A Unified Distribution Theory for Robustness Studies of Tests in Multivariate Analysis[1,2]

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A UNIFIED DISTRIBUTION THEORY FOR ROBUSTNESS STUDIES OF TESTS IN MULTIVARIATE ANALYSIS[1,2]

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1. Introduction. Consider the test of the following two hypotheses: 1) equality of covariance matrices in two p-variate normal populations and 2) equality of p-dimensional mean vectors in ℓ p-variate normal populations having a common covariance matrix. In order to carry out some exact investigations of robustness of tests of 1) when the assumption of normality is violated and of 2) when that of a common covariance matrix is disturbed, a distribution problem has been studied by the author [9], namely, that of the density of the characteristic roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$, where $\underset{\sim}{S}_1$ is distributed $W(p,n_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ and $\underset{\sim}{S}_2$, $W(p,n_1,\underset{\sim}{\Sigma}_2,\underset{\sim}{0})$, under an assumption on $\underset{\sim}{\Omega}$ or $\underset{\sim}{\Sigma}_1 \underset{\sim}{\Sigma}_2^{-1}$. Further the results are extended to the case of $\underset{\sim}{\Omega}$ random. Again, using the density under violations are obtained [15] the density function (under a condition), the moments of T (a constant times Hotelling's T_0^2); the m.g.f. of Pillai's trace; the distribution of Wilks' Λ and two expressions for the density function of the largest root.

The above results are presented first in this paper and then a new distributional form is suggested for T as a series in powers of (T/p)/(1+T/p) which for the two-roots case reduces to the exact form of

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Hotelling [2]. Some tabulations are presented to study the usefulness of this distributional form.

Finally a numerical study is attempted of the robustness of tests of hypothesis 1) and 2) based on the four criteria above for the two-roots case following methods similar to those contained in Pillai and Jayachandran [13], [14] Pillai and Dotson [11] and Pillai and Al-Ani [10] as applied to the density of violations.

2. Preliminaries. In this section we state some results which will be needed in the sequal.

Lemma 1. Let $S(p \times p)$ and $T(p \times p)$ be positive definite symmetric matrices. Then

$$(2.1) \qquad \int_{O(p)} {}_qF_r(a_1,\ldots,a_q;\ b_1,\ldots,b_r;\ \underset{\sim}{S}\ \underset{\sim}{H}\ \underset{\sim}{T}\ \underset{\sim}{H}')(d\ \underset{\sim}{H}) = {}_qF_r(a_1,\ldots,a_q;\ b_1,\ldots,b_r;\ \underset{\sim}{S},\ \underset{\sim}{T}) \qquad \text{(James [5])}$$

where the hypergeometric function ${}_qF_r$ of matrix argument is defined by James [ ] as

$$(2.2) \qquad {}_qF_r(a_1,\ldots,a_q;\ b_1,\ldots,b_r;\ \underset{\sim}{S},\ \underset{\sim}{T}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a_1)_\kappa \cdots (a_q)_\kappa}{(b_1)_\kappa \cdots (b_r)_\kappa} \cdot \frac{C_\kappa(\underset{\sim}{S})\ C_\kappa(\underset{\sim}{T})}{C\ (I_p)k!},$$

and where $a_1,\ldots,a_q,\ b_1,\ldots,b_r$ are real or complex constants and the multivariate coefficient $(a)_\kappa$ is given by

$$(2.3) \qquad (a)_\kappa = \prod_{i=1}^{p} (a - \tfrac{1}{2}(i-1))_{k_i}$$

where $(a)_k = a(a+1)\ldots(a+k-1)$ and $(a)_0 = 1$. The partition κ of k is such that $\kappa = (k_1,k_2,\ldots,k_p)$ into not more than p components with

$k_1 \geq k_2 \geq \ldots \geq k_p \geq 0$ such that $k = k_1 + \ldots + k_p$. The zonal polynomials $C_\kappa(\underset{\sim}{S})$ are expressible in terms of elementary symmetric functions of the latent roots of $\underset{\sim}{S}$, (James [5]), and $(d\underset{\sim}{H})$ stands for the invariant or Haar measure on the orthogonal group $O(p)$ normalized so that the measure of the whole group is unity. Note that with this measure we have

$$\int_{O(p)} (d\underset{\sim}{H}) = 2^p \pi^{\frac{1}{2}p^2} / \Gamma_p(\tfrac{1}{2}p), \qquad \text{(Constantine [1])} \tag{2.4}$$

where

$$\Gamma_p(a) = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(a - \tfrac{1}{2}(i-1)). \tag{2.5}$$

Lemma 2. Let $\underset{\sim}{S}(p \times p)$ be a symmetric matrix, and $C_\kappa(\underset{\sim}{S})$ and $C_\sigma(\underset{\sim}{S})$ be zonal polynomials of degree k and s respectively corresponding to the partition $\kappa = (k_1 \geq k_2 \geq \ldots \geq k_p \geq 0)$ and $\sigma = (s_1 \geq s_2 \geq \ldots \geq s_p \geq 0)$. Then

$$C_\kappa(\underset{\sim}{S})\, C_\sigma(\underset{\sim}{S}) = \sum_\delta g_{\kappa,\sigma}^{\delta}\, C_\sigma(\underset{\sim}{S}) \tag{2.6}$$

where $\delta = (d_1 \geq d_2 \geq \ldots \geq d_p \geq 0)$ such that $\sum_{i=1}^{p} d_1 = k + s$ and $g_{\kappa,\sigma}^{\delta}$ are constants. (Constantine [2], (Pillai and Sugiyama [16]).

Tables of the coefficients $g_{\kappa,\sigma}^{\delta}$ for various values of k and s can be found, for instance, in [7].

Lemma 3. Let $\underset{\sim}{R}(p \times p)$ be a complex symmetric matrix whose real part is positive definite, and let $\underset{\sim}{T}(p \times p)$ be an arbitrary complex symmetric matrix. Then

$$\int_{\underset{\sim}{S}>0} e^{-\mathrm{tr}\, \underset{\sim}{R}\, \underset{\sim}{S}} |\underset{\sim}{S}|^{t - \frac{1}{2}(p+1)} C_\kappa(\underset{\sim}{S}\, \underset{\sim}{T}) dS \tag{2.7}$$

$$= \Gamma_p(t,\kappa)\, |\underset{\sim}{R}|^{-t}\, C_\kappa(\underset{\sim}{T}\, \underset{\sim}{R}^{-1}), \qquad \text{(Constantine [1])}$$

the integration being over the space of positive definite p x p matrices, and valid for all complex numbers t such that $\mathrm{Re}(t) > \frac{1}{2}(p-1)$. The constant

$\Gamma_p(t,\kappa)$ is given by

$$(2.8) \qquad \Gamma_p(t,\kappa) = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(t+k_i - \tfrac{1}{2}(i-1)) = (t)_\kappa \, \Gamma_p(t).$$

Lemma 4. Let $\underset{\sim}{S}(p \times p)$ be a positive definite symmetric matrix and $\underset{\sim}{T}(p \times p)$ a complex matrix whose real part is positive definite symmetric. Then

$$(2.9) \quad \frac{2^{\frac{1}{2}p(p-1)} \Gamma_p(b)}{(2\pi i)^{\frac{1}{2}p(p+1)}} \int_{Re(\underset{\sim}{T})=\underset{\sim}{X}_0>0} e^{tr \underset{\sim}{T}} |\underset{\sim}{T}|^{-b} {}_qF_r(a_1,\ldots,a_q;b_1,\ldots,b_r;\underset{\sim}{T}^{-1}\underset{\sim}{S})d\underset{\sim}{T}$$

$$= {}_qF_{r+1}\,(a_1,\ldots,a_q;\ b_1,\ldots,b_r,b;\ \underset{\sim}{S}), \qquad \text{(James [5])}\cdot$$

Lemma 5. Let $\underset{\sim}{S}(p \times p)$ be a matrix and $\underset{\sim}{T}(p \times p)$ be a complex matrix whose real part is positive definite. Then for $\gamma > -1$.

$$(2.1 \quad L_\kappa^\delta(\underset{\sim}{S}) = \Gamma_p(\gamma+m,\kappa) \cdot \frac{2^{\frac{1}{2}p(p-1)}}{(2\pi i)^{\frac{1}{2}p(p+1)}} \int_{Re(\underset{\sim}{T})>0} e^{tr \underset{\sim}{T}} |\underset{\sim}{T}|^{-\gamma-m} C_\kappa(\underset{\sim}{I}-\underset{\sim}{S}\,\underset{\sim}{T}^{-1})d\underset{\sim}{T},$$

(Constantine [2]),

where $m = \frac{1}{2}(p+1)$ and $L_\kappa^\gamma(\underset{\sim}{S})$ is the generalized Laguerre polynomials defined by Constantine [2] as

$$(2.11) \qquad e^{-tr\,\underset{\sim}{S}} L_\kappa^\gamma(\underset{\sim}{S}) = \int_{R>0} e^{-tr\,\underset{\sim}{R}} |\underset{\sim}{R}|^\gamma C_\kappa(\underset{\sim}{R})\, A_\gamma(\underset{\sim}{R}\,\underset{\sim}{S})d\,\underset{\sim}{R},$$

where $A_\gamma(\underset{\sim}{R})$ is the Bessel function of matrix argument defined by:

$$(2.12) \quad A_\gamma(\underset{\sim}{R}) = \frac{2^{\frac{1}{2}p(p-1)}}{(2\pi i)^{\frac{1}{2}p(p+1)}} \int_{Re(\underset{\sim}{T})>0} e^{tr\,\underset{\sim}{T}} e^{-tr\,\underset{\sim}{R}\,\underset{\sim}{T}^{-1}} |\underset{\sim}{T}|^{-\gamma-m}\, d\,\underset{\sim}{T}\ .$$

3. <u>The distribution of latent roots of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$</u>. The distribution of the latent roots of $\underset{\sim}{S}_1\ \underset{\sim}{S}_2^{-1}$ will be derived in this section when $\underset{\sim}{S}_1(p \times p)$ is distributed $W(p,n_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ i.e. non-central Wishart distribution on n_1 d.f. with non-centrality $\underset{\sim}{\Omega}$ and covariance matrix $\underset{\sim}{\Sigma}_1$ and $\underset{\sim}{S}_2(p \times p)$ central Wishart $W(p,n_2,\underset{\sim}{\Sigma}_2,\underset{\sim}{0})$,

(3.1) where $n_1, n_2 \geq p$.

Theorem 1. Let the p x p matrices $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be independently distributed, $\underset{\sim}{S}_1$ having $W(p,n_1,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ and $\underset{\sim}{S}_2$ having $W(p,n_2,\underset{\sim}{\Sigma}_2,\underset{\sim}{0})$. Let $\underset{\sim}{R} = \mathrm{diag}(r_1,\ldots,r_p)$, where $r_1,\ldots,r_p$ are the roots of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ such that $0 < r_1 < r_2 < \ldots < r_p < \infty$. Then the joint density function of $r_1,\ldots,r_p$, when $\underset{\sim}{\Omega}$ is "random"*, is given by

$$(3.2) \qquad C(p,n_1,n_2)\, e^{-\mathrm{tr}\underset{\sim}{\Omega}} |\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} |\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)} \prod_{i>j} (r_i - r_j)$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_\kappa C_\kappa(\underset{\sim}{I}_p) k!} \sum_{n=0}^{\infty} \sum_{\nu,\delta} \frac{(-1)^n g^\delta_{\kappa,\nu} (\frac{n_1+n_2}{2})_\delta C_\delta(\underset{\sim}{\Lambda}^{-1}) C_\delta(\underset{\sim}{R})}{C_\delta(\underset{\sim}{I}_p) n!}$$

where

$$(3.3) \qquad C(p,n_1,n_2) = \pi^{\frac{1}{2}p^2} [\Gamma_p(\tfrac{n_1+n_2}{2})]/[\Gamma_p(\tfrac{n_1}{2})\, \Gamma_p(\tfrac{n_2}{2})\, \Gamma_p(\tfrac{p}{2})] \text{ and } \underset{\sim}{\Lambda} = \underset{\sim}{\Sigma}_1^{\frac{1}{2}} \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{\Sigma}_1^{\frac{1}{2}} .$$

Proof. Joint density of $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ is

$$\{\Gamma_p(\tfrac{n_1}{2})\, \Gamma_p(\tfrac{n_2}{2})\, |2\underset{\sim}{\Sigma}_1|^{\frac{1}{2}n_1} |2\underset{\sim}{\Sigma}_2|^{\frac{1}{2}n_2}\}^{-1} e^{-\mathrm{tr}\underset{\sim}{\Omega}}\, e^{-\frac{1}{2}\mathrm{tr}\, \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{S}_1} |\underset{\sim}{S}_1|^{\frac{1}{2}(n_1-p-1)}$$

$$e^{-\frac{1}{2}\mathrm{tr}\, \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{S}_2} |\underset{\sim}{S}_2|^{\frac{1}{2}(n_2-p-1)}\; {}_0F_1(\tfrac{1}{2}n_1; \tfrac{1}{2} \underset{\sim}{\Sigma}_1^{-1} \underset{\sim}{\Omega}\, \underset{\sim}{S}_1) .$$

Since the roots are invariant under the simultaneous transformations $\underset{\sim}{A}_1 = \frac{1}{2} \underset{\sim}{\Sigma}_1^{-\frac{1}{2}} \underset{\sim}{S}_1 \underset{\sim}{\Sigma}_1^{-\frac{1}{2}}$ and $\underset{\sim}{A}_2 = \frac{1}{2} \underset{\sim}{\Sigma}_1^{-\frac{1}{2}} \underset{\sim}{S}_2 \underset{\sim}{\Sigma}_1^{-\frac{1}{2}}$, we have now the joint density of $\underset{\sim}{A}_1$ and $\underset{\sim}{A}_2$ ($\sum_1^{-\frac{1}{2}}$(p x p) being symmetric positive definite like other matrices of the form $\underset{\sim}{A}^{1/2}$ defined later)

$$(3.4) \quad C_1(p,n_1,n_2) e^{-\mathrm{tr}\underset{\sim}{\Omega}} |\underset{\sim}{\Lambda}|^{\frac{1}{2}n_2} e^{-\mathrm{tr}\underset{\sim}{A}_1} |\underset{\sim}{A}_1|^{\frac{1}{2}(n_1-p-1)}\, e^{-\mathrm{tr}\underset{\sim}{\Lambda}\underset{\sim}{A}_2} |\underset{\sim}{A}_2|^{\frac{1}{2}(n_2-p-1)}\; {}_0F_1(\tfrac{1}{2}n_1; \underset{\sim}{\Omega}\underset{\sim}{A}_1),$$

where

$$\underset{\sim}{\Lambda} = \underset{\sim}{\Sigma}_1^{\frac{1}{2}} \underset{\sim}{\Sigma}_2^{-1} \underset{\sim}{\Sigma}_1^{\frac{1}{2}} \text{ and } C_1(p,n_1,n_2) = \{\Gamma_p(\tfrac{n_1}{2})\, \Gamma_p(\tfrac{n_2}{2})\}^{-1} .$$

Now transform $\underset{\sim}{A}_1 = \underset{\sim}{A}_2^{\frac{1}{2}} \underset{\sim}{R}\, \underset{\sim}{A}_2^{\frac{1}{2}}$ and $\underset{\sim}{A}_2 = \underset{\sim}{A}_2$. (The same notation is used

*"random" implies diagonalization by orthogonal transformation $\underset{\sim}{H}$ and integrating of H over O(p).

to denote the matrix $\underset{\sim}{R}$ both before and after diagonalization). The Jacobian is $|\underset{\sim}{A}_2|^{\frac{1}{2}(p+1)}$, and hence the joint density of $\underset{\sim}{R}$ and $\underset{\sim}{A}_2$ is

$$C_1(p,n_1,n_2)\ e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Lambda}|^{\frac{1}{2}n_2}\ e^{-\mathrm{tr}\ \underset{\sim}{R}\ \underset{\sim}{A}_2}|\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot\ e^{-\mathrm{tr}\underset{\sim}{\Lambda}\ \underset{\sim}{A}_2}|\underset{\sim}{A}_2|^{\frac{1}{2}(n_1+n_2-p-1)}\ {}_0F_1(\tfrac{1}{2}n_1;\underset{\sim}{\Omega}\ \underset{\sim}{A}_2^{\frac{1}{2}}\ \underset{\sim}{R}\ \underset{\sim}{A}_2^{\frac{1}{2}})\ .$$

We transform $\underset{\sim}{\Omega} \to \underset{\sim}{H}\ \underset{\sim}{\Omega}\ \underset{\sim}{H}'$ where $\underset{\sim}{H} \in O(p)$, and integrating over $O(p)$ using Lemma 1, all the factors in the above expression remain the same except the hypergeometric function which now becomes ${}_0F_1(\frac{1}{2}n_1;\ \underset{\sim}{\Omega},\ \underset{\sim}{R}\ \underset{\sim}{A}_2)$. Further, expand $e^{-\mathrm{tr}\ \underset{\sim}{R}\ \underset{\sim}{A}_2} = {}_0F_0(-\underset{\sim}{R}\ \underset{\sim}{A}_2)$ and ${}_0F_1(\frac{1}{2}n_1;\ \underset{\sim}{\Omega},\ \underset{\sim}{R}\ \underset{\sim}{A}_2)$ in zonal polynomials using (2.2) and apply Lemma 2. The joint density of $\underset{\sim}{R}$ and $\underset{\sim}{A}_2$ becomes

$$C_1(p,n_1,n_2)\ e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Lambda}|^{\frac{1}{2}n_2}|\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)}\ e^{-\mathrm{tr}\underset{\sim}{\Lambda}\underset{\sim}{A}_2}|\underset{\sim}{A}_2|^{\frac{1}{2}(n_1+n_2-p-1)}$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_\kappa\ C_\kappa(\underset{\sim}{I}_p)k!}\sum_{n=0}^{\infty}\sum_{\nu,\delta}\frac{(-1)^n\ g^{\delta}_{\kappa,\nu}C_\delta(\underset{\sim}{R}\ \underset{\sim}{A}_2)}{n!}\ .$$

Now integrate out $\underset{\sim}{A}_2$ using Lemma 3 we have the density of $\underset{\sim}{R}$ of the form

$$C_1(p,n_1,n_2)\ e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}\ |\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_\kappa C_\kappa(\underset{\sim}{I}_p)k!}\sum_{n=0}^{\infty}\sum_{\nu,\delta}\frac{(-1)^n\ g^{\delta}_{\kappa,\nu}\Gamma_p(\frac{n_1+n_2}{2},\delta)\ C_\delta(\underset{\sim}{R}\ \underset{\sim}{\Lambda}^{-1})}{n!}\ .$$

$\underset{\sim}{R}$ being symmetric, can be diagonalized by an orthogonal transformation $\underset{\sim}{H}$ such that $\underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}' = \mathrm{diag}(r_1,\ldots,r_p)$ where $r_1,\ldots,r_p$ are roots of $\underset{\sim}{R}$. For uniqueness, we assume that the elements in the first row of H are positive and the roots are arranged in the order $0 < r_1 < r_2 <\ldots< r_p < \infty$. The volume element $d\underset{\sim}{R}$ becomes (Constantine [1], James [5])

$$(3.5)\qquad d\underset{\sim}{R} = \prod_{i>j} (r_i - r_j) \prod_{i=1}^{p} dr_i (d\underset{\sim}{H}) .$$

Substituting $\underset{\sim}{R}$ in the above expression and integrating over $\underset{\sim}{H}$, we have the joint density of $r_1,\ldots,r_p$ in the form

$$C_1(p,n_1,n_2)\ e^{-\mathrm{tr}\underset{\sim}{\Omega}} |\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} |\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)} \prod_{i>j} (r_i - r_j)$$

$$\cdot \sum_{k=0} \sum_{\kappa} \frac{C_\kappa(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_\kappa C_\kappa(\underset{\sim}{I}_p) k!} \sum_{k=0}^{\infty} \sum_{\nu,\delta} \frac{(-1)^n\ g^\delta_{\kappa,\nu}\ \Gamma_p(\frac{n_1+n_2}{2},\delta)}{n!}$$

$$2^{-p} \int_{0(p)} C_\delta(\underset{\sim}{H}\underset{\sim}{R}\underset{\sim}{H}'\underset{\sim}{\Lambda}^{-1}) (d\underset{\sim}{H}) .$$

The factor 2^{-p} multiplying the integral arises from the restriction that the elements in the first row of $\underset{\sim}{H}$ are positive. Finally we use the property of the zonal polynomials given by James [5] for normalized measure

$$(3.6)\qquad \int_{0(p)} C_\kappa(\underset{\sim}{H}\ \underset{\sim}{S}\ \underset{\sim}{H}'\underset{\sim}{T}) (d\ \underset{\sim}{H}) = [C_\kappa(\underset{\sim}{S})\ C_\kappa(\underset{\sim}{T}) / C_\kappa(\underset{\sim}{I}_p)$$

and making use of (2.4) and (2.8) we have the result as stated in the theorem. Q.E.D.

For $\underset{\sim}{\Omega} = \underset{\sim}{0}$, since $g^\delta_{0,\nu} = 1$ and $\delta = \nu$, the expression (3.2) gives the result stated by James [5] page 484 as a special case.

It seems that, the expression (3.2) is not convenient for further development and may not converge for all values of $\underset{\sim}{R}$, $\underset{\sim}{\Lambda}$ and $\underset{\sim}{\Omega}$. For further development we shall prove the following theorem.

<u>Theorem 2</u>. Let $\underset{\sim}{S}_1$, $\underset{\sim}{S}_2$, $\underset{\sim}{R}$ and r_i's be as stated in Theorem 1. Then the joint density function of $r_1,\ldots,r_p$ is given by

$$(3.7)\ C(p,n_1,n_2)\ e^{-\mathrm{tr}\underset{\sim}{\Omega}} |\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} |\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)} |\underset{\sim}{I}+\lambda\underset{\sim}{R}|^{-\frac{1}{2}(n_1+n_2)} \prod_{i>j} (r_i - r_j)$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \left(\frac{n_1+n_2}{2}\right)_\kappa \frac{C_\kappa(\lambda\underset{\sim}{R}(\underset{\sim}{I}+\lambda\underset{\sim}{R})^{-1})}{k!}$$

$$\sum_{d=0}^{k} \sum_{\delta} \frac{a_{\kappa,\delta}\, C_\delta(-\lambda^{-1}\Lambda^{-1})\, L^{\frac{1}{2}(n_1-p-1)}_{\delta}(\Omega)}{(\frac{n_1}{2})_\delta\, C_\delta(I)\, C_\delta(I)} ,$$

where $\Lambda = \Sigma_1^{\frac{1}{2}} \Sigma_2^{-1} \Sigma_1^{\frac{1}{2}}$, λ, a positive real number, $C(p,n_1,n_2)$ is as defined in (3.3), $L_\delta^\gamma(S)$ is as in (2.11) and $a_{\kappa,\delta}$ are constants (Constantine [2], Pillai and Jouris [12]), and Λ is "random".

Proof. We start from the expression (3.4). Now apply Lemma 4 to ${}_0F_1(\frac{1}{2}n_1;\Omega^{\frac{1}{2}}A_1\Omega^{\frac{1}{2}})$ to get the joint density of A_1 and A_2 as

$$C_2(p,n_2)\, e^{-tr\Omega}|\Lambda|^{\frac{1}{2}n_2}\, e^{-tr\Lambda A_2}|A_2|^{\frac{1}{2}(n_2-p-1)}|A_1|^{\frac{1}{2}(n_1-p-1)}$$
$$\cdot\int_{Re(T)=X_0>0} e^{trT}|T|^{-\frac{1}{2}n_1}\, e^{-tr(I-W)A_1}(d\,T),$$

where

$$C_2(p,n_2) = 2^{\frac{1}{2}p(p-1)}/[(2\pi i)^{\frac{1}{2}p(p+1)}\,\Gamma_p(\tfrac{n_2}{2})],\quad W = \Omega^{\frac{1}{2}}T^{-1}\Omega^{\frac{1}{2}}$$

and $T = X_0 + i\,Y$ with X_0 positive definite symmetric matrix and Y a real symmetric matrix. The roots are invariant under the simultaneous transformations

$$B_1 = \Lambda^{\frac{1}{2}} A_1 \Lambda^{\frac{1}{2}}$$

and

$$B_2 = \Lambda^{\frac{1}{2}} A_2 \Lambda^{\frac{1}{2}} .$$

Making substitutions in the above density and after taking the Jacobians into consideration, we obtain the joint density of B_1 and B_2 as follows:

$$C_2(p,n_2)\, e^{-tr\Omega}|\Lambda|^{-\frac{1}{2}n_1}\, e^{-trB_2}|B_1|^{\frac{1}{2}(n_1-p-1)}|B_2|^{\frac{1}{2}(n_2-p-1)}$$
$$\cdot\int_{Re(T)>0} e^{trT}|T|^{-\frac{1}{2}n_1}\, e^{-tr(\Lambda^{-\frac{1}{2}}(I-W)\Lambda^{-\frac{1}{2}}B_1)}(dT).$$

Apply the transformation $B_2 = B_1^{\frac{1}{2}} R_1 B_1^{\frac{1}{2}}$ and $B_1 = B_1$ and integrate out B_1

using (2.7) with $\kappa = (0)$ and also (2.8), then we have the density of $\underset{\sim}{R}$ in the form

(3.8) $$C_3(p,n_1,n_2)\ e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}|\underset{\sim}{R}_1|^{\frac{1}{2}(n_2-p-1)}$$

$$\cdot \int_{\mathrm{Re}(\underset{\sim}{T})>\underset{\sim}{0}} e^{\mathrm{tr}\underset{\sim}{T}}|\underset{\sim}{T}|^{-\frac{1}{2}n_1}|\underset{\sim}{R}_1 + \Lambda^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{\Lambda}^{-\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)}(d\underset{\sim}{T}),$$

where

$$C_3(p,n_1,n_2) = C_2(p,n_2)\ \Gamma_p(\frac{n_1+n_2}{2}) .$$

Transform now $\underset{\sim}{R} = \underset{\sim}{R}_1^{-1}$, (the Jacobian of this transformation is $|\underset{\sim}{R}|^{-(p+1)}$), then (3.8) becomes

(3.9) $$C_3(p,n_1,n_2)\ e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}|\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot \int_{\mathrm{Re}(\underset{\sim}{T})>\underset{\sim}{0}} e^{\mathrm{tr}\underset{\sim}{T}}|\underset{\sim}{T}|^{-\frac{1}{2}n_1}|\underset{\sim}{I} + \underset{\sim}{R}\underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{\Lambda}^{-\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)}(d\underset{\sim}{T}).$$

Since $\underset{\sim}{R}$ is symmetric we can diagonalize by an orghogonal transformation $\underset{\sim}{H}$ and can use the same technique as before. After substitutions and making use of (3.5) and integrating over $\underset{\sim}{H}$, we will get the joint density of roots $r_1,\ldots,r_p$ of $\underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ in the form

(3.10) $$C_3(p,n_1,n_2)\ e^{-\mathrm{tr}\underset{\sim}{\Omega}}|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}|\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)} \prod_{i>j}(r_i-r_j)$$

$$\cdot \int_{\mathrm{Re}(\underset{\sim}{T})>\underset{\sim}{0}} e^{\mathrm{tr}\underset{\sim}{T}}|\underset{\sim}{T}|^{-\frac{1}{2}n_1}\int_{O(p)} 2^{-p}|\underset{\sim}{I}+\underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'\ \underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{\Lambda}^{-\frac{1}{2}}|^{-\frac{1}{2}(n_1+n_2)}$$

$$(d\underset{\sim}{H})(d\underset{\sim}{T}) .$$

Following Khatri [ ], we can write

$$|\underset{\sim}{I} + \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'\underset{\sim}{A}| = |\underset{\sim}{I} + \lambda\underset{\sim}{R}|\ |\underset{\sim}{I} - (\underset{\sim}{I}-\lambda^{-1}\underset{\sim}{A})\ \underset{\sim}{H}\ (\lambda\underset{\sim}{R})\ (\underset{\sim}{I} + \lambda\underset{\sim}{R})^{-1}\ \underset{\sim}{H}'|$$

where λ is a positive real number and in our case $\underset{\sim}{A} = \underset{\sim}{\Lambda}^{-\frac{1}{2}}(\underset{\sim}{I}-\underset{\sim}{W})\underset{\sim}{\Lambda}^{-\frac{1}{2}}$.

After making use of James [5]

$$\int_{O(p)}|\underset{\sim}{I} - \underset{\sim}{H}\ \underset{\sim}{R}\ \underset{\sim}{H}'\underset{\sim}{A}|^{-a}\ (d\underset{\sim}{H}) = {}_1F_0(a;\ \underset{\sim}{A},\ \underset{\sim}{R})$$

and the formula (2.4), the expression (3.10) now becomes

$$(3.11)\qquad C_4(p,n_1,n_2)\, e^{-\mathrm{tr}\Omega}|\Lambda|^{-\frac{1}{2}n_1}|R|^{\frac{1}{2}(n_1-p-1)}|I+\lambda R|^{-\frac{1}{2}(n_1+n_2)}\prod_{i>j}(r_i-r_j)$$

$$\cdot\int_{\mathrm{Re}(T)>0} e^{\mathrm{tr}T}|T|^{-\frac{1}{2}n_1}\,{}_1F_0\left(\frac{n_1+n_2}{2};\ I-\lambda^{-1}\Lambda^{-\frac{1}{2}}(I-W)\Lambda^{-\frac{1}{2}},\lambda R(I+\lambda R)^{-1}\right)dT,$$

where

$$C_4(p,n_1,n_2) = [C_3(p,n_1,n_2)\ \pi^{\frac{1}{2}p^2}]/\Gamma_p(\tfrac{p}{2})\ .$$

Now we can make use of (2.2) to expand ${}_1F_0$ and integrate term by term. The expression (3.11) involves the integral of the form

$$(3.12)\qquad \int_{\mathrm{Re}(T)>0} e^{\mathrm{tr}T}|T|^{-\frac{1}{2}n_1}\frac{C_\kappa(I-\lambda^{-1}\Lambda^{-\frac{1}{2}}(I-W)\Lambda^{-\frac{1}{2}})}{C_\kappa(I)}\ dT\ .$$

Let us use the relation (Constantine [2])

$$(3.13)\qquad \frac{C_\kappa(I-S)}{C_\kappa(I)} = \sum_{n=0}^{k}\sum_{\nu}(-1)^n a_{\kappa,\nu}\frac{C_\nu(S)}{C_\nu(I)}\ ,$$

where $a_{\kappa,\nu}$ are constants discussed earlier. Then (3.12) becomes

$$(3.14)\qquad \sum_{n=0}^{k}\sum_{\nu}\frac{(-\lambda^{-1})^n a_{\kappa,\nu}}{C_\nu(I)}\int_{\mathrm{Re}(T)>0} e^{\mathrm{tr}T}|T|^{-\frac{1}{2}n_1}C_\nu(\Lambda^{-1}(I-W))\ dT\ .$$

Λ being symmetric we can transform $\Lambda \rightarrow H\,\Lambda\,H'$ by an orthogonal transformation H and integrate over H using (3.6). Then (3.14) becomes

$$(3.15)\qquad \sum_{n=0}^{k}\sum_{\nu}\frac{(-\lambda^{-1})^n a_{\kappa,\nu}C_\nu(\Lambda^{-1})}{C_\nu(I)\ C_\nu(I)}\int_{\mathrm{Re}(T)>0} e^{\mathrm{tr}T}|T|^{-\frac{1}{2}n_1}C_\nu(I-T^{-1}\Omega)\ dT.$$

Applying Lemma 5 to (3.15) gives

$$(3.16)\qquad \frac{(2\pi i)^{\frac{1}{2}p(p+1)}}{2^{\frac{1}{2}p(p-1)}}\sum_{n=0}^{k}\sum_{\nu}\frac{a_{\kappa,\nu}C_\nu(-\lambda^{-1}\Lambda^{-1})\ L_\nu^{\frac{1}{2}(n_1-p-1)}(\Omega)}{\Gamma_p(\frac{n_1}{2},\nu)\ C_\nu(I)\ C_\nu(I)}\ .$$

Combining (3.11) - (3.16) and making use of (2.8) we have the result as stated in the theorem. Q.E.D.

Formula (3.7) will give special cases:

a) For $\Omega = 0$, it is seen from (2.12) that $A_\gamma(0) = \{\Gamma_p(\gamma + \frac{p+1}{2})\}^{-1}$ and from (2.11) and (2.7) we have $L_\kappa^\delta(0) = (\gamma + \frac{p+1}{2})_\kappa C_\kappa(I)$. Substituting $L_\delta^{\frac{1}{2}(n_1-p-1)}(0)$ into (3.7) and making use of (3.13), we have the result of Khatri [6].

b) For $\Lambda = I$ and $\lambda = 1$ and using the relation (Constantine [2])

$$\text{(3.17)} \qquad \frac{L_\delta^\gamma(S)}{(\gamma+m)_\kappa C_\kappa(I)} = \sum_{n=0}^{k} \sum_{\nu} \frac{(-1)^n a_{\kappa,\nu} C_\nu(S)}{(\gamma+m)_\nu C_\nu(I)} ,$$

where $m = \frac{1}{2}(p+1)$ whenever S is (p x p) matrix and setting $\nu = \delta = \kappa$ we have the result of Constantine [1], James [5].

Let us now consider Ω as a random matrix $\frac{1}{2}\Sigma_1^{\frac{1}{2}} MYY'M'\Sigma_1^{-\frac{1}{2}}$ where YY' is distributed as a central Wishart $W(p,n_3,\Sigma_3,0)$, i.e.,:

$$\text{(3.18)} \qquad \{\Gamma_p(\tfrac{1}{2} n_3)|2\Sigma_3|^{\frac{1}{2} n_3}\}^{-1} |YY'|^{\frac{1}{2}(n_3-p-1)} \exp[tr(-\tfrac{1}{2}\Sigma_3^{-1} YY')].$$

Multiplying (3.2) by (3.18) and integrating term by term with respect to YY' using (2.7) we get the joint density of $r_1,\ldots,r_p$ of the form:

$$\text{(3.19)} \qquad C(p,n_1,n_2)|\Lambda|^{-\frac{1}{2} n_1} |I+M'\Sigma_1^{-1}M\Sigma_3|^{-\frac{1}{2} n_3} |R|^{\frac{1}{2}(n_1-p-1)}$$

$$\cdot \prod_{i>j} (r_i-r_j) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2} n_3)_\kappa C_\kappa[(I + \Omega_1)^{-1} \Omega_1]}{(\frac{1}{2} n_1)_\kappa C_\kappa(I) k!}$$

$$\cdot \sum_{n=0}^{\infty} \sum_{\nu,\delta} \frac{(-1)^n g_{\kappa,\nu}^\delta (\frac{1}{2}(n_1+n_2))_\delta C_\delta(\Lambda^{-1}) C_\delta(R)}{C_\delta(I) n!}, \text{ where } \Omega_1 = \Sigma_3^{\frac{1}{2}} M'\Sigma_1^{-1} M\Sigma_3^{\frac{1}{2}} .$$

Now, alternately, consider (3.7). Expand the generalized Laguerre polynomial in (3.7) using the expansion in (3.17). After multiplying this new form of

(3.7) by (3.18) and integrating $\underset{\sim}{Y}\underset{\sim}{Y}'$ and using (2.7) we have the joint density of $r_1,\ldots,r_p$ of the form.

(3.20)
$$C(p,n_1,n_2)|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}|\underset{\sim}{I}+\underset{\sim}{M}'\underset{\sim}{\Sigma}_1^{-1}\underset{\sim}{M}\underset{\sim}{\Sigma}_3|^{-\frac{1}{2}n_3}$$
$$\cdot\ |\underset{\sim}{R}|^{\frac{1}{2}(n_1-p-1)}|\underset{\sim}{I}+\lambda\underset{\sim}{R}|^{-\frac{1}{2}(n_1+n_2)}\prod_{i>j}(r_i-r_j)$$
$$\cdot\ \sum_{k=0}^{\infty}\sum_{\kappa}(\tfrac{1}{2}(n_1+n_2))_\kappa\frac{C_\kappa(\lambda\underset{\sim}{R}(\underset{\sim}{I}+\lambda\underset{\sim}{R})^{-1})}{k!}\sum_{d=0}^{k}\sum_{\delta}\frac{a_{\kappa,\delta}C_\delta(-\lambda^{-1}\underset{\sim}{\Lambda}^{-1})}{C_\delta(\underset{\sim}{I})}$$
$$\sum_{n=0}^{d}\sum_{\nu}\frac{(-1)^n a_{\delta,\nu}(\frac{1}{2}n_3)_\nu C_\nu[(\underset{\sim}{I}+\underset{\sim}{\Omega}_1)^{-1}\underset{\sim}{\Omega}_1]}{(\frac{1}{2}n_1)_\nu C_\nu(\underset{\sim}{I})}\ .$$

It is easy to see that the distribution of the characteristic roots for the canonical correlation problem is a special case of (3.20).

4. <u>Distribution problems of test criteria under violations</u>. In this section we state some results concerning distributions of some test statistics derived in the light of Section 3. Details are available in Pillai and Sudjana [15].

i. <u>The density function of $T=\lambda\ \mathrm{tr}\ \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$</u>, for $\lambda>0$, and $\underset{\sim}{\Omega}$ random (convergent for $|T/\lambda\lambda_1|<1$, where λ_1 is the smallest latent root of $\underset{\sim}{\Lambda}$) is given by

(4.1)
$$f(T)=[\Gamma_p(\tfrac{1}{2}(n_1+n_2))/\Gamma_p(\tfrac{1}{2}n_2)]|\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1}e^{-\mathrm{tr}\ \underset{\sim}{\Omega}}T^{\frac{1}{2}pn_1-1}$$
$$\cdot\ \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\frac{1}{2}(n_1+n_2))_\kappa\ (-T)^k}{k!\ \Gamma_p(\frac{1}{2}pn_1+k)}\ \frac{C_\kappa(\lambda^{-1}\underset{\sim}{\Lambda}^{-1})\ L_\kappa^{\frac{1}{2}(n_1-p-1)}(\underset{\sim}{\Omega})}{C_\kappa(\underset{\sim}{I})}$$

Formula (4.3) will give special cases as follows:

a). For $\underset{\sim}{\Omega}=\underset{\sim}{0}$ which implies $L_\kappa^\gamma(\underset{\sim}{0})=(\frac{1}{2}n_1)_\kappa\ C_\kappa(\underset{\sim}{I})$ where $\gamma=\frac{1}{2}(n_1-p-1)$, we have the result of Khatri [6] formula (9).

b). For $\Lambda = I$ and $\lambda = 1$ we have Theroem 4 of Constantine [2].

ii. <u>The moments of T.</u> For Λ "random"

$$(4.2) \qquad E(T^k) = [\Gamma_p(\tfrac{1}{2}n_2)]^{-1} \sum_{\kappa} \Gamma_p(\tfrac{1}{2}n_2,-\kappa)\; C_\kappa(\lambda\Lambda) L_\kappa^{\frac{1}{2}(n_1-p-1)}(\Omega)/C_\kappa(I).$$

Using the fact that $\Gamma_p(t,-\kappa) = [(-1)^k\Gamma_p(t)]/(-t+\frac{1}{2}(p+1))_\kappa$, we get the k^{th} moment of T as

$$(4.3) \qquad E(T^k) = (-1)^k \sum_{\kappa} \frac{C_\kappa(\lambda\Lambda)\; L_\kappa^{\frac{1}{2}(n_1-p-1)}(-\Omega)}{(\frac{1}{2}(p+1-n_2))_\kappa\; C_\kappa(I)},$$

which exists only for $n_2 > 2k+p-1$.

Formula (4.3) will give special cases for special values of Ω and Λ. If we let $\Omega = 0$, (4.3) gives

$$E(T^k) = (-1)^k \sum_{\kappa} (\tfrac{1}{2}n_1)_\kappa\; C_\kappa(\lambda\Lambda)/(\tfrac{1}{2}(p+1-n_2))_\kappa,$$

which is the result of Khatri [6]; expect that his formula contains an error in the denominator giving $(\frac{1}{2}(p-n_2-1))_J$ instead of $(\frac{1}{2}(p-n_2+1))_J$. Substitution of $\Lambda = I$, $\lambda = 1$ in (4.3) will give the result of Constantine [2] formula (38).

iii. <u>The moment generating function of $V^{(p)}$.</u> The m.g.f. of Pillai's trace defined by $V^{(p)} = tr[(\lambda R)(I+\lambda R)^{-1}]$ for Λ "random" is given by

$$(4.4) \qquad E(\exp(t\,V^{(p)})) = e^{-tr\Omega}|\lambda\Lambda|^{-\frac{1}{2}n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n_1)_\kappa\; C_\kappa(I)}{k!\,(\frac{1}{2}(n_1+n_2))_\kappa}$$

$$\cdot \sum_{d=0}^{k} \sum_{\delta} a_{\kappa,\delta}\; (\tfrac{1}{2}(n_1+n_2))_\delta\; t^{k-d} \sum_{n=0}^{d} \sum_{\nu} \frac{a_{\kappa,\nu}\; C_\nu(-\lambda^{-1}\Lambda^{-1}) L_\nu^{\frac{1}{2}(n_1-p-1)}(\Omega)}{(\frac{1}{2}n_1)_\nu\; C_\nu(I)\; C_\nu(I)}.$$

For $\Omega = 0$, (4.4) gives the result of Khatri [6] with a correction of his expression given by Pillai [8]. Another special case which can be derived from (4.4) is formula (3.5) of Pillai [8] if in (4.4) we let $\Lambda = I$, $\lambda = 1$

and we use (Constantine [ ])

$$(4.5)\qquad L_{\nu}^{\frac{1}{2}(n_1-p-1)}(\underset{\sim}{\Omega}) = (\tfrac{1}{2}n_1)_{\nu}\, C_{\nu}(\underset{\sim}{I}) \sum_{s=0}^{n} \sum_{\sigma} \frac{(-1)^s\, a_{\nu,\delta}\, C_{\sigma}(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_{\sigma}\, C_{\sigma}(\underset{\sim}{I})}$$

and finally we let $\delta = \nu = \sigma$.

iv. The density function of $W^{(p)}$. The distribution of Wilks' Λ denoted here by $W^{(p)} = |\underset{\sim}{I} + \lambda\underset{\sim}{R}|^{-1}$ has been derived starting from Theorem 2 and is given by

$$(4.6)\qquad f(W^{(p)}) = [\Gamma_p(\tfrac{1}{2}(n_1+n_2))/\Gamma_p(\tfrac{1}{2}n_2)]\, e^{-\mathrm{tr}\,\underset{\sim}{\Omega}} |\lambda\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} (W^{(p)})^{\frac{1}{2}(n_2-p-1)}$$

$$\cdot \sum_{k=0} \sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_{\kappa}\, (\frac{1}{2}n_1)_{\kappa}\, C_{\kappa}(\underset{\sim}{I})}{k!}\, G_{p,p}^{p,0}\left(W^{(p)} \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_p \end{matrix}\right)$$

$$\cdot \sum_{d=0}^{k} \sum_{\delta} \frac{a_{\kappa,\delta}\, C_{\delta}(-\lambda^{-1}\Lambda^{-1}) L_{\delta}^{\frac{1}{2}(n_1-p-1)}(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_{\delta}\, C_{\delta}(\underset{\sim}{I})\, C_{\delta}(\underset{\sim}{I})} ,$$

where $G(\cdot|\because)$ denotes Meijer's G-function, $a_i = \frac{1}{2}n_1 + k_{p-i+1}+b_i$ and $b_i = \frac{1}{2}(i-1)$.

As in the previous sections, formula (4.6) will give special cases as follows:

a) If we let $\underset{\sim}{\Omega} = \underset{\sim}{0}$, $L_{\delta}^{\gamma}(\underset{\sim}{0}) = (\frac{1}{2}n_1)_{\delta}\, C_{\delta}(\underset{\sim}{I})$ where $\gamma = \frac{1}{2}(n_1-p-1)$, and after making use of (3.13) we obtain formula (4.7) of Pillai, Al-Ani and Jouris [18] for testing the hypothesis H_0: $\gamma\underset{\sim}{\Lambda} = \underset{\sim}{I}$, $\gamma > 0$ being given.

b) For $\lambda = 1$ and $\underset{\sim}{\Lambda} = \underset{\sim}{I}$, and after making use of (4.5) we let $\kappa = \delta = \sigma$ (σ being the partition of s in the expansion of L_{δ}), then we have formula (5.2) of [18].

v. <u>The density function of the largest root</u>. Two expressions for the density function of the largest root r_p of $\underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ under violations are given below.

For $\underset{\sim}{\Lambda}$ "random" the density of the largest root is given by

$$(4.7) \qquad C(p,n_1,n_2)\, e^{-\operatorname{tr}\underset{\sim}{\Omega}} |\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}p+1)_\kappa (\frac{1}{2}(p-1))_\kappa}{k!(\frac{1}{2}(n_1+p+1))_\kappa (\frac{1}{2}p)_\kappa}$$

$$\cdot \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g_{\kappa,\nu} \frac{(\frac{1}{2}(n_1+n_2))_\delta C_\delta(r_p \underset{\sim}{\Lambda}^{-1})\, L_\delta^{\frac{1}{2}(n_1-p-1)}(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_\delta\, C_\delta(\underset{\sim}{I})} ,$$

where $C(p,n_1,n_2) = \dfrac{\Gamma(\frac{1}{2})\ \Gamma_p(\frac{n_1+n_2}{2})\Gamma_{p-1}(\frac{p}{2}+1)}{\Gamma(\frac{p}{2})\ \Gamma(\frac{n_1}{2})\ \Gamma_p(\frac{n_2}{2})\ \Gamma_{p-1}(\frac{n_1+p+1}{2})}$.

For $\underset{\sim}{\Omega} = \underset{\sim}{0}$ and $\kappa = \delta$, the density of r_p is

$$(4.8) \qquad f(r_p) = C(p,n_1,n_2)|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}(n_1+n_2))_\kappa (\frac{1}{2}p+1)_\kappa (\frac{1}{2}(p-1))_\kappa C_\kappa(r_p \underset{\sim}{\Lambda}^{-1})}{k!\ (\frac{1}{2}(n_1+p+1))_\kappa (\frac{1}{2}p)_\kappa}$$

$$= C(p,n_1,n_2)|\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} r_p^{\frac{1}{2}pn_1-1}\ {}_3F_2(\tfrac{1}{2}(n_1+n_2),\ \tfrac{1}{2}p+1, \tfrac{1}{2}(p-1);$$

$$\tfrac{1}{2}(n_1+p+1),\ \tfrac{1}{2}p;\ r_p \underset{\sim}{\Lambda}^{-1}) .$$

An alternate expression for the density function of r_p for $\underset{\sim}{\Lambda}$ "random" is further obtained as

$$(4.9) \qquad C(p,n_1,n_2)\ e^{-\operatorname{tr}\underset{\sim}{\Omega}} |\underset{\sim}{\Lambda}|^{-\frac{1}{2}n_1} |\underset{\sim}{I} + r_p \underset{\sim}{\Lambda}^{-1}|^{-\frac{1}{2}(n_1+n_2)} r_p^{\frac{1}{2}pn_1-1}$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}p+1)_\kappa (\frac{1}{2}(p-1))_\kappa}{k!(\frac{1}{2}(n_1+p+1))_\kappa (\frac{1}{2}p)_\kappa} \sum_{n=0}^{\infty} \sum_{\nu} \frac{(-1)^n}{n!} \sum_{\delta} g_{\kappa,\nu}^{\delta} \frac{L_\delta^{\frac{1}{2}(n_1-p-1)}(\underset{\sim}{\Omega})}{(\frac{1}{2}n_1)_\delta C_\delta(\underset{\sim}{I})}$$

$$\cdot \sum_{t=0}^{\infty} \sum_{\tau} \frac{1}{t!} \sum_{\mu} g_{\delta,\tau}^{\mu}\ (\tfrac{1}{2}(n_1+n_2))_\mu\ C_\mu[(\underset{\sim}{I} + r_p^{-1}\underset{\sim}{\Lambda})^{-1}] .$$

The density in (7.5) will give (16) of Khatri [6] if we let $\underset{\sim}{\Omega} = \underset{\sim}{0}$ and $n = t = 0$.

5. The density of T under equality of covariance matrices.

The density of T given in section 3(i) when $\underset{\sim}{\Lambda} = \underset{\sim}{I}$ is not quite useful since the series converges only for $|T| < 1$. Hence, on the basis of that density, the following form is suggested for the density of T.

$$(5.1)\quad \{e^{-\mathrm{tr}\,\underset{\sim}{\Omega}}\Gamma_p(\tfrac{1}{2}\nu)/\Gamma(\tfrac{1}{2}pn_1)\,\Gamma_p(\tfrac{1}{2}n_2)\}\,\frac{T^{\frac{1}{2}pn_1-1}}{(1+T/p)^{\frac{1}{2}\nu p}}\sum_{k=0}^{\infty}(-1)^k\,C_k\frac{(T/p)^k}{(1+T/p)^k},$$

where

$$(5.2)\qquad C_k = D_k - \sum_{r=0}^{k-1}\Big[\Big\{\prod_{j=r}^{k-1}(\tfrac{1}{2}\nu p+j)/(k-r)!\Big\}\,C_r\Big],\quad \nu = n_1+n_2$$

and where

$$(5.3)\qquad D_k = [p^k/\{(\tfrac{1}{2}pn_1)_k\;k!\}]\sum_{\kappa}(\tfrac{1}{2}\nu)_\kappa\,L_\kappa^\gamma(\underset{\sim}{\Omega}).$$

It may be observed that D_k is the coefficient of $(-T/p)^k$ in Constantine's form of the density of T [2]. The density (5.1) gives for p = 2 the exact distributional form of Hotelling (see Constantine [2]). Some further indication of the usefulness of the density (5.1) may be had from the comparison of percentage points in Table 1.

Table 1

Comparison of percentage points using terms up to k=8 or k=10 in Eq. (5.1) and exact points

p=3	n_2	Upper 5% Points		Upper 1% Points	
n_1=4 (k=8)		Eq. (5.1)	Exact	Eq. (5.1)	Exact
	24	1.1502	1.1540	1.5238	1.5581
	34	0.74668	0.74702	0.97765	0.98145
	44	0.55169	0.55174	0.71450	0.71518
	64	0.36208	0.36208	0.46310	0.46316
	84	0.26939	0.26939	0.34234	0.34235
	104	0.21447	0.21447	0.27149	0.27150
n_1=10 (k=10)	44	1.1637	1.1642	1.4104	1.4103
	64	0.76063	0.76070	0.90841	0.90860
	84	0.56470	0.56471	0.66981	0.66986
	104	0.44897	0.44897	0.53038	0.53039

p=4	n_2	Upper 5% Points		Upper 1% Points	
n_1=5 (k=8)		Eq. (5.1)	Exact	Eq. (5.1)	Exact
n_1=5 (k=8)	45	0.82538	0.82732	1.0214	1.0272
	65	0.54205	0.54221	0.66351	0.66455
	85	0.40312	0.40315	0.49077	0.49100
	105	0.32083	0.32083	0.38924	0.38929
	125	0.26643	0.26643	0.32246	0.32248
n_1=9 (k=10)	65	0.88312	0.88405	1.0369	1.0416
	85	0.65646	0.65663	0.76774	0.76869
	105	0.52220	0.52224	0.60887	0.60905
	125	0.43348	0.43349	0.50422	0.50430

The percentage points generally agree to the number of decimals the total probability is accurate. For p=5, n_1=6, n_2=56 probability has three decimals accuracy using terms up to k=8 in Eq. (5.1) and so also for n_1=10 and n_2=56 using up to k=10. For p=6, n_1=6, n_2 should be about 80 to give the same accuracy in probability. The exact points in Table 1 have been taken from Pillai and Young [17].

Table 2

Powers of T for p=3 (taking k upto 6) for testing $(\omega_1, \omega_2, \omega_3) = (0,0,0)$ against different simple alternative hypotheses $\alpha = 0.05$ and $\alpha = 0.01$.

ω_1	ω_2	ω_3	Power 5%	Power 1%
		$n_1 = 4, n_2 = 84$		
0	0	0.00001	0.050	0.010
0	0	0.001	0.050	0.010
0	0.005	0.005	0.050	0.010
0	0	0.003	0.050	0.010
0	0.0015	0.0015	0.050	0.010
0	0.0015	0.0015	0.050	0.010
0	0	0.111	0.056	0.012
0.001	0.01	0.1	0.056	0.012
		$n_1 = 4, n_2 = 124$		
0	0	0.00001	0.0500	0.0100
0	0	0.001	0.0500	0.0100
0	0	0.0005	0.0500	0.0100
0	0	0.003	0.0501	0.0100
0	0.0015	0.0015	0.0501	0.0100
0	0	0.111	0.056	0.012
0.001	0.01	0.1	0.056	0.012
		$n_1 = 4, n_2 = 300$		
0	0	0.00001	0.05000	0.01000
0	0	0.001	0.05005	0.01002
0	0.0005	0.0005	0.05005	0.01002
0	0	0.003	0.05015	0.01004
0	0.0015	0.0015	0.05015	0.01004
0	0	0.111	0.0557	0.0116
0.001	0.01	0.1	0.0557	0.0116
0	0	0.5	0.078	0.019
0	0.25	0.25	0.078	0.019
0.1	0.1	0.2	0.072	0.017
		$n_1 = 8, n_2 = 124$		
0	0	0.00001	0.050	0.010
0	0	0.001	0.050	0.010
0	0.0005	0.0005	0.050	0.010
0	0	0.003	0.050	0.010
0	0.0015	0.0015	0.050	0.010
0	0	0.111	0.054	0.011
0.001	0.01	0.1	0.054	0.011
		$n_1 = 8, n_2 = 300$		
0	0	0.00001	0.0500	0.0100
0	0	0.001	0.0500	0.0100
0	0.0005	0.0005	0.0500	0.0100
0	0	0.003	0.0501	0.0100
0	0.0015	0.0015	0.0501	0.0100
0	0	0.111	0.0538	0.0111
0.001	0.01	0.1	0.0538	0.0111
0	0	0.5	0.068	0.015
0	0.25	0.25	0.068	0.015
0.1	0.1	0.2	0.064	0.014

6. Numerical Study of robustness of tests of 1) and 2) based on four criteria. In this section are provided tables of upper tail probabilities in the two-roots case of $U^{(2)}$ (=T), $V^{(2)}$, $W^{(2)}$, and the larger root, using the upper 5 per cent points of the respective criteria under the null hypotheses of 1) and 2), for different values of ω_1, ω_2, λ_1 and λ_2 i.e. characteristic roots of $\underset{\sim}{\Omega}$ and $\underset{\sim}{\Lambda}$ respectively, and various values of $m = (n_1-p-1)/2$ and $n = (n_2-p-1)/2$. Table 3 presents the upper tail probabilities for m = 0 and Table 4 for m = 2, and for n = 5, 15 and 40. The c.d.f.'s of the criteria were obtained from (3.7) using methods similar to those available in Pillai and Jayachandran [13], [14], Pillai and Dotson [11] and Pillai and Al-Ani [10].

From the tabulations it appears that

1. For hypothesis 1) i.e. H_0: $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, the powers of tests based on all four criteria show considerable change even for small changes of ω_1 and ω_2. This probably is indicative that the tests are not robust against non-normality.
2. For hypothesis 2), the powers of tests based on all four criteria show modest changes for small deviations of λ_1 and λ_2 from unity but changes become pronounced as λ's deviate more from unity.
3. Tabulations do not reveal any advantage of one test statistic over the others in regard to either hypothesis. It is likely that tabulations for larger deviations may bring more light on this problem.

The above results are of an exact nature except for some of the assumptions made in the model unlike the large sample investigation of earlier authors, for example, Ito [3] and Ito and Schull [4].

I wish to acknowledge the assistance of Sudjana in the preparation of the paper. The tabulations in this paper have been taken from a joint paper under preparation with Sudjana.

Upper tail probabilities from the cdf's of four criteria under violations

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0	0.001	1	1.001	0.050106	0.050112	0.050109	0.050100
				0.050079	0.050084	0.050082	0.050073
				0.050026	0.050028	0.050027	0.050026
		1	1.01	0.050823	0.050869	0.050852	0.050769
				0.050796	0.050840	0.050824	0.050744
		1.025	1.025	0.054066	0.054301	0.054213	0.053783
				0.054038	0.054272	0.054184	0.053757
		1	1.05	0.054085	0.054280	0.054217	0.053821
				0.054057	0.054250	0.054188	0.053795
		1	1.1	0.058328	0.058646	0.058571	0.057811
				0.058298	0.058614	0.058538	0.05778
		1	1.3	0.0770	0.0771	0.0774	0.0756
				0.0769	0.0771	0.0774	0.0755
				m = 0, n = 15			
0	0.001	1	1.001	0.050133	0.050134	0.050134	0.050121
				0.050099	0.050100	0.050100	0.050093
				0.050033	0.050033	0.050033	0.050027
		1	1.01	0.051035	0.051043	0.051041	0.050967
				0.051001	0.051009	0.051007	0.050935
		1.025	1.025	0.055132	0.055179	0.055165	0.054795
				0.055096	0.055143	0.055130	0.054762
		1	1.05	0.055162	0.055190	0.055187	0.054858
				0.055127	0.055154	0.055151	0.054825
		1	1.1	0.060585	0.060604	0.060619	0.060013
				0.060546	0.060565	0.060580	0.05998
		1	1.3	0.0849	0.0845	0.0847	0.0835
				0.0848	0.0845	0.0847	0.0835
				m = 0, n = 40			
0	0.001	1	1.001	0.050145	0.050145	0.050145	0.050132
				0.050108	0.050109	0.050109	0.050102
				0.050036	0.050036	0.050036	0.050029
		1	1.01	0.051131	0.051132	0.051132	0.051063
				0.051094	0.051095	0.051095	0.051028
		1.025	1.025	0.055617	0.055625	0.055623	0.055285
				0.055577	0.055586	0.055584	0.055248
		1	1.05	0.055652	0.055652	0.055654	0.055361
				0.055613	0.055613	0.055615	0.055324
		1	1.1	0.061617	0.061601	0.061613	0.061087
				0.061574	0.061558	0.061571	0.06105
		1	1.3	0.0885	0.0883	0.0883	0.0874
				0.0885	0.0882	0.0883	0.0874

Entries in 2nd row denote powers of the test H_o: $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ assuming $\underset{\sim}{\Omega} = \underset{\sim}{0}$

Entries in 3rd row denote powers of the test H_o: $\underset{\sim}{\Omega} = \underset{\sim}{0}$ assuming $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$.

Table 3 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0	0.01	1	1.001	0.050344	0.050364	0.050356	0.050322
				0.050079	0.050084	0.050082	0.050073
				0.050264	0.050280	0.050274	0.050248
		1	1.01	0.051064	0.051123	0.051101	0.050993
				0.050796	0.050840	0.050824	0.050744
		1.025	1.025	0.054319	0.054569	0.054475	0.054018
				0.054038	0.054272	0.054184	0.053757
		1	1.05	0.054338	0.054547	0.054479	0.054056
				0.054057	0.054250	0.054188	0.053795
		1	1.1	0.058596	0.058930	0.058849	0.058060
				0.058298	0.058614	0.058538	0.05778
		1	1.3	0.0773	0.0775	0.0778	0.0759
				0.0769	0.0771	0.0774	0.0755
				m = 0, n = 15			
0	0.01	1	1.001	0.050432	0.050436	0.050435	0.050401
				0.050099	0.050100	0.050100	0.050093
				0.050332	0.050335	0.050334	0.050307
		1	1.01	0.051338	0.051349	0.051346	0.051251
				0.051001	0.051009	0.051007	0.050935
		1.025	1.025	0.055453	0.055503	0.055489	0.055095
				0.055096	0.055143	0.055130	0.054762
		1	1.05	0.055483	0.055514	0.055510	0.055158
				0.055127	0.055154	0.055151	0.054825
		1	1.1	0.050928	0.060951	0.060965	0.060333
				0.060546	0.060565	0.060580	0.05998
		1	1.3	0.0853	0.0850	0.0852	0.0839
				0.0848	0.0845	0.0847	0.0835
				m = 0, n = 40			
0	0.01	1	1.001	0.050472	0.050473	0.050472	0.050440
				0.050108	0.050109	0.050109	0.050102
				0.050363	0.050363	0.050363	0.050337
		1	1.01	0.051462	0.051464	0.051464	0.051376
				0.051094	0.051095	0.051095	0.051028
		1.025	1.025	0.055969	0.055978	0.055975	0.055616
				0.055577	0.055586	0.055584	0.055248
		1	1.05	0.056004	0.056005	0.056007	0.055693
				0.055613	0.055613	0.055615	0.055324
		1	1.1	0.061995	0.061980	0.061993	0.061442
				0.061574	0.061558	0.061571	0.06105
		1	1.3	0.0890	0.0888	0.0888	0.0879
				0.0885	0.0882	0.0883	0.0874

Table 3 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				$m = 0, n = 5$			
0	0.1	1	1.001	0.052750	0.052897	0.052844	0.052567
				0.050079	0.050084	0.050082	0.050073
				0.052667	0.052809	0.052758	0.052490
		1	1.01	0.053497	0.053686	0.053618	0.053262
				0.050796	0.050840	0.050824	0.050744
		1.025	1.025	0.056874	0.057262	0.057120	0.056394
				0.054038	0.054272	0.054184	0.053757
		1	1.05	0.056893	0.057239	0.057124	0.056432
				0.054057	0.054250	0.054188	0.053795
		1	1.1	0.06131	0.06178	0.06166	0.06057
				0.058298	0.058615	0.058538	0.05778
		1	1.3	0.081	0.081	0.081	0.080
				0.0769	0.0771	0.0774	0.0755
				$n = 0, n = 15$			
0	0.1	1	1.001	0.053462	0.053488	0.053482	0.053246
				0.050099	0.050100	0.050100	0.050093
				0.053357	0.053383	0.053377	0.053149
		1	1.01	0.054408	0.054442	0.054434	0.054132
				0.051001	0.051009	0.051007	0.050935
		1.025	1.025	0.058703	0.058778	0.058757	0.058135
				0.055096	0.055143	0.055130	0.054762
		1	1.05	0.058733	0.058788	0.058779	0.058199
				0.055127	0.055154	0.055151	0.054825
		1	1.1	0.06440	0.06445	0.06446	0.06358
				0.060546	0.060565	0.060580	0.05998
		1	1.3	0.090	0.090	0.090	0.088
				0.0848	0.0845	0.0847	0.0835
				$m = 0, n = 40$			
0	0.1	1	1.001	0.053785	0.053788	0.053788	0.053575
				0.050108	0.050109	0.050109	0.050102
				0.053670	0.053673	0.053673	0.053467
		1	1.01	0.054821	0.054826	0.054826	0.054552
				0.051094	0.051095	0.051095	0.051028
		1.025	1.025	0.059536	0.059548	0.059546	0.058980
				0.055577	0.055586	0.055584	0.055248
		1	1.05	0.059572	0.059575	0.059577	0.059058
				0.055613	0.055613	0.055615	0.055324
		1	1.1	0.06582	0.06581	0.06582	0.06504
				0.06157	0.06156	0.06157	0.06105
		1	1.3	0.094	0.094	0.094	0.092
				0.089	0.088	0.088	0.087

Table 3 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0	0.5	1	1.001	0.063932	0.064464	0.064338	0.063065
				0.050079	0.050084	0.050082	0.050073
				0.063836	0.064362	0.064237	0.062977
		1	1.01	0.064803	0.065382	0.065241	0.063869
				0.050796	0.050840	0.050824	0.050744
		1.025	1.025	0.06873	0.06954	0.06932	0.06748
				0.054038	0.054272	0.054184	0.053757
		1	1.05	0.06875	0.06951	0.06932	0.06752
				0.05406	0.05425	0.05419	0.05379
		1	1.1	0.0739	0.0748	0.0746	0.0723
				0.0583	0.0586	0.0585	0.0578
		1	1.3	0.096	0.096	0.096	0.093
				0.077	0.077	0.077	0.076
				m = 0, n = 15			
0	0.5	1	1.001	0.067712	0.067744	0.067770	0.066757
				0.050099	0.050100	0.050100	0.050093
				0.067588	0.067619	0.067644	0.066642
		1	1.01	0.068839	0.068880	0.068904	0.067804
				0.051001	0.051009	0.051007	0.050935
		1.025	1.025	0.07394	0.07402	0.07404	0.07453
				0.055096	0.055143	0.055130	0.054762
		1	1.05	0.07397	0.07403	0.07406	0.07259
				0.05513	0.05515	0.05515	0.05482
		1	1.1	0.0806	0.0807	0.0808	0.0789
				0.0605	0.0606	0.0606	0.0600
		1	1.3	0.110	0.110	0.110	0.107
				0.085	0.085	0.085	0.0835
				m = 0, n = 40			
0	0.5	1	1.001	0.069442	0.069415	0.069437	0.068559
				0.050108	0.050109	0.050109	0.050102
				0.069304	0.069277	0.069299	0.068430
		1	1.01	0.070688	0.070662	0.070683	0.069726
				0.051094	0.051095	0.051095	0.051028
		1.025	1.025	0.07633	0.07631	0.07633	0.07500
				0.05558	0.05559	0.05558	0.05525
		1	1.05	0.07637	0.07634	0.07636	0.07508
				0.05561	0.05561	0.05562	0.05532
		1	1.1	0.0838	0.0837	0.0838	0.0821
				0.0616	0.0616	0.0616	0.0611
		1	1.3	0.116	0.115	0.115	0.114
				0.089	0.088	0.088	0.087

Table 3 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
0	1	1	1.001	0.07899	0.07958	0.07967	0.07733
				0.050079	0.050084	0.050082	0.050073
				0.078873	0.079463	0.079538	0.077229
		1	1.01	0.08001	0.08066	0.08074	0.07828
				0.05080	0.05084	0.05082	0.05074
		1.025	1.025	0.0846	0.0855	0.0855	0.0825
				0.0540	0.0543	0.0542	0.0538
		1	1.05	0.0847	0.0855	0.0855	0.0826
				0.0541	0.0543	0.0542	0.0538
		1	1.1	0.091	0.091	0.091	0.088
				0.058	0.059	0.059	0.058
				m = 0, n = 15			
0	1	1	1.001	0.08723	0.08706	0.08725	0.08554
				0.050099	0.050100	0.050100	0.050093
				0.087084	0.086913	0.087101	0.085403
		1	1.01	0.08859	0.08843	0.08862	0.08679
				0.05100	0.05101	0.05101	0.05094
		1.025	1.025	0.0947	0.0946	0.0948	0.0924
				0.0551	0.0551	0.0551	0.0548
		1	1.05	0.0947	0.0946	0.0948	0.0925
				0.0551	0.0552	0.0552	0.0548
		1	1.	0.103	0.103	0.103	0.100
				0.061	0.061	0.061	0.060
				m = 0, n = 40			
0	1	1	1.001	0.09104	0.09088	0.09098	0.08958
				0.050108	0.050109	0.050109	0.050102
				0.090873	0.090717	0.090823	0.089426
		1	1.01	0.09255	0.09239	0.09249	0.09099
				0.051094	0.051095	0.051095	0.051028
		1.025	1.025	0.0994	0.0992	0.0993	0.0973
				0.0556	0.0556	0.0556	0.0552
		1	1.05	0.0994	0.0992	0.0993	0.0974
				0.0556	0.0556	0.0556	0.0553
		1	1.1	0.108	0.108	0.108	0.106
				0.062	0.062	0.062	0.061

Table 3 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 0, n = 5			
1	1	1	1.001	0.1124	0.1203	0.1164	0.1049
				0.0501	0.0501	0.0501	0.0501
				0.1122	0.1201	0.1164	0.1048
		1	1.01	0.1138	0.1217	0.1178	0.1062
				0.0508	0.0508	0.0508	0.0507
		1.025	1.025	0.128	0.128	0.124	0.112
				0.054	0.054	0.054	0.054
		1	1.05	0.120	0.128	0.124	0.112
				0.054	0.054	0.054	0.054
		1	1.1	0.128	0.138	0.132	0.119
				0.058	0.059	0.059	0.058
				m = 0, n = 15			
1	1	1	1.001	0.1314	0.1341	0.1327	0.1214
				0.0501	0.0501	0.0501	0.0501
				0.1312	0.1339	0.1326	0.1213
		1	1.01	0.1332	0.1359	0.1345	0.1231
				0.0510	0.0510	0.0510	0.0509
		1.025	1.025	0.141	0.144	0.143	0.130
				0.055	0.055	0.055	0.055
		1	1.05	0.141	0.144	0.143	0.130
				0.055	0.055	0.055	0.055
		1	1.1	0.152	0.153	0.153	0.140
				0.061	0.061	0.061	0.060
				m = 0, n = 40			
1	1	1	1.001	0.1402	0.1412	0.1406	0.1295
				0.0501	0.0501	0.0501	0.0501
				0.1400	0.1409	0.1405	0.1293
		1	1.01	0.1422	0.1432	0.1426	0.1314
				0.0511	0.0511	0.0511	0.0510
		1.025	1.025	0.151	0.152	0.152	0.140
				0.056	0.056	0.056	0.055
		1	1.05	0.151	0.152	0.152	0.104
				0.056	0.056	0.056	0.055
		1	1.1	0.163	0.163	0.163	0.151
				0.062	0.062	0.062	0.061

Upper tail probabilities from the cdf's of four criteria under violations

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 5			
0	0.001	1	1.001	0.050109	0.050122	0.050118	0.050100
				0.050095	0.050107	0.050103	0.050084
				0.050014	0.050015	0.050015	0.050012
		1	1.01	0.050970	0.051089	0.051048	0.050862
				0.050956	0.051074	0.051033	0.050850
		1.025	1.025	0.054894	0.055526	0.055306	0.054310
				0.054879	0.055510	0.055290	0.054297
		1	1.05	0.054915	0.055454	0.055287	0.054371
				0.054901	0.055437	0.055271	0.054358
		1	1.1	0.06012	0.06106	0.06082	0.05904
				0.060100	0.061036	0.060793	0.05901
		1	1.3	0.085	0.086	0.085	0.081
				0.084	0.086	0.085	0.081
				m = 2, n = 15			
0	0.001	1	1.001	0.050150	0.050153	0.050152	0.050137
				0.050130	0.050134	0.050133	0.050115
				0.050019	0.050019	0.050019	0.050017
		1	1.01	0.051339	0.051368	0.051360	0.051192
				0.051320	0.051349	0.051341	0.051175
		1.025	1.025	0.056812	0.056980	0.056931	0.056005
				0.056791	0.056958	0.056910	0.055987
		1	1.05	0.056847	0.056949	0.056936	0.056129
				0.056827	0.056928	0.056915	0.056111
		1	1.1	0.06424	0.06433	0.06437	0.06287
				0.064218	0.064302	0.064347	0.06284
		1	1.3	0.100	0.099	0.098	0.096
				0.100	0.098	0.098	0.096
				m = 2, n = 40			
0	0.001	1	1.001	0.050171	0.050172	0.050171	0.050160
				0.050149	0.050150	0.050150	0.050134
				0.050021	0.050021	0.050021	0.050019
		1	1.01	0.051530	0.051535	0.051534	0.051383
				0.051508	0.051513	0.051512	0.051363
		1.025	1.025	0.057816	0.057850	0.057841	0.056983
				0.057792	0.057826	0.057817	0.056962
		1	1.05	0.057859	0.057859	0.057868	0.057152
				0.057835	0.057835	0.057844	0.057130
		1	1.1	0.06643	0.06636	0.06641	0.06513
				0.066400	0.066336	0.066388	0.06509
		1	1.3	0.108	0.107	0.106	0.105
				0.108	0.107	0.106	0.105

Entries in 2^{nd} row denote powers of the test H_o: $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$ assuming $\underset{\sim}{\Omega} = \underset{\sim}{0}$

Entries in 3^{rd} row denote powers of the test H_o: $\underset{\sim}{\Omega} = 0$ assuming $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$.

Table 4 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 5			
0	0.01	1	0.001	0.050231	0.050260	0.050250	0.050208
				0.050095	0.050107	0.050103	0.050084
				0.050136	0.050153	0.050147	0.050124
		1	1.01	0.051094	0.051229	0.051183	0.050971
				0.050956	0.051074	0.051033	0.050850
		1.025	1.025	0.055026	0.05567	0.055449	0.054426
				0.054879	0.055510	0.055290	0.054297
		1	1.05	0.055048	0.055603	0.055430	0.054487
				0.054901	0.055437	0.055271	0.054358
		1	1.1	0.06026	0.06122	0.06097	0.05916
				0.060100	0.06104	0.06079	0.05901
		1	1.3	0.085	0.086	0.086	0.082
				0.084	0.086	0.085	0.081
				m = 2, n = 15			
0	0.01	1	1.001	0.050318	0.050326	0.050324	0.050287
				0.050130	0.050134	0.050133	0.050115
				0.050187	0.050192	0.050190	0.050170
		1	1.01	0.051510	0.051544	0.051535	0.051344
				0.051320	0.051349	0.051341	0.051175
		1.025	1.025	0.056998	0.057170	0.057120	0.056168
				0.056791	0.056958	0.056910	0.055987
		1	1.05	0.057033	0.057139	0.057125	0.056292
				0.056827	0.056928	0.056915	0.056111
		1	1.1	0.06445	0.06454	0.06458	0.06305
				0.064218	0.064302	0.064347	0.06284
		1	1.3	0.100	0.099	0.099	0.097
				0.100	0.098	0.098	0.096
				m = 2, n = 40			
0	0.01	1	1.001	0.050363	0.050365	0.050365	0.050333
				0.050149	0.050150	0.050150	0.050134
				0.050214	0.050214	0.050214	0.050199
		1	1.01	0.051726	0.051732	0.051731	0.051559
				0.051508	0.051513	0.051512	0.051363
		1.025	1.025	0.058030	0.058065	0.058056	0.057173
				0.057792	0.057826	0.057817	0.056962
		1	1.05	0.058073	0.058074	0.058083	0.057342
				0.057835	0.057834	0.057844	0.057130
		1	1.1	0.06664	0.06660	0.06665	0.06534
				0.06640	0.06634	0.06639	0.06509
		1	1.3	0.108	0.107	0.106	0.106
				0.108	0.107	0.106	0.105

Table 4 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 5			
0	0.1	1	1.001	0.051464	0.051644	0.051582	0.051299
				0.050095	0.050107	0.050103	0.050084
				0.051367	0.051534	0.051477	0.051213
		1	1.01	0.052345	0.052634	0.052535	0.052076
				0.050956	0.051074	0.051033	0.050850
		1.025	1.025	0.05636	0.05718	0.05689	0.05559
				0.05488	0.05551	0.05529	0.05430
		1	1.05	0.05638	0.05710	0.05687	0.05565
				0.05490	0.05544	0.05527	0.05436
		1	1.1	0.0616	0.0628	0.0625	0.0604
				0.0601	0.0610	0.0608	0.0590
		1	1.3	0.086	0.088	0.087	0.083
				0.084	0.086	0.085	0.081
				m = 2, n = 15			
0	0.1	1	1.001	0.052022	0.052066	0.052054	0.051798
				0.050130	0.050134	0.050133	0.050115
				0.051887	0.051928	0.051917	0.051679
		1	1.01	0.053245	0.053317	0.053298	0.052881
				0.051320	0.051349	0.051341	0.051175
		1.025	1.025	0.05887	0.05909	0.05903	0.05782
				0.05679	0.05696	0.05691	0.05599
		1	1.05	0.05891	0.05905	0.05903	0.05794
				0.05683	0.05693	0.05692	0.05611
		1	1.1	0.0665	0.0666	0.0667	0.0648
				0.0642	0.0643	0.0643	0.0628
		1	1.3	0.103	0.103	0.103	0.099
				0.100	0.098	0.098	0.096
				m = 2, n = 40			
0	0.1	1	1.001	0.052311	0.052318	0.052317	0.052085
				0.050149	0.050150	0.050150	0.050134
				0.052157	0.052164	0.052162	0.051947
		1	1.01	0.053713	0.053725	0.053724	0.053343
				0.051508	0.051513	0.051512	0.051363
		1.025	1.025	0.05019	0.06024	0.06023	0.05910
				0.05779	0.05783	0.05782	0.05696
		1	1.05	0.06024	0.06024	0.06025	0.05927
				0.05784	0.05784	0.05784	0.05713
		1	1.1	0.0691	0.0690	0.0690	0.0675
				0.0664	0.0663	0.0664	0.0651
		1	1.3	0.112	0.111	0.110	0.109
				0.108	0.107	0.106	0.105

Table 4 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 5			
0	0.5	1	1.001	0.057131	0.057902	0.057666	0.056342
				0.050095	0.050107	0.050103	0.050084
				0.057025	0.057782	0.057550	0.056249
		1	1.01	0.058091	0.058985	0.058709	0.057182
				0.050956	0.051074	0.051033	0.050850
		1.025	1.025	0.06246	0.06385	0.06345	0.06098
				0.05488	0.05551	0.05529	0.05430
		1	1.05	0.06248	0.06387	0.06347	0.06105
				0.05490	0.05544	0.05527	0.05436
		1	1.1	0.0683	0.0701	0.0696	0.0662
				0.0601	0.0610	0.0608	0.0590
		1	1.3	0.096	0.098	0.097	0.092
				0.084	0.086	0.085	0.081
				m = 2, n = 15			
0	0.5	1	1.001	0.059949	0.060087	0.060074	0.058910
				0.050130	0.050134	0.050133	0.050115
				0.059800	0.059934	0.059921	0.058778
		1	1.01	0.061313	0.061482	0.061461	0.060105
				0.051320	0.051349	0.051341	0.051175
		1.025	1.025	0.06757	0.06790	0.06784	0.06555
				0.05679	0.05696	0.05691	0.05599
		1	1.05	0.06761	0.06787	0.06784	0.06568
				0.05683	0.05693	0.05692	0.05611
		1	1.1	0.0760	0.0763	0.0763	0.0733
				0.0642	0.0643	0.0643	0.0628
		1	1.3	0.116	0.115	0.115	0.111
				0.100	0.098	0.098	0.096
				m = 2, n = 40			
0	0.5	1	1.001	0.061426	0.061422	0.061437	0.060405
				0.050149	0.050150	0.050150	0.050134
				0.061252	0.061248	0.061262	0.060251
		1	1.01	0.063006	0.063007	0.063021	0.061811
				0.051508	0.051513	0.051512	0.051363
		1.025	1.025	0.07032	0.07032	0.07033	0.06823
				0.05779	0.05783	0.05782	0.05696
		1	1.05	0.07033	0.07033	0.07035	0.06841
				0.05784	0.05784	0.05784	0.05713
		1	1.1	0.0802	0.0801	0.0802	0.0775
				0.0664	0.0663	0.0664	0.0651
		1	1.3	0.127	0.126	0.126	0.123
				0.108	0.107	0.106	0.105

Table 4 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 5			
0	1	1	1.001	0.06464	0.06596	0.06564	0.06309
				0.05010	0.05011	0.05010	0.05008
				0.064523	0.065823	0.065504	0.06299
		1	1.01	0.06570	0.06716	0.06680	0.06401
				0.05096	0.05107	0.05103	0.05085
		1.025	1.025	0.0705	0.0727	0.0721	0.0682
				0.0549	0.0555	0.0553	0.0543
		1	1.05	0.0706	0.0726	0.0721	0.0683
				0.0549	0.0554	0.0553	0.0544
		1	1.1	0.077	0.080	0.079	0.074
				0.060	0.060	0.061	0.060
				m = 2, n = 15			
0	1	1	1.001	0.07067	0.07075	0.07084	0.06870
				0.05013	0.05013	0.05013	0.05012
				0.070496	0.070579	0.070666	0.06855
		1	1.01	0.07221	0.07233	0.07241	0.07004
				0.05132	0.05135	0.05134	0.05118
		1.025	1.025	0.0793	0.0796	0.0796	0.0763
				0.0568	0.0570	0.0569	0.0560
		1	1.05	0.0793	0.0796	0.0796	0.0763
				0.0568	0.0569	0.0569	0.0561
		1	1.1	0.089	0.089	0.089	0.085
				0.064	0.064	0.064	0.063
				m = 2, n = 40			
0	1	1	1.001	0.07387	0.07375	0.07384	0.07203
				0.05012	0.05015	0.05015	0.05013
				0.073667	0.073556	0.073641	0.07185
		1	1.01	0.07568	0.07556	0.07565	0.07362
				0.05151	0.05151	0.05151	0.05136
		1.025	1.025	0.0840	0.0839	0.0840	0.0809
				0.0578	0.0578	0.0578	0.0570
		1	1.05	0.0840	0.0839	0.0840	0.0809
				0.0578	0.0578	0.0578	0.0571
		1	1.1	0.095	0.095	0.095	0.091
				0.066	0.066	0.066	0.065

Table 4 (Contd.)

ω_1	ω_2	λ_1	λ_2	$U^{(2)}$	$V^{(2)}$	$W^{(2)}$	Largest Root
				m = 2, n = 5			
1	1	1	1.001	0.0809	0.0861	0.0840	0.0764
				0.0501	0.0501	0.0501	0.0501
				0.0808	0.0858	0.0839	0.0763
		1	1.01	0.0822	0.0875	0.0854	0.0775
				0.0510	0.0511	0.0510	0.0509
		1.025	1.025	0.088	0.094	0.092	0.083
				0.055	0.056	0.055	0.054
		1	1.05	0.088	0.094	0.092	0.083
				0.055	0.055	0.055	0.054
		1	1.1	0.096	0.099	0.099	0.090
				0.060	0.060	0.061	0.060
				m = 2, n = 15			
1	1	1	1.001	0.0947	0.0964	0.0957	0.0878
				0.0501	0.0501	0.0501	0.0501
				0.0944	0.0961	0.0955	0.0876
		1	1.01	0.0966	0.0984	0.0976	0.0895
				0.0513	0.0514	0.0513	0.0512
		1.025	1.025	0.105	0.107	0.106	0.097
				0.057	0.057	0.057	0.056
		1	1.05	0.106	0.108	0.106	0.097
				0.057	0.057	0.057	0.056
		1	1.1	0.117	0.117	0.117	0.108
				0.064	0.064	0.064	0.063
				m = 2, n = 40			
1	1	1	1.001	0.1021	0.1026	0.1023	0.0946
				0.0501	0.0502	0.0502	0.0501
				0.1018	0.1023	0.1021	0.0944
		1	1.01	0.1044	0.1049	0.1046	0.0966
				0.0515	0.0515	0.0515	0.0514
		1.025	1.025	0.115	0.115	0.115	0.106
				0.058	0.058	0.058	0.057
		1	1.05	0.115	0.115	0.115	0.106
				0.058	0.058	0.058	0.057
		1	1.1	0.129	0.129	0.129	0.118
				0.066	0.066	0.066	0.065

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13. ABSTRACT

The paper deals with robustness studies of tests of two hypotheses, namely 1) equality of covariance matrices in two p-variate normal populations and 2) equality of p-dimensional mean vectors in ℓ p-variate normal populations having a common unknown covariance matrix, based on four test criteria a) Hotelling's trace b) Pillai's trace c) Wilks' Λ and d) Roy's largest root. The robustness studied for 1) is against non-normality and for 2) against unequal covariance matrices. In this connection, Pillai's distribution of the characteristic roots of $S_1 S_2^{-1}$ under violations is used to derive the moments or distributions of criteria, where S_1 is $W(p,n_1,\Sigma_1,\Omega)$ and S_2, $W(p,n_1,\Sigma_2,0)$, under a condition on Ω or $\Sigma_1 \Sigma_2^{-1}$. Numerical comparisons of powers of the tests of the two hypotheses based on the four criteria are made for the two-roots case. In addition, a new distributional form is suggested for Hotelling's trace (T) as a series in powers of (T/p)/(1+T/p) which is exact in the two-roots case.

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