0.9901	0.9965	0.9958	0.9951
	0.8522	0.9192	0.9112	0.9657	0.9632	0.9609
	0.9211	0.9439	0.9309	0.9719	0.9686	0.9657

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Mimeograph Series #457	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) On Subset Selection Procedures for Poisson Processes and Some Applications to the Binomial and Multinomial Problems		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER Mimeo. Series #457
7. AUTHOR(s) Shanti S. Gupta and Wing-Yue Wong		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0455
9. PERFORMING ORGANIZATION NAME AND ADDRESS Purdue University Department of Statistics W. Lafayette, IN 47907		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR 042-243
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Washington, DC		12. REPORT DATE July 1976
		13. NUMBER OF PAGES 36
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Poisson processes, subset selection, correct selection, waiting times, expected subset size, slippage configuration.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper deals with selection procedures for k ($\geq$ 2) Poisson processes. Selection rule R_1 is proposed and its properties and efficiency are investigated. In this rule the population π_i is selected if $X_i(t) \geq N-c_1$ where N is a fixed positive integer and all processes are observed until time t such that $\max_{1\leq i\leq k} X_i(t) = N$. The nonnegative constants $c_1 = c_1(k,P^*,N)$ have been computed and tabulated (Table I). Also, the actual value of the probability of a

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correct selection, the probability of selecting a non-best population, and the expected proportion in the selected subset are tabulated for certain slippage configurations (Table II). Another rule R_3 based on the number of arrivals in a fixed time and R_4, a conditional version of R_3, are proposed and studied. Applications to the selection of binomial populations and multinomial cells are described.

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