Note on a Theorem of Passow*

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Note on a Theorem of Passow*

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Consider a system of functions $u_i \in C^n[a,b]$, $i = 0, 1, \ldots, m$. We are interested in Hermite interpolation. Thus if $a \le x_0 < x_1 < \ldots < x_k \le b$ and $y_j^{(i)}$, $i = 0, \ldots, a_j - 1$, $j = 0, 1, \ldots, k$ are arbitrarily specified with $\sum_{j=0}^{k} a_j = m + 1$ and $\max_j a_j \le n - 1$ we are interested in when there exists a unique $u(x) = \sum_{i=0}^{m} c_i u_i(x)$ such that $u^{(r)}(x_j) = y_j^{(r)}$, $j = 0, 1, \ldots, k$; $r = 0, 1, \ldots a_j - 1$. Under such conditions the system $\{u_i\}$ is called an extended Tchebycheff system (ETS) of order $n+1$. In the case $n = 0$ the system is referred to as simply a Tchebycheff system (T S) and if $n = m$ the reference to the order is omitted. Note that an ETS of order $n+1$ is a ETS of any lower order.

It is well known that x^{t_k}, $k = 0, 1, \ldots m$ where $t_0 = 0 < t_1 < \ldots < t_m$ is an ETS on any interval $[a,b]$ for $0 \le a < b$. If $t_0 = 0$ and for all k, t_{2k} is even and t_{2k+1} is odd, then $t_0, t_1, \ldots t_m$ is said to have the alternating parity property (APP). Recently E. Passow [2] [3] proved the following:

<u>Theorem 1</u>. The system $\{x^{t_k}\}_0^m$ is an ETS of order $n + 1$ if and only if $t_i = i$, $i = 0, \ldots, n$ and $\{t_i\}$ has APP.

The purpose of this note is to generalize this result slightly to a larger class of systems $\{u_i\}$. Let w_k, $k = 0, 1, \ldots$, be strictly positive or $(-\infty, \infty)$ and $r - k$ times differentiable. Then define

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$$u_0(x) = w_0(x)$$

$$u_1(x) = w_0(x) \int_0^x w_1(\xi_1)\, d\xi_1 \qquad (1)$$

$$u_2(x) = w_0(x) \int_0^x w_1(\xi_1) \int_0^{\xi_1} w_2(\xi_2)\, d\xi_2\, d\xi_1$$

$$\vdots \qquad\qquad \vdots$$

$$u_r(x) = w_0(x) \int_0^x w_1\ (\xi_1) \cdots \int_0^{\xi_{r-1}} w_r(\xi_r)\, d\xi_r \cdots d\xi_1$$

For x negative the integrals are assumed oriented in the obvious sense so that $u_k(x)$ is negative iff k is odd. The system (1) is a basis for any r<u>th</u> order differential operator for which successive Wronskians do not vanish; see [1]. The value r here corresponds to t_m. We will show;

<u>Theorem 1</u>. The system $\{u_{t_k}\}_0^m$ is an ETS of order n+1 if and only if $t_i = i$, $i = 0, 1, \ldots, n$ and $\{t_i\}$ has APP provided $\left|\frac{u_i(x)}{u_j(x)}\right| \to 0$ for $|x| \to \infty$ when $i < j$.

The proof follows [2] and [3] however, some changes are necessary. The proof is divided into a three lemmas. The first of which is nearly obvious.

<u>Lemma A</u>. If f(x) is continuous on $(-\infty,\infty)$ and has at most n distinct zeros, then $g(x) = \int_0^x f(\xi)\, d\xi$

(a) has at most n zeros if x = 0 is a zero of f.

(b) always has at most n + 1 distinct zeros.

<u>Lemma B</u>. The system $u_{k_1}, \ldots, u_{k_r}$ is a T.S. on [a,b] for $0 < a < b$.

<u>Proof</u>. The proof is by induction on r. For $r = 1$, u_{k_1} has no zeros on [a,b]. Assuming the result true for r - 1 we consider

$$\sum_{i=1}^{r} a_i\, u_{k_i}(x) = U(x).$$

We write U(x) in the form

$$U(x) = w_1(x) \int_0^x w_1(\xi) \cdots \int_0^{\xi_{k_1-1}} [V(\xi)]d\xi$$

where

$$V(\xi) = a_1 w_{k_1}(\xi) + a_2 w_{k_1}(\xi) \int_0^{\xi} w_{k_1+1} \qquad \text{etc.}$$

Now $V(\xi)$ can have at most r-1 zeros, since otherwise

$$\frac{d}{d\xi}\left(\frac{V(\xi)}{w_{k_1}(\xi)}\right)$$

would have at least r-1 zeros violating the induction hypothesis. By Lemma A, U(x) has at most r zeros one of which is zero so that U(x) has at most r-1 zeros on [a,b].

<u>Lemma C</u>.

(a) If $\{t_i\}_0^m$ has APP then $\{u_{t_i}\}_0^m$ is a TS.

(b) If $\left|\frac{u_i(x)}{u_j(x)}\right| \to 0$ as $|x| \to \infty$ for $i < j$ then $\{u_{t_i}\}_0^m$ is a TS

implies that $\{t_i\}_0^m$ has APP.

<u>Proof</u>. We first show (a) that APP implies TS. The proof is by induction. For m = 0 we have $t_0 = 0$ and $u_0(x)$ is assumed to be positive on $(-\infty,\infty)$. Assuming the result for m-1 we consider

$$U(x) = \sum_{i=0}^{m} a_i u_{t_i}(x).$$

If U(x) has at least m+1 distinct zeros then we consider

$$D_0 U(x) = \frac{d}{dx}\frac{U(x)}{w_0(x)} = \sum_{i=1}^{m} a_i v_{t_i}(x)$$

where

$$v_{t_i}(x) = D_0\, u_{t_i}(x)$$

As in the proof of Lemma B we write $D_0U(x)$ in the form

$$w_1(x) \int_0^x w_2(\xi_2) \cdots \int_0^{\xi_{t_1-2}} w_{t_1-1}(\xi_{t_1-1}) \int_0^{\xi_{t_1-1}} V(\xi)\, d\xi \qquad (2)$$

where

$$V(\xi) = a_1\, w_{t_1}(\xi) + a_2\, w_{t_1}(\xi) \int_0^{\xi} \cdots$$

is a linear combination of m functions <u>again</u> satisfying the APP. Therefore, by the induction $V(\xi)$ has at most m-1 zeros. As in [2]; if $a_1 = 0$ then $D_0U(x)$ has at most m-1 zeros by Lemma A part (a). If $a_1 \neq 0$, then since the number of integrals t_1-1 in (2) is even $x = 0$ is not a separating zero of $D_0U(x)$ and $U(x)$ has at most m zeros.

We turn now to the converse (b), i.e. TS implies APP. The case m = 0 and m = 1 are easily checked. We then assume the result for m-1 and suppose $t_0, t_1 \ldots, t_m$ does not have APP. The two cases. (i) $t_0 \ldots t_{n-1}$ has APP and (ii) $t_0 \ldots t_{n-1}$ does not have APP can be handled as in [2]. For case (i) we assume that t_{n-1} and t_n are both odd and consider a polynomial $U_{n-1} = \sum_0^{n-1} a_i\, u_{t_i}$ with n-1 distinct simple zeros on [a,b] with $a_{n-1} > 0$. Then $U_{n-1} - \epsilon\, u_{t_n}$ has n+1 zeros for ϵ sufficiently small using the assumptions in (b), i.e. $U_{m-1} - \epsilon\, u_{t_n}$ will have a zero near every simple zero of U_{n-1} and will gain two more zeros for large x. Case (ii) is again handled as in [2].

<u>Proof of Theorem I</u>. If $\{t_i\}$ does not have APP then by Lemma C, $\{u_{k_i}\}$ is not a TS so it is also not an ETS.

Suppose that $t_j > j$ for some $j \leq n$ and consider the minimalsuch j.

We then take $x_0 = 0, y_0^{(j)} \neq 0, y_0^{(j)} = 0, i = 0,\ldots,j-1$ (see introduction) and consider any $U(x) = \sum_0 a_i u_{t_i}(x)$. A little reflection shows that any function of this form has the jth derivative at $x = 0$ equal to zero. This follows from the fact that if we define

$$D_i f(x) = \frac{d}{dx} \frac{f(x)}{w_i(x)} \quad i = 0,1,\ldots$$

then $f(0) = 0$ and $D_i D_{i-1} \ldots D_0 f(0) = 0, i = 0,\ldots,j-1$ if and only if $f^{(i)}(0) = 0,1,\ldots,j$. The proof of the converse also follows [3] using the operators $D_0,\ldots,D_{n-1}$ instead of the ordinary derivatives.

1. Karlin, S. and Studden, W. J. Tchebycheff Systems, Interscience, New York, 1966.

2. Passow, Eli. Alternating Parity of Tchebycheff Systems, J. Approximation Theory (to appear).

3. Passow, Eli. Extended Tchebycheff Systems on $(-\infty,\infty)$